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SOME NEW GENERALIZED HERMITE-HADAMARD INEQUALITY FOR GENERALIZED CONVEX FUNCTION AND APPLICATIONS

HÜSEYIN BUDAK AND MEHMET ZEKI SARIKAYA

ABSTRACT. In this paper, some new inequality for generalized convex functions are obtained. Some applications for some generalized special means are also given.

1. Introduction

Theorem 1 (Hermite-Hadamard inequality). *Let $f : I \subseteq R \to R$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. If f is a convex function then the following double inequality, which is well known in the literature as the Hermite–Hadamard inequality, holds* [5]

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In [2], authors gave the following lemma:

Lemma 1. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If $f'' \in L_1[a, b]$, then*

$$\frac{1}{2}\int_0^1 (x-a)(b-x)f''(x)dx = \frac{b-a}{2}(f(a)+f(b)) - \int_a^b f(x)dx. \tag{1.2}$$

Also,they obtained following inequality:

Theorem 2. *With the above assumptions, given that $k \leq f''(x) \leq K$ on $[a, b]$, we have the inequality*

$$k\tfrac{(b-a)^2}{12} \leq \tfrac{f(a)+f(b)}{2} - \tfrac{1}{b-a}\int_a^b f(x)dx \leq K\tfrac{(b-a)^2}{12}. \tag{1.3}$$

2. Preliminaries

Recall the set R^α of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [11, 12] and so on.

Recently, the theory of Yang's fractional sets [11] was introduced as follows.

For $0 < \alpha \leq 1$, we have the following α-type set of element sets:

Z^α : The α-type set of integer is defined as the set $\{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, ..., \pm n^\alpha, ...\}$.

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Q^{α} : The α-type set of the rational numbers is defined as the set $\{m^{\alpha}=\left(\frac{p}{q}\right)^{\alpha}:$ $p,q\in Z,\ q\neq 0\}$.

J^{α} : The α-type set of the irrational numbers is defined as the set $\{m^{\alpha}\neq\left(\frac{p}{q}\right)^{\alpha}:$ $p,q\in Z,\ q\neq 0\}$.

R^{α} : The α-type set of the real line numbers is defined as the set $R^{\alpha}=Q^{\alpha}\cup J^{\alpha}$.

If a^{α},b^{α} and c^{α} belongs the set R^{α} of real line numbers, then

(1) $a^{\alpha}+b^{\alpha}$ and $a^{\alpha}b^{\alpha}$ belongs the set R^{α};
(2) $a^{\alpha}+b^{\alpha}=b^{\alpha}+a^{\alpha}=(a+b)^{\alpha}=(b+a)^{\alpha}$;
(3) $a^{\alpha}+(b^{\alpha}+c^{\alpha})=(a+b)^{\alpha}+c^{\alpha}$;
(4) $a^{\alpha}b^{\alpha}=b^{\alpha}a^{\alpha}=(ab)^{\alpha}=(ba)^{\alpha}$;
(5) $a^{\alpha}(b^{\alpha}c^{\alpha})=(a^{\alpha}b^{\alpha})c^{\alpha}$;
(6) $a^{\alpha}(b^{\alpha}+c^{\alpha})=a^{\alpha}b^{\alpha}+a^{\alpha}c^{\alpha}$;
(7) $a^{\alpha}+0^{\alpha}=0^{\alpha}+a^{\alpha}=a^{\alpha}$ and $a^{\alpha}1^{\alpha}=1^{\alpha}a^{\alpha}=a^{\alpha}$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 1. [11] *A non-differentiable function $f:R\rightarrow R^{\alpha}$, $x\rightarrow f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon>0$, there exists $\delta>0$, such that*

$$|f(x)-f(x_0)|<\varepsilon^{\alpha}$$

holds for $|x-x_0|<\delta$, where $\varepsilon,\delta\in R$. If $f(x)$ is local continuous on the interval (a,b), we denote $f(x)\in C_{\alpha}(a,b)$.

Definition 2. [11] *The local fractional derivative of $f(x)$ of order α at $x=x_0$ is defined by*

$$f^{(\alpha)}(x_0)=\left.\frac{d^{\alpha}f(x)}{dx^{\alpha}}\right|_{x=x_0}=\lim_{x\rightarrow x_0}\frac{\Delta^{\alpha}\left(f(x)-f(x_0)\right)}{(x-x_0)^{\alpha}},$$

where $\Delta^{\alpha}\left(f(x)-f(x_0)\right)\cong\Gamma(\alpha+1)\left(f(x)-f(x_0)\right)$.

