Received 15/06/15

INEQUALITIES OF HERMITE-HADAMARD TYPE FOR *HH*-CONVEX FUNCTIONS

S. S. DRAGOMIR[1,2]

Abstract. Some inequalities of Hermite-Hadamard type for HH-convex functions defined on positive intervals are given. Applications for special means are also provided.

1. Introduction

Following [4] (see also [42]) we say that the function $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is *HA-convex* if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le (1-t) f(x) + tf(y) \tag{1.1}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.1) is reversed, then f is said to be *HA-concave*.

If $I \subset (0,\infty)$ and f is convex and nondecreasing function then f is HA-convex and if f is HA-convex and nonincreasing function then f is convex.

If $[a,b] \subset I \subset (0,\infty)$ and if we consider the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$, defined by $g(t) = f\left(\frac{1}{t}\right)$, then we can state the following fact [4]:

Lemma 1. *The function f is HA-convex (concave) on $[a,b]$ if and only if g is convex (concave) in the usual sense on $\left[\frac{1}{b}, \frac{1}{a}\right]$.*

Therefore, as examples of HA-convex functions we can take $f(t) = g\left(\frac{1}{t}\right)$, where g is any convex function on $\left[\frac{1}{b}, \frac{1}{a}\right]$.

In the recent paper [27] we obtained the following characterization result as well:

Lemma 2. *Let $f, h : [a,b] \subset (0,\infty) \to \mathbb{R}$ be so that $h(t) = tf(t)$ for $t \in [a,b]$. Then f is HA-convex (concave) on the interval $[a,b]$ if and only if h is convex (concave) on $[a,b]$.*

Following [4] (see also [42]) we say that the function $f : I \subset \mathbb{R} \setminus \{0\} \to (0,\infty)$ is *HH-convex* if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le \frac{f(x) f(y)}{(1-t) f(y) + tf(x)} \tag{1.2}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.2) is reversed, then f is said to be *HH-concave*.

We observe that the inequality (1.2) is equivalent to

$$(1-t)\frac{1}{f(x)} + t\frac{1}{f(y)} \le \frac{1}{f\left(\frac{xy}{tx+(1-t)y}\right)} \tag{1.3}$$

1991 *Mathematics Subject Classification.* 26D15; 25D10.

Key words and phrases. Convex functions, Integral inequalities, HH-Convex functions.

for all $x, y \in I$ and $t \in [0,1]$.

Therefore we have the following fact:

Lemma 3. *The function* $f : I \subset \mathbb{R} \setminus \{0\} \to (0, \infty)$ *is HH-convex (concave) on* I *if and only if* $g : I \subset \mathbb{R} \setminus \{0\} \to (0, \infty)$, $g(x) = \frac{1}{f(x)}$ *is HA-concave (convex) on* I.

Taking into account the above lemmas, we can state the following result:

Proposition 1. *Let* $f : [a, b] \subset (0, \infty) \to (0, \infty)$ *and define the related functions* $P_f : \left[\frac{1}{b}, \frac{1}{a}\right] \subset (0, \infty) \to (0, \infty)$, $P_f(x) = \frac{1}{f\left(\frac{1}{x}\right)}$ *and* $Q_f : [a, b] \subset (0, \infty) \to (0, \infty)$, $Q_f(x) = \frac{x}{f(x)}$. *The following statements are equivalent:*

(i) The function f *is HH-convex (concave) on* $[a, b]$;

(ii) The function P_f *is concave (convex) on* $\left[\frac{1}{b}, \frac{1}{a}\right]$;

(iii) The function Q_f *is concave (convex) on* $[a, b]$.

For a convex function $h : [c, d] \to \mathbb{R}$, the following inequality is well known in the literature as the *Hermite-Hadamard inequality*

$$h\left(\frac{c+d}{2}\right) \le \frac{1}{d-c} \int_c^d h(t)\, dt \le \frac{h(c) + h(d)}{2}. \tag{1.4}$$

For related results, see [1]-[20], [23]-[28], [29]-[38] and [39]-[50].

Motivated by the above results, we establish in this paper some inequalities of Hermite-Hadamard type for *HH*-convex functions defined on positive intervals. Applications for special means are also provided.

2. The Results

We have the following result that can be obtained by the use of the regular Hermite-Hadamard inequality (1.4):

Theorem 1. *Let* $f : [a, b] \subset (0, \infty) \to (0, \infty)$ *be a HH-convex (concave) function on* $[a, b]$. *Then we have*

$$f\left(\frac{2ab}{a+b}\right) \ge (\le) \frac{ab}{b-a} \int_a^b \frac{1}{t^2 f(t)} dt \ge (\le) \frac{f(b) + f(a)}{2} \tag{2.1}$$

and

$$\frac{\frac{a+b}{2}}{f\left(\frac{a+b}{2}\right)} \ge (\le) \frac{1}{b-a} \int_a^b \frac{t}{f(t)} dt \ge (\le) \frac{af(b) + bf(a)}{2f(a)f(b)}. \tag{2.2}$$

Proof. Since f is *HH*-convex (concave) on $[a, b]$, then by the Proposition 1 we have that P_f is concave (convex) on $\left[\frac{1}{b}, \frac{1}{a}\right]$. By Hermite-Hadamard inequality (1.4) for P_f we have

$$f\left(\frac{1}{\frac{\frac{1}{a}+\frac{1}{b}}{2}}\right) \ge (\le) \frac{1}{\frac{1}{a} - \frac{1}{b}} \int_{\frac{1}{b}}^{\frac{1}{a}} \frac{1}{f\left(\frac{1}{s}\right)} ds \ge (\le) \frac{f\left(\frac{1}{\frac{1}{b}}\right) + f\left(\frac{1}{\frac{1}{a}}\right)}{2},$$

which is equivalent to

$$f\left(\frac{2ab}{a+b}\right) \ge (\le) \frac{ab}{b-a} \int_{\frac{1}{b}}^{\frac{1}{a}} \frac{1}{f\left(\frac{1}{s}\right)} ds \ge (\le) \frac{f(b) + f(a)}{2}. \tag{2.3}$$

If we make the change of variable $\frac{1}{s} = t$, then $s = \frac{1}{t}$ and $ds = -\frac{dt}{t^2}$ and from (2.3) we get (2.1).

Since f is HH-convex (concave) on $[a,b]$, then by the Proposition 1 we also have that Q_f is concave (convex) on $[a,b]$. By Hermite-Hadamard inequality (1.4) for Q_f we have

$$\frac{\frac{a+b}{2}}{f\left(\frac{a+b}{2}\right)} \geq (\leq) \frac{1}{b-a}\int_a^b \frac{t}{f(t)}dt \geq (\leq) \frac{\frac{a}{f(a)}+\frac{b}{f(b)}}{2},$$

which is equivalent to (2.2). □

We use the following result obtained by the author in [21] and [22]:

Lemma 4. *Let $h:[\alpha,\beta]\to\mathbb{R}$ be a convex (concave) function on $[\alpha,\beta]$. Then we have the inequalities*

$$0 \leq (\geq) \frac{h(\alpha)+h(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,dt \leq (\geq) \frac{1}{8}\left[h'_-(\beta)-h'_+(\alpha)\right](\beta-\alpha) \tag{2.4}$$

and

$$0 \leq (\geq) \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,dt - h\left(\frac{\alpha+\beta}{2}\right) \leq (\geq) \frac{1}{8}\left[h'_-(\beta)-h'_+(\alpha)\right](\beta-\alpha). \tag{2.5}$$

The constant $\frac{1}{8}$ is best possible in (2.4) and (2.5).

We have the following reverse inequalities:

Theorem 2. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a HH-convex (concave) function on $[a,b]$. Then we have*

$$\begin{aligned} 0 &\geq (\leq) \frac{f(b)+f(a)}{2} - \frac{ab}{b-a}\int_a^b \frac{1}{t^2 f(t)}dt \\ &\geq (\leq) \frac{1}{8ab}\left(\frac{a^2}{f^2(a)}f'_+(a) - \frac{b^2}{f^2(b)}f'_-(b)\right)(b-a), \end{aligned} \tag{2.6}$$