If there exists $f^{(k+1)\alpha}(x)=\overbrace{D_x^{\alpha}...D_x^{\alpha}}^{k+1\text{ times}}f(x)$ for any $x\in I\subseteq R$, then we denoted $f\in D_{(k+1)\alpha}(I)$, where $k=0,1,2,...$

Definition 3. [11] *Let $f(x)\in C_{\alpha}[a,b]$. Then the local fractional integral is defined by,*

$${}_aI_b^{\alpha}f(x)=\frac{1}{\Gamma(\alpha+1)}\int_a^b f(t)(dt)^{\alpha}=\frac{1}{\Gamma(\alpha+1)}\lim_{\Delta t\rightarrow 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^{\alpha},$$

with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\{\Delta t_1,\Delta t_2,...,\Delta t_{N-1}\}$, where $[t_j,t_{j+1}]$, $j=0,...,N-1$ and $a=t_0<t_1<...<t_{N-1}<t_N=b$ is partition of interval $[a,b]$.

Here, it follows that ${}_aI_b^{\alpha}f(x)=0$ if $a=b$ and ${}_aI_b^{\alpha}f(x)=-{}_bI_a^{\alpha}f(x)$ if $a<b$. If for any $x\in[a,b]$, there exists ${}_aI_x^{\alpha}f(x)$, then we denoted by $f(x)\in I_x^{\alpha}[a,b]$.

Definition 4 (Generalized convex function)**.** [11] *Let $f:I\subseteq R\rightarrow R^{\alpha}$. For any $x_1,x_2\in I$ and $\lambda\in[0,1]$, if the following inequality*

$$f(\lambda x_1+(1-\lambda)x_2)\leq\lambda^{\alpha}f(x_1)+(1-\lambda)^{\alpha}f(x_2)$$

holds, then f is called a generalized convex function on I.

Here are two basic examples of generalized convex functions:
(1) $f(x) = x^{\alpha p}$, $x \geq 0$, $p > 1$;
(2) $f(x) = E_\alpha(x^\alpha)$, $x \in R$ where $E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Leffer function.

Theorem 3. *Let $f \in D_\alpha(I)$, then the following conditions are equivalent*
a) f is a generalized convex function on I
b) $f^{(\alpha)}$ is an increasing function on I
c) for any $x_1, x_2 \in I$,

$$f(x_2) - f(x_1) \geq \frac{f^{(\alpha)}(x_1)}{\Gamma(1+\alpha)} (x_2 - x_1)^\alpha .$$

Corollary 1. *Let $f \in D_{2\alpha}(a,b)$. Then f is a generalized convex function (or a generalized concave function) if and only if*

$$f^{(2\alpha)}(x) \geq 0 \left(or\ f^{(2\alpha)}(x) \leq 0 \right)$$

for all $x \in (a,b)$.

Lemma 2. [11]
(1) (Local fractional integration is anti-differentiation) Suppose that $f(x) = g^{(\alpha)}(x) \in C_\alpha [a,b]$, then we have

$${}_aI_b^\alpha f(x) = g(b) - g(a).$$

(2) (Local fractional integration by parts) Suppose that $f(x), g(x) \in D_\alpha [a,b]$ and $f^{(\alpha)}(x)$, $g^{(\alpha)}(x) \in C_\alpha [a,b]$, then we have

$${}_aI_b^\alpha f(x) g^{(\alpha)}(x) = f(x)g(x)|_a^b - {}_aI_b^\alpha f^{(\alpha)}(x) g(x).$$

Lemma 3. [11]

$$\frac{d^\alpha x^{k\alpha}}{dx^\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha};$$

$$\frac{1}{\Gamma(\alpha+1)} \int_a^b x^{k\alpha} (dx)^\alpha = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha} \right), \ k \in R.$$

Lemma 4 (Generalized Hölder's inequality)**.** [11] *Let $f, g \in C_\alpha [a,b]$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)g(x)| (dx)^\alpha \leq \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)|^p (dx)^\alpha \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |g(x)|^q (dx)^\alpha \right)^{\frac{1}{q}}.$$