$$\begin{aligned} 0 &\geq (\leq) \frac{ab}{b-a}\int_a^b \frac{1}{t^2 f(t)}dt - f\left(\frac{2ab}{a+b}\right) \geq (\leq) \\ &\geq (\leq) \frac{1}{8ab}\left(\frac{a^2}{f^2(a)}f'_+(a) - \frac{b^2}{f^2(b)}f'_-(b)\right)(b-a), \end{aligned} \tag{2.7}$$

$$\begin{aligned} 0 &\geq (\leq) \frac{af(b)+bf(a)}{2f(a)f(b)} - \frac{1}{b-a}\int_a^b \frac{t}{f(t)}dt \\ &\geq (\leq) \frac{1}{8}\left(\frac{f(b)-bf'_-(b)}{f^2(b)} - \frac{f(a)-af'_+(a)}{f^2(a)}\right)(b-a) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} 0 &\geq (\leq) \frac{1}{b-a}\int_a^b \frac{t}{f(t)}dt - \frac{\frac{a+b}{2}}{f\left(\frac{a+b}{2}\right)} \\ &\geq (\leq) \frac{1}{8}\left(\frac{f(b)-bf'_-(b)}{f^2(b)} - \frac{f(a)-af'_+(a)}{f^2(a)}\right)(b-a). \end{aligned} \tag{2.9}$$

Proof. The first part in all inequalities (2.6)-(2.9) follow from Theorem 1.

Now, if we take the derivative of $P_f(x)$, then we have

$$\begin{aligned} P_f'(x) &= \left(\frac{1}{f\left(\frac{1}{x}\right)}\right)' = \left(f^{-1}\left(\frac{1}{x}\right)\right)' = -f^{-2}\left(\frac{1}{x}\right)\left(f\left(\frac{1}{x}\right)\right)' \\ &= -f^{-2}\left(\frac{1}{x}\right)f'\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = f^{-2}\left(\frac{1}{x}\right)f'\left(\frac{1}{x}\right)\left(\frac{1}{x^2}\right). \end{aligned}$$

Therefore we have

$$P_{+f}'\left(\frac{1}{b}\right) = b^2 f^{-2}(b)\, f_-'(b) = \frac{b^2}{f^2(b)} f_-'(b)$$

and

$$P_{-f}'\left(\frac{1}{a}\right) = a^2 f^{-2}(a)\, f_+'(a) = \frac{a^2}{f^2(a)} f_+'(a)$$

and by the right hand side inequalities in Lemma 4 we get the corresponding inequalities in (2.6) and (2.7).

If we take the derivative of Q_f, we have

$$Q_f'(x) = \left(\frac{x}{f(x)}\right)' = \frac{f(x) - xf'(x)}{f^2(x)}.$$

Therefore

$$Q_{+f}'(a) = \frac{f(a) - af_+'(a)}{f^2(a)} \text{ and } Q_{-f}'(b) = \frac{f(b) - bf_-'(b)}{f^2(b)}$$

and by the right hand side inequalities in Lemma 4 we get the corresponding inequalities in (2.8) and (2.9). □

We have the following result:

Theorem 3. *Let $f : [a, b] \subset (0, \infty) \to (0, \infty)$ be a HH-convex (concave) function on $[a, b]$. Then we have*

$$\frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt \le (\ge) \frac{G^2(f(a), f(b))}{L(f(a), f(b))}. \tag{2.10}$$

Proof. By the definition of *HH*-convex (concave) function, we have by integrating on $[0, 1]$ over λ, that

$$\int_0^1 f\left(\frac{ab}{(1-\lambda)b + \lambda a}\right)d\lambda \le (\ge) \int_0^1 \frac{f(a) f(b)}{(1-\lambda) f(b) + \lambda f(a)}d\lambda. \tag{2.11}$$

Consider the change of variable $\frac{ab}{(1-\lambda)b+\lambda a} = t$. Then $(1-\lambda)b + \lambda a = \frac{ab}{t}$ and $(b-a)\,d\lambda = \frac{ab}{t^2}dt$. Using this change of variable, we have

$$\int_0^1 f\left(\frac{ab}{(1-\lambda)b + \lambda a}\right)d\lambda = \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt.$$

If $f(b) = f(a)$, then

$$\int_0^1 \frac{f(a) f(b)}{(1-\lambda) f(b) + \lambda f(a)}d\lambda = f(a).$$

If $f(b) \neq f(a)$, then by the change of variable $(1-\lambda) f(b) + \lambda f(a) = s$, then we have

$$\int_0^1 \frac{f(a) f(b)}{(1-\lambda) f(b) + \lambda f(a)} d\lambda = \frac{f(a) f(b)}{f(a) - f(b)} \int_{f(b)}^{f(a)} \frac{ds}{s}$$
$$= \frac{f(a) f(b)}{L(f(a), f(b))} = \frac{G^2(f(a), f(b))}{L(f(a), f(b))}.$$

By making use of (2.11) we deduce the desired result (2.10). □

We also have:

Theorem 4. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a HH-convex (concave) function on $[a,b]$. Then we have*

$$f\left(\frac{2ab}{a+b}\right) \leq (\geq) \frac{\int_a^b \frac{1}{t^2} f(t) f\left(\frac{abt}{(a+b)t-ab}\right) dt}{\int_a^b \frac{f(t)}{t^2} dt}. \tag{2.12}$$

Proof. From the definition of *HH*-convex (concave) function we have

$$f\left(\frac{2xy}{x+y}\right) \leq (\geq) \frac{2f(x) f(y)}{f(x) + f(y)} \tag{2.13}$$

for any $x, y \in [a,b]$.

If we take

$$x = \frac{ab}{(1-\lambda) b + \lambda a},\ y = \frac{ab}{(1-\lambda) a + \lambda b} \in [a,b],$$

then

$$\frac{2xy}{x+y} = \frac{2\frac{ab}{(1-\lambda)b+\lambda a} \cdot \frac{ab}{(1-\lambda)a+\lambda b}}{\frac{ab}{(1-\lambda)b+\lambda a} + \frac{ab}{(1-\lambda)a+\lambda b}}$$
$$= \frac{2\frac{1}{(1-\lambda)b+\lambda a} \cdot \frac{ab}{(1-\lambda)a+\lambda b}}{\frac{1}{(1-\lambda)b+\lambda a} + \frac{1}{(1-\lambda)a+\lambda b}}$$
$$= \frac{2\frac{1}{(1-\lambda)b+\lambda a} \cdot \frac{ab}{(1-\lambda)a+\lambda b}}{\frac{(1-\lambda)a+\lambda b+(1-\lambda)b+\lambda a}{((1-\lambda)b+\lambda a)((1-\lambda)a+\lambda b)}} = \frac{2ab}{a+b}$$

and by (2.13) we get

$$f\left(\frac{2ab}{a+b}\right) \leq (\geq) \frac{2f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right) f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)}{f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right) + f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)},$$

which is equivalent to

$$f\left(\frac{2ab}{a+b}\right)\left[f\left(\frac{ab}{(1-\lambda) b + \lambda a}\right) + f\left(\frac{ab}{(1-\lambda) a + \lambda b}\right)\right] \tag{2.14}$$
$$\leq (\geq)\, 2f\left(\frac{ab}{(1-\lambda) b + \lambda a}\right) f\left(\frac{ab}{(1-\lambda) a + \lambda b}\right),$$

for any $\lambda \in [0,1]$.

If we integrate the inequality over λ on $[0,1]$ we get

$$f\left(\frac{2ab}{a+b}\right)\left[\int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)d\lambda+\int_0^1 f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)d\lambda\right] \leq (\geq)\, 2\int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)d\lambda. \tag{2.15}$$

Now, we observe that

$$\int_0^1 f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)d\lambda=\int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)d\lambda = \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt \tag{2.16}$$

and

$$\begin{aligned}&\int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)d\lambda \\ &=\int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)f\left(\frac{ab}{a+b-((1-\lambda)b+\lambda a)}\right)d\lambda \\ &=\int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)f\left(\frac{1}{\frac{1}{b}+\frac{1}{a}-\frac{((1-\lambda)b+\lambda a)}{ab}}\right)d\lambda.\end{aligned} \tag{2.17}$$