In [3], Mo et al. proved the following generalized Hermite-Hadamard inequality for generalized convex function:

Theorem 4 (Gereneralized Hermite-Hadamard's inequality)**.** *Let $f(x) \in I_x^\alpha [a,b]$ be generalized convex function on $[a,b]$ with $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_aI_b^\alpha f(x) \leq \frac{f(a)+f(b)}{2^\alpha}.$$

In [9], Sarikaya et al. gave the the following generalized Bullen inequality for generalized convex function

Theorem 5 (Generalized Bullen inequality)**.** *Let $f(x) \in I_x^{(\alpha)}[a,b]$ be a generalized convex function on $[a,b]$ with $a<b$. Then we have the inequality*

$$\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(x) \leq \frac{1}{2^{\alpha}}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2^{\alpha}}\right]. \tag{2.1}$$

For more information and recent developments on local freactional theory, please refer to [1],[3],[4],[6]-[15].

The aim of the paper is to establish some new generalized Hermite- Hadamard inequalities for generalized convex functions and we give some applications for some generalized special means.

3. Main Results

Theorem 6. *Let $I \subseteq R$ be an interval, $f: I^0 \subseteq R \to R^{\alpha}$ (I^0 is the interior of I) such that $f \in D_{2\alpha}(I^0)$ and $f^{(2\alpha)} \in C_{2\alpha}[a,b]$ for $a,b \in I^0$ with $a<b$ Then, we have the identity:*

$$\begin{aligned} & \frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(x) \\ = & \frac{1}{2^{\alpha}(b-a)^{\alpha}[\Gamma(1+\alpha)]^2}\int_a^b (x-a)^{\alpha}(b-x)^{\alpha}f^{(2\alpha)}(x)\,(dx)^{\alpha}. \end{aligned} \tag{3.1}$$

Proof. Using the local fractional integration by parts twice (Lemma 2), we obtain

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)}\int_a^b (x-a)^{\alpha}(b-x)^{\alpha}f^{(2\alpha)}(x)\,(dx)^{\alpha} \\ = & (x-a)^{\alpha}(b-x)^{\alpha}f^{(\alpha)}(x)\Big|_a^b \\ & -\frac{1}{\Gamma(1+\alpha)}\int_a^b \Gamma(1+\alpha)(-2x+a+b)^{\alpha}f^{(\alpha)}(x)\,(dx)^{\alpha} \\ = & -\Gamma(1+\alpha)(-2x+a+b)^{\alpha}f(x)\big|_a^b \\ & +\frac{1}{\Gamma(1+\alpha)}\int_a^b [\Gamma(1+\alpha)]^2(-2)^{\alpha}f(x)\,(dx)^{\alpha} \end{aligned} \tag{3.2}$$

$$= -\Gamma(1+\alpha)(a-b)^{\alpha} f(b) + \Gamma(1+\alpha)(b-a)^{\alpha} f(a)$$

$$-\frac{2^{\alpha}[\Gamma(1+\alpha)]^2}{\Gamma(1+\alpha)} \int_a^b f(x)\,(dx)^{\alpha}$$

$$= \Gamma(1+\alpha)(b-a)^{\alpha}[f(a)+f(b)] - 2^{\alpha}[\Gamma(1+\alpha)]^2 \ {}_aI_b^{\alpha} f(x).$$

If we divide the resulting equality (3.2) by $2^{\alpha}(b-a)^{2\alpha}\Gamma(1+\alpha)$, then we obtain the desired result. □

Theorem 7. *With the above assumptions, given that $k^{\alpha} \le f^{(2\alpha)}(x) \le K^{\alpha}$ for all $x \in [a,b]$, $k, K \in R$, we have the following inequality*

$$k^{\alpha}.\frac{(b-a)^{2\alpha}}{2^{\alpha}\Gamma(1+\alpha)}\left[\frac{\Gamma(1+1\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right] \tag{3.3}$$

$$\le \frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} \ {}_aI_b^{\alpha} f(x)$$

$$\le K^{\alpha}.\frac{(b-a)^{2\alpha}}{2^{\alpha}\Gamma(1+\alpha)}\left[\frac{\Gamma(1+1\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right].$$

Proof. From assumptions, we have

$$k^{\alpha}(x-a)^{\alpha}(b-x)^{\alpha} \le (x-a)^{\alpha}(b-x)^{\alpha} f^{(2\alpha)}(x) \le K^{\alpha}(x-a)^{\alpha}(b-x)^{\alpha} \tag{3.4}$$

for all $x \in [a,b]$. Dividing (3.4) by $2^{\alpha}(b-a)^{\alpha}[\Gamma(1+\alpha)]^2$, then integrating the resulting inequality with respect to x over $[a,b]$, we get

$$\frac{k^{\alpha}}{2^{\alpha}(b-a)^{\alpha}[\Gamma(1+\alpha)]^2}\int_a^b (x-a)^{\alpha}(b-x)^{\alpha}\,(dx)^{\alpha}$$

$$\le \frac{1}{2^{\alpha}(b-a)^{\alpha}[\Gamma(1+\alpha)]^2}\int_a^b (x-a)^{\alpha}(b-x)^{\alpha} f^{(2\alpha)}(x)\,(dx)^{\alpha}$$

$$\le \frac{K^{\alpha}}{2^{\alpha}(b-a)^{\alpha}[\Gamma(1+\alpha)]^2}\int_a^b (x-a)^{\alpha}(b-x)^{\alpha} K^{\alpha}(x)\,(dx)^{\alpha}.$$

From Teorem 6, we have

$$\frac{1}{2^{\alpha}(b-a)^{\alpha}[\Gamma(1+\alpha)]^2}\int_a^b (x-a)^{\alpha}(b-x)^{\alpha} f^{(2\alpha)}(x)\,(dx)^{\alpha}.$$

$$= \frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} \ {}_aI_b^{\alpha} f(x)$$

and a simple calculating shows that

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b (x-a)^{\alpha}(b-x)^{\alpha}\,(dx)^{\alpha} = (b-a)^{3\alpha}\left[\frac{\Gamma(1+1\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right].$$

This completes the proof. □

Corollary 2. *With the assumptions of Theorem 7, given that* $\left\| f^{(2\alpha)} \right\|_{\infty} := \sup_{x\in[a,b]} \left| f^{(2\alpha)}(x) \right|$, *then we have the following inequality .*

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} \ {}_{a}I_{b}^{\alpha} f(x) \right| \\ \leq & \frac{(b-a)^{2\alpha}}{2^{\alpha}\Gamma(1+\alpha)} \left[\frac{\Gamma(1+1\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \right] \left\| f^{(2\alpha)} \right\|_{\infty}. \end{aligned}$$

Theorem 8. *The assumptions of Theorem 6 are satisfied. If we assume that the new mapping* $\varphi : I^{0} \subseteq R \rightarrow R^{\alpha}$, $\varphi(x) = (x-a)^{\alpha}(b-x)^{\alpha} f^{(2\alpha)}(x)$ *is generalized convex on* $[a,b]$, *then we have the inequality*

$$\begin{aligned} & \frac{(b-a)^{2\alpha}}{8^{\alpha}\left[\Gamma(1+\alpha)\right]^{2}} f^{(2\alpha)}\left(\frac{a+b}{2}\right) \\ \leq & \frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} \ {}_{a}I_{b}^{\alpha} f(x) \\ \leq & \frac{(b-a)^{2\alpha}}{16^{\alpha}\left[\Gamma(1+\alpha)\right]^{2}} f^{(2\alpha)}\left(\frac{a+b}{2}\right). \end{aligned} \tag{3.5}$$

Proof. Because of the generalized convexity of φ, applying the first inequality of generalized Hermite-Hadamard (Theorem 4) we can state:

$$\frac{1}{(b-a)^{\alpha}} \int_{a}^{b} \varphi(x)\,(dx)^{\alpha} \geq \varphi\left(\frac{a+b}{2}\right) = \frac{(b-a)^{2\alpha}}{4^{\alpha}} f^{(2\alpha)}\left(\frac{a+b}{2}\right). \tag{3.6}$$