If we change the variable $t=\frac{ab}{(1-\lambda)b+\lambda a}$, then we have

$$\begin{aligned}&\int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)f\left(\frac{1}{\frac{1}{b}+\frac{1}{a}-\frac{((1-\lambda)b+\lambda a)}{ab}}\right)d\lambda \\ &=\frac{ab}{b-a}\int_a^b \frac{1}{t^2}f(t)\,f\left(\frac{1}{\frac{1}{b}+\frac{1}{a}-\frac{1}{t}}\right)dt \\ &=\frac{ab}{b-a}\int_a^b \frac{1}{t^2}f(t)\,f\left(\frac{abt}{(a+b)t-ab}\right)dt.\end{aligned} \tag{2.18}$$

On making use of (2.15) - (2.18) we deduce the desired result (2.12). □

Remark 1. *By Cauchy-Bunyakowsky-Schwarz integral inequality we have*

$$\begin{aligned}&\int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)d\lambda \\ &\leq\left(\int_0^1 f^2\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)d\lambda\right)^{1/2}\left(\int_0^1 f^2\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)d\lambda\right)^{1/2} \\ &=\int_0^1 f^2\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)d\lambda=\int_a^b\frac{f^2(t)}{t^2}dt.\end{aligned} \tag{2.19}$$

Now, if $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ is a HH-convex function on $[a,b]$, then by (2.12) and (2.19) we get

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{\int_a^b\frac{1}{t^2}f(t)\,f\left(\frac{abt}{(a+b)t-ab}\right)dt}{\int_a^b\frac{f(t)}{t^2}dt}\leq\frac{\int_a^b\frac{f^2(t)}{t^2}dt}{\int_a^b\frac{f(t)}{t^2}dt}. \tag{2.20}$$

The following lemma is of interest as well.

Lemma 5. *If $f:[a,b]\subset(0,\infty)\rightarrow(0,\infty)$ is HH-convex on $[a,b]$, then the associated function $R_f:[a,b]\subset(0,\infty)\rightarrow(0,\infty)$, $R_f(x)=\frac{f(x)}{x}$ is convex on $[a,b]$. The reverse is not true.*

Proof. Let $\alpha,\beta>0$ with $\alpha+\beta=1$ and $x,y\in[a,b]$.

By the HH-convexity of f we have

$$R_f(\alpha x+\beta y)=\frac{f(\alpha x+\beta y)}{\alpha x+\beta y}=\frac{f\left(\frac{1}{\frac{1}{\alpha x+\beta y}}\right)}{\alpha x+\beta y} \tag{2.21}$$

$$=\frac{f\left(\frac{1}{\frac{\alpha x\frac{1}{x}+\beta y\frac{1}{y}}{\alpha x+\beta y}}\right)}{\alpha x+\beta y}\leq\frac{\frac{1}{\frac{\alpha x\frac{1}{f(x)}+\beta y\frac{1}{f(y)}}{\alpha x+\beta y}}}{\alpha x+\beta y}$$

$$=\frac{\alpha x+\beta y}{\alpha x\frac{1}{f(x)}+\beta y\frac{1}{f(y)}}\cdot\frac{1}{\alpha x+\beta y}$$

$$=\frac{1}{\alpha\frac{x}{f(x)}+\beta\frac{y}{f(y)}}.$$

By the weighted Cauchy-Bunyakowsky-Schwarz inequality we have

$$\left(\alpha\frac{x}{f(x)}+\beta\frac{y}{f(y)}\right)\left(\alpha\frac{f(x)}{x}+\beta\frac{f(y)}{y}\right)$$

$$=\left(\alpha\left(\sqrt{\frac{x}{f(x)}}\right)^2+\beta\left(\sqrt{\frac{y}{f(y)}}\right)^2\right)$$

$$\times\left(\alpha\left(\sqrt{\frac{f(x)}{x}}\right)^2+\beta\left(\sqrt{\frac{f(y)}{y}}\right)^2\right)$$

$$\geq(\alpha+\beta)^2=1,$$

which implies that

$$\frac{1}{\alpha\frac{x}{f(x)}+\beta\frac{y}{f(y)}}\leq\alpha\frac{f(x)}{x}+\beta\frac{f(y)}{y}$$

for any $\alpha,\beta>0$ with $\alpha+\beta=1$ and $x,y\in[a,b]$.