Similarliy, applying the generalized generalized Bullen inequality (Theorem 5) for φ, we get

$$\begin{aligned} \frac{1}{(b-a)^{\alpha}} \int_{a}^{b} \varphi(x)\,(dx)^{\alpha} & \leq \frac{1}{2^{\alpha}} \left[\varphi\left(\frac{a+b}{2}\right) + \frac{\varphi(a)+\varphi(b)}{2^{\alpha}} \right] \\ & = \frac{(b-a)^{2\alpha}}{8^{\alpha}} f^{(2\alpha)}\left(\frac{a+b}{2}\right). \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we have

$$\frac{(b-a)^{2\alpha}}{4^{\alpha}} f^{(2\alpha)}\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)^{\alpha}} \int_{a}^{b} \varphi(x)\,(dx)^{\alpha} \leq \frac{(b-a)^{2\alpha}}{8^{\alpha}} f^{(2\alpha)}\left(\frac{a+b}{2}\right) \tag{3.8}$$

If we divide the inequalities (3.8) by $2^{\alpha}\left[\Gamma(1+\alpha)\right]^{2}$, we obtain desired result, which completes the proof. □

Theorem 9. *The assumptions of Theorem 6 are satisfied. Then we have the inequality*

$$\left|\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_{a}I_{b}^{\alpha}f(x)\right| \tag{3.9}$$

$$\leq \frac{(b-a)^{\left(1+\frac{1}{p}\right)\alpha}}{2^{\alpha}}\left[B(p+1,p+1)\right]^{\frac{1}{p}}\left\|f^{(2\alpha)}\right\|_{q}$$

where, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\left\|f^{(2\alpha)}\right\|_{q}$ *is defined by*

$$\left\|f^{(2\alpha)}\right\|_{q}=\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}\left|f^{(2\alpha)}(x)\right|^{q}(dx)^{\alpha}\right)^{\frac{1}{q}}$$

and $B(x,y)$ *is defined by*

$$B(x,y)=\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}t^{(x-1)\alpha}(1-t)^{(y-1)\alpha}(dt)^{\alpha}.$$

Proof. Taking modulus in (3.1) and using the generalized Holder's inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_{a}I_{b}^{\alpha}f(x)\right| \\
\leq\ & \frac{1}{2^{\alpha}(b-a)^{\alpha}\left[\Gamma(1+\alpha)\right]^{2}}\int_{a}^{b}\left|(x-a)^{\alpha}(b-x)^{\alpha}\right|\left|f^{(2\alpha)}(x)\right|(dx)^{\alpha} \\
\leq\ & \frac{1}{2^{\alpha}(b-a)^{\alpha}\Gamma(1+\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_{0}^{b}\left|f^{(2\alpha)}(x)\right|^{q}(dx)^{\alpha}\right)^{\frac{1}{q}} \\
&\times\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}(x-a)^{p\alpha}(b-x)^{p\alpha}(dx)^{\alpha}\right)^{\frac{1}{p}}. \\
=\ & \frac{\left\|f^{(2\alpha)}\right\|_{q}}{2^{\alpha}(b-a)^{\alpha}\Gamma(1+\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}(x-a)^{p\alpha}(b-x)^{p\alpha}(dx)^{\alpha}\right)^{\frac{1}{p}}.
\end{aligned}$$

Using the changing variable $x=(1-t)a+tb$, we have

$$\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}(x-a)^{p\alpha}(b-x)^{p\alpha}(dx)^{\alpha}=(b-a)^{(2p+1)\alpha}B(p+1,p+1)$$

which completes the proof. □

Theorem 10. *The assumptions of Theorem 6 are satisfied. If $\left|f^{(2\alpha)}\right|$ is generalized convex, then we have the inequality*

$$\left|\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(x)\right| \leq \frac{(b-a)^{2\alpha}}{\Gamma(1+\alpha)}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right)\left[\frac{\left|f^{(2\alpha)}(a)\right|+\left|f^{(2\alpha)}(b)\right|}{2^{\alpha}}\right]. \tag{3.10}$$

Proof. Taking modulus in (3.1) we get

$$\left|\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(x)\right| \leq \frac{1}{2^{\alpha}(b-a)^{\alpha}\left[\Gamma(1+\alpha)\right]^2}\int_a^b (x-a)^{\alpha}(b-x)^{\alpha}\left|f^{(2\alpha)}(x)\right|(dx)^{\alpha}.$$

Since generalized convexity of $\left|f^{(2\alpha)}\right|$, we have

$$\begin{aligned}\left|f^{(2\alpha)}(x)\right| &= \left|f^{(2\alpha)}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)\right| \\ &\leq \left(\frac{x-a}{b-a}\right)^{\alpha}\left|f^{(2\alpha)}(b)\right|+\left(\frac{b-x}{b-a}\right)^{\alpha}\left|f^{(2\alpha)}(a)\right|.\end{aligned}$$

Then, it follows that

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(x)\right| \\ &\leq \frac{1}{2^{\alpha}(b-a)^{\alpha}\Gamma(1+\alpha)}\left[\frac{\left|f^{(2\alpha)}(b)\right|}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_a^b (x-a)^{2\alpha}(b-x)^{\alpha}(dx)^{\alpha}\right. \\ &\left.+\frac{\left|f^{(2\alpha)}(a)\right|}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_a^b (x-a)^{\alpha}(b-x)^{2\alpha}(dx)^{\alpha}\right].\end{aligned} \tag{3.11}$$

Using the changing variable $x=(1-t)a+tb$, we have

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\int_a^b (x-a)^{2\alpha}(b-x)^{\alpha}(dx)^{\alpha} \\ &= (b-a)^{4\alpha}\int_a^b t^{2\alpha}(1-t)^{\alpha}(dt)^{\alpha} \\ &= (b-a)^{4\alpha}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right),\end{aligned} \tag{3.12}$$

and similarly
(3.13)

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b (x-a)^{\alpha}(b-x)^{2\alpha}(dx)^{\alpha} = (b-a)^{4\alpha}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} - \frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right).$$

Putting (3.12) and (3.13) in (3.11), we obtain required result. □

4. Applications to Some Special Means

We consider some generalized means as in [7]:

$$A(a,b) = \frac{a^{\alpha}+b^{\alpha}}{2^{\alpha}};$$

$$L_n(a,b) = \left[\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)}\left[\frac{b^{(n+1)\alpha}-a^{(n+1)\alpha}}{(b-a)^{\alpha}}\right]\right]^{\frac{1}{n}},\ n\in Z\backslash\{-1,0\},\ a,b\in R,\ a\neq b.$$

Proposition 1. *Let $a,b\in R$, $0<a<b$, $0\notin[a,b]$ and $n\in Z$, $|n(n-1)|\geq 3$. Then, we have the inequality*

$$\begin{aligned}&|A(a^n,b^n)-\Gamma(1+\alpha)[L_n(a,b)]^n|\\ =\ & \frac{(b-a)^{\left(1+q+\frac{1}{p}\right)\alpha}}{2^{\alpha}}[B(p+1,p+1)]^{\frac{1}{p}}\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|^{\frac{1}{q}}\left[L_{q(n-2)}^{n-2}(a,b)\right].\end{aligned}$$

Proof. Let us reconsider the inequality (3.9):

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(x)\right|\\ \leq\ & \frac{(b-a)^{\left(1+\frac{1}{p}\right)\alpha}}{2^{\alpha}}[B(p+1,p+1)]^{\frac{1}{p}}\left\|f^{(2\alpha)}\right\|_q\end{aligned}$$

Consider the mapping $f:(0,\infty)\to R^{\alpha}$, $f(x)=x^{n\alpha}$, $n\in Z\backslash\{-1,0\}$. Then, $0<a<b$, we have

$$\begin{aligned}\frac{f(a)+f(b)}{2^{\alpha}} &= A(a^n,b^n)\ ,\frac{{}_aI_b^{\alpha}f(x)}{(b-a)^{\alpha}} = [L_n(a,b)]^n,\\ \left|f^{(2\alpha)}(x)\right| &= \left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|x^{(n-2)\alpha}\end{aligned}$$

and

$$\begin{aligned}\left\|f^{(2\alpha)}\right\|_q &= \left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|^{\frac{1}{q}}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b x^{q(n-2)\alpha}(dx)^{\alpha}\right)^{\frac{1}{q}}\\ &= \left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|^{\frac{1}{q}}\left(\frac{\Gamma(1+q(n-2)\alpha)}{\Gamma(1+(q(n-2)+1)\alpha)}\left(b^{(q(n-2)+1)\alpha}-a^{(q(n-2)+1)\alpha}\right)\right)^{\frac{1}{q}}\\ &= \left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|^{\frac{1}{q}}(b-a)^{q\alpha}\left[L_{q(n-2)}(a,b)\right]^{n-2}.\end{aligned}$$