By (2.21) we have

$$R_f(\alpha x+\beta y)\leq\alpha\frac{f(x)}{x}+\beta\frac{f(y)}{y}=\alpha R_f(x)+\beta R_f(y)$$

for any $\alpha,\beta>0$ with $\alpha+\beta=1$ and $x,y\in[a,b]$, which shows that R_f is convex on $[a,b]$.

Consider the function $f:[a,b]\subset(0,\infty)\rightarrow(0,\infty)$, $f(x)=x^p$, $p\neq0$. The function $R_f(x)=x^{p-1}$ is convex iff $p\in(-\infty,1)\cup[2,\infty)$. Since $Q_f(x)=x^{1-p}$ which is concave iff $p\in(0,1)$. By Proposition 1 we have that the function $f:[a,b]\subset(0,\infty)\rightarrow(0,\infty)$, $f(x)=x^p$ is HH-convex iff Q_f is concave, namely $p\in(0,1)$. Therefore R_f is convex and not HH-convex if $p\in(-\infty,0)\cup[2,\infty)$. □

If we denote by $\mathcal{C}_I[a,b]$ the class of all positive functions f for which R_f is convex, then the class of HH-convex functions $f:[a,b]\subset(0,\infty)\rightarrow(0,\infty)$ on $[a,b]$ is strictly enclosed in $\mathcal{C}_I[a,b]$.

We have the following inequalities of Hermite-Hadamard type.

Theorem 5. *If $f \in \mathcal{C}_I[a,b]$, then we have*

$$\frac{f\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}} \leq \frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt \leq \frac{f(a)b+f(b)a}{2ab}, \tag{2.22}$$

$$\begin{aligned} 0 &\leq \frac{f(a)b+f(b)a}{2ab} - \frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt \\ &\leq \frac{1}{8}\left[\frac{f'_-(b)b-f(b)}{b^2} - \frac{f'_+(a)a-f(a)}{a^2}\right](b-a) \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} 0 &\leq \frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt - \frac{f\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}} \\ &\leq \frac{1}{8}\left[\frac{f'_-(b)b-f(b)}{b^2} - \frac{f'_+(a)a-f(a)}{a^2}\right](b-a). \end{aligned} \tag{2.24}$$

Proof. By the Hermite-Hadamard inequalities (1.4) for R_f we have

$$\frac{f\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}} \leq \frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt \leq \frac{\frac{f(a)}{a}+\frac{f(b)}{b}}{2}$$

and the inequality (2.22) is proved.

We have

$$R'_f(t) = \left(\frac{f(t)}{t}\right)' = \frac{f'(t)t-f(t)}{t^2}$$

and then

$$R'_{-f}(b) = \frac{f'_-(b)b-f(b)}{b^2} \text{ and } R'_{+f}(a) = \frac{f'_+(a)a-f(a)}{a^2}.$$

By Lemma 4 we have

$$\begin{aligned} 0 &\leq \frac{\frac{f(a)}{a}+\frac{f(b)}{b}}{2} - \frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt \\ &\leq \frac{1}{8}\left[\frac{f'_-(b)b-f(b)}{b^2} - \frac{f'_+(a)a-f(a)}{a^2}\right](b-a) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt - \frac{f\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}} \\ &\leq \frac{1}{8}\left[\frac{f'_-(b)b-f(b)}{b^2} - \frac{f'_+(a)a-f(a)}{a^2}\right](b-a), \end{aligned}$$

which are equivalent to the desired inequalities (2.23) and (2.24). □

3. Applications

Consider the function $f : [a,b] \subset (0,\infty) \to (0,\infty)$, $f(x) = x^p$. Observe that $Q_f(x) = x^{p-1}$ is convex iff $p \in (-\infty,1) \cup [2,\infty)$ and concave iff $p \in (1,2)$. By Proposition 1 we have that the function f is HH-convex (concave) on $[a,b]$ iff Q_f is concave (convex) on $[a,b]$, namely $p \in (1,2)$ $(p \in (-\infty,1) \cup [2,\infty))$.