Then, we obtain

$$\begin{aligned}
& \left|A(a^n,b^n)-\Gamma(1+\alpha)\left[L_n(a,b)\right]^n\right| \\
\leq\ & \frac{(b-a)^{\left(1+\frac{1}{p}\right)\alpha}}{2^\alpha}\left[B(p+1,p+1)\right]^{\frac{1}{p}}\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|^{\frac{1}{q}}(b-a)^{q\alpha}\left[L_{q(n-2)}(a,b)\right]^{n-2} \\
=\ & \frac{(b-a)^{\left(1+q+\frac{1}{p}\right)\alpha}}{2^\alpha}\left[B(p+1,p+1)\right]^{\frac{1}{p}}\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|^{\frac{1}{q}}\left[L_{q(n-2)}^{n-2}(a,b)\right].
\end{aligned}$$

This completes the proof. □

Proposition 2. *Let $a,b\in R$, $0<a<b$, $0\notin[a,b]$ and $n\in Z$, $|n(n-1)|\geq 3$. Then, we have the inequality*

$$\begin{aligned}
& \left|A(a^n,b^n)-\Gamma(1+\alpha)\left[L_n(a,b)\right]^n\right| \\
=\ & \frac{(b-a)^{2\alpha}}{\Gamma(1+\alpha)}\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right)A(a^{n-2},b^{n-2}).
\end{aligned}$$

Proof. The proof is obvious from Theorem 10 applied $f(x)=x^{n\alpha}$, $n\in Z\setminus\{-1,0\}$. □

References

[1] G-S. Chen, *Generalizations of Hölder's and some related integral inequalities on fractal space,* Journal of Function Spaces and Applications Volume 2013, Article ID 198405.

[2] S. S. Dragomir and C. E. M. Pearce, *Selected topics on Hermite-Hadamard inequalities and applications*, RGMIA Monographs, Victoria University, 2000.

[3] H. Mo, X Sui and D Yu, *Generalized convex functions on fractal sets and two related inequalities*, Abstract and Applied Analysis, Volume 2014, Article ID 636751, 7 pages.

[4] H. Mo, *Generalized Hermite-Hadamard inequalities involving local fractional integral,* arXiv:1410.1062.

[5] J.E. Pečarić, F. Proschan and Y.L. Tong, *Convex functions, partial orderings and statistical applications*, Academic Press, Boston, 1992.

[6] M. Z. Sarikaya and H Budak, *Generalized Ostrowski type inequalities for local fractional integrals*, RGMIA Research Report Collection, 18(2015), Article 62, 11 pp.

[7] M. Z. Sarikaya, S.Erden and H. Budak, *Some generalized Ostrowski type inequalities involving local fractional integrals and applications*, RGMIA Research Report Collection, 18(2015), Article 63, 12 pp.

[8] M. Z. Sarikaya H. Budak, *On generalized Hermite-Hadamard inequality for generalized convex function*, RGMIA Research Report Collection, 18(2015), Article 64, 15 pp.

[9] M. Z. Sarikaya, S.Erden and H. Budak, *Some integral inequalities for local fractional integrals*, RGMIA Research Report Collection, 18(2015), Article 65, 12 pp.

[10] M. Z. Sarikaya, H. Budak and S.Erden, *On new inequalities of Simpson's type for generalized convex functions*, RGMIA Research Report Collection, 18(2015), Article 66, 13 pp.

[11] X. J. Yang, *Advanced Local Fractional Calculus and Its Applications*, World Science Publisher, New York, 2012.

[12] J. Yang, D. Baleanu and X. J. Yang, *Analysis of fractal wave equations by local fractional Fourier series method*, Adv. Math. Phys. , 2013 (2013), Article ID 632309.

[13] X. J. Yang, *Local fractional integral equations and their applications*, Advances in Computer Science and its Applications (ACSA) 1(4), 2012.

[14] X. J. Yang, *Generalized local fractional Taylor's formula with local fractional derivative*, Journal of Expert Systems, 1(1) (2012) 26-30.

[15] X. J. Yang, *Local fractional Fourier analysis,* Advances in Mechanical Engineering and its Applications 1(1), 2012 12-16.

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY
E-mail address: hsyn.budak@gmail.com

E-mail address: sarikayamz@gmail.com