We introduce the L_q-*harmonic mean* for $q \neq 0, -1$

$$L_q(a,b) := \begin{cases} \left(\frac{b^{q+1}-a^{q+1}}{(q+1)(b-a)}\right)^{\frac{1}{q}} \text{ if } b \neq a \\ \\ b \text{ if } b = a, \end{cases}$$

the *logarithmic mean* by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a \\ \\ b \text{ if } b = a, \end{cases}$$

and the *identric mean* by

$$I(a,b) := \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} \text{ if } b \neq a \\ \\ b \text{ if } b = a. \end{cases}$$

If we set $L_0(a,b) := I(a,b)$ and $L_{-1}(a,b) := L(a,b)$, then we have that the function $\mathbb{R} \ni q \mapsto L_q(a,b)$ is monotonic increasing as a function of q. We also have the inequalities

$$H(a,b) \leq G(a,b) \leq L(a,b) \leq I(a,b) \leq A(a,b).$$

By making use of Theorem 1 we have for $p \in (1,2)$ $(p \in (-\infty,1) \cup [2,\infty))$ that

$$\frac{H^p(a,b)}{G^2(a,b)} \geq (\leq) L_{-p-2}^{-p-2}(a,b) \geq (\leq) \frac{A(a^p,b^p)}{G^2(a,b)},\ p \neq 0 \tag{3.1}$$

and

$$A^{1-p}(a,b) \geq (\leq) L_{1-p}^{1-p}(a,b) \geq (\leq) A\left(a^{p-1},b^{p-1}\right). \tag{3.2}$$

If we take $p = -1$ in (3.1), then we get

$$\frac{1}{G^2(a,b)\,H(a,b)} \leq \frac{1}{L(a,b)} \leq \frac{A\left(a^{-1},b^{-1}\right)}{G^2(a,b)}.$$

By Theorem 3 we have for $p \in (1,2)$ $(p \in (-\infty,1) \cup [2,\infty))$

$$G^2(a,b)\,L_{p-2}^{p-2}(a,b) \leq (\geq) \frac{G^2(a^p,b^p)}{L(a^p,b^p)}. \tag{3.3}$$

Observe that

$$L(a^p,b^p) = \frac{b^p - a^p}{p(\ln b - \ln a)} = \frac{b^p - a^p}{p(b-a)} \cdot \frac{b-a}{\ln b - \ln a} = L_{p-1}^{p-1}(a,b)\,L(a,b)$$

and by (3.3) we get

$$L_{p-2}^{p-2}(a,b)\,L_{p-1}^{p-1}(a,b) \leq (\geq) \frac{G^2(a^p,b^p)}{G^2(a,b)\,L(a,b)}, \tag{3.4}$$

for $p \in (1,2)$ $(p \in (-\infty,1) \cup (2,\infty))$.

Now, consider the function $f : [a,b] \subset (1,\infty) \to (0,\infty)$, $f(t) = \frac{t}{\ln t}$. Then $Q_f(x) = \frac{x}{\frac{x}{\ln x}} = \ln x$ is concave on $[a,b]$, therefore f is HH-convex on $[a,b] \subset (1,\infty)$. If we use the inequality (2.2), then we get the well known inequality

$$A(a,b) \geq I(a,b) \geq G(a,b).$$

If we use the inequality (2.10) for $f:[a,b]\subset(1,\infty)\to(0,\infty)$, $f(t)=\frac{t}{\ln t}$, then we get

$$\frac{ab}{b-a}\int_a^b\frac{1}{t\ln t}dt\le\frac{G^2\left(\frac{a}{\ln a},\frac{b}{\ln b}\right)}{L\left(\frac{a}{\ln a},\frac{b}{\ln b}\right)}. \tag{3.5}$$

Since

$$\int_a^b\frac{1}{t\ln t}dt=\int_a^b\frac{1}{\ln t}d(\ln t)=\ln(\ln b)-\ln(\ln a),$$

and

$$G^2\left(\frac{a}{\ln a},\frac{b}{\ln b}\right)=\frac{G^2(a,b)}{G^2(\ln a,\ln b)},$$

then from (3.5) we have

$$G^2(\ln a,\ln b)\,L\left(\frac{a}{\ln a},\frac{b}{\ln b}\right)\le L(a,b)\,L(\ln b,\ln a). \tag{3.6}$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa