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GENERALIZED WEIGHTED OSTROWSKI TYPE INEQUALITIES FOR LOCAL FRACTIONAL INTEGRALS

HÜSEYIN BUDAK AND MEHMET ZEKI SARIKAYA

Abstract. In this paper, we establish the generalized weighted Ostrowski inequality for local fractional integrals on fractal sets R^{α} $(0<\alpha\leq 1)$ of real line numbers. The results presented here would provide extensions of those given in earlier works.

1. Introduction

In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives [14]:

Theorem 1 (Ostrowski inequality)**.** *Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) whoose derivative $f':(a,b)\to R$ is bounded on (a,b), i.e. $\|f'\|_{\infty}:=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_{\infty}, \tag{1.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

This inequality is well known in the literature as the *Ostrowski inequality.* For more information recent development on Ostrowski inequality, please refer to [1]-[5],[7]-[11],[15]-[20] *and so on.*

2. Preliminaries

Recall the set R^{α} of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [26, 27] and so on.

Recently, the theory of Yang's fractional sets [26] was introduced as follows.

For $0<\alpha\leq 1$, we have the following α-type set of element sets:

Z^{α} : The α-type set of integer is defined as the set $\{0^{\alpha},\pm 1^{\alpha},\pm 2^{\alpha},...,\pm n^{\alpha},...\}$.

Q^{α} : The α-type set of the rational numbers is defined as the set $\{m^{\alpha}=\left(\frac{p}{q}\right)^{\alpha}:p,q\in Z,\ q\neq 0\}$.

J^{α} : The α-type set of the irrational numbers is defined as the set $\{m^{\alpha}\neq\left(\frac{p}{q}\right)^{\alpha}:p,q\in Z,\ q\neq 0\}$.

R^{α} : The α-type set of the real line numbers is defined as the set $R^{\alpha}=Q^{\alpha}\cup J^{\alpha}$.

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If a^{α}, b^{α} and c^{α} belongs the set R^{α} of real line numbers, then
(1) $a^{\alpha}+b^{\alpha}$ and $a^{\alpha}b^{\alpha}$ belongs the set R^{α};
(2) $a^{\alpha}+b^{\alpha}=b^{\alpha}+a^{\alpha}=(a+b)^{\alpha}=(b+a)^{\alpha}$;
(3) $a^{\alpha}+(b^{\alpha}+c^{\alpha})=(a+b)^{\alpha}+c^{\alpha}$;
(4) $a^{\alpha}b^{\alpha}=b^{\alpha}a^{\alpha}=(ab)^{\alpha}=(ba)^{\alpha}$;
(5) $a^{\alpha}(b^{\alpha}c^{\alpha})=(a^{\alpha}b^{\alpha})c^{\alpha}$;
(6) $a^{\alpha}(b^{\alpha}+c^{\alpha})=a^{\alpha}b^{\alpha}+a^{\alpha}c^{\alpha}$;
(7) $a^{\alpha}+0^{\alpha}=0^{\alpha}+a^{\alpha}=a^{\alpha}$ and $a^{\alpha}1^{\alpha}=1^{\alpha}a^{\alpha}=a^{\alpha}$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 1. [26] *A non-differentiable function $f:R\to R^{\alpha}$, $x\to f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon>0$, there exists $\delta>0$, such that*

$$|f(x)-f(x_0)|<\varepsilon^{\alpha}$$

holds for $|x-x_0|<\delta$, where $\varepsilon,\delta\in R$. If $f(x)$ is local continuous on the interval (a,b), we denote $f(x)\in C_{\alpha}(a,b)$.

Definition 2. [26] *The local fractional derivative of $f(x)$ of order α at $x=x_0$ is defined by*

$$f^{(\alpha)}(x_0)=\left.\frac{d^{\alpha}f(x)}{dx^{\alpha}}\right|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^{\alpha}\left(f(x)-f(x_0)\right)}{(x-x_0)^{\alpha}},$$

where $\Delta^{\alpha}\left(f(x)-f(x_0)\right)\cong\Gamma(\alpha+1)\left(f(x)-f(x_0)\right)$.

If there exists $f^{(k+1)\alpha}(x)=\overbrace{D_x^{\alpha}...D_x^{\alpha}}^{k+1\text{ times}}f(x)$ for any $x\in I\subseteq R$, then we denoted $f\in D_{(k+1)\alpha}(I)$, where $k=0,1,2,...$

Definition 3. [26] *Let $f(x)\in C_{\alpha}[a,b]$. Then the local fractional integral is defined by,*

$${}_aI_b^{\alpha}f(x)=\frac{1}{\Gamma(\alpha+1)}\int_a^b f(t)(dt)^{\alpha}=\frac{1}{\Gamma(\alpha+1)}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^{\alpha},$$

with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\{\Delta t_1,\Delta t_2,...,\Delta t_{N-1}\}$, where $[t_j,t_{j+1}]$, $j=0,...,N-1$ and $a=t_0<t_1<...<t_{N-1}<t_N=b$ is partition of interval $[a,b]$.

Here, it follows that ${}_aI_b^{\alpha}f(x)=0$ if $a=b$ and ${}_aI_b^{\alpha}f(x)=-{}_bI_a^{\alpha}f(x)$ if $a<b$. If for any $x\in[a,b]$, there exists ${}_aI_x^{\alpha}f(x)$, then we denoted by $f(x)\in I_x^{\alpha}[a,b]$.

Definition 4 (Generalized convex function)**.** [26] *Let $f:I\subseteq R\to R^{\alpha}$. For any $x_1,x_2\in I$ and $\lambda\in[0,1]$, if the following inequality*

$$f(\lambda x_1+(1-\lambda)x_2)\leq\lambda^{\alpha}f(x_1)+(1-\lambda)^{\alpha}f(x_2)$$

holds, then f is called a generalized convex function on I.

Here are two basic examples of generalized convex functions:
(1) $f(x)=x^{\alpha p}$, $x\geq 0$, $p>1$;
(2) $f(x)=E_{\alpha}(x^{\alpha})$, $x\in R$ where $E_{\alpha}(x^{\alpha})=\sum\limits_{k=0}^{\infty}\frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Leffer function.

Theorem 2. [12] *Let $f \in D_\alpha(I)$, then the following conditions are equivalent*

a) f is a generalized convex function on I

b) $f^{(\alpha)}$ is an increasing function on I

c) for any $x_1, x_2 \in I$,

$$f(x_2) - f(x_1) \geq \frac{f^{(\alpha)}(x_1)}{\Gamma(1+\alpha)} (x_2 - x_1)^\alpha .$$

Corollary 1. [12] *Let $f \in D_{2\alpha}(a,b)$. Then f is a generalized convex function (or a generalized concave function) if and only if*

$$f^{(2\alpha)}(x) \geq 0 \left(or\ f^{(2\alpha)}(x) \leq 0 \right)$$

for all $x \in (a,b)$.

Lemma 1. [26]

(1) (Local fractional integration is anti-differentiation) Suppose that $f(x) = g^{(\alpha)}(x) \in C_\alpha [a,b]$, then we have

$${}_aI_b^\alpha f(x) = g(b) - g(a).$$

(2) (Local fractional integration by parts) Suppose that $f(x), g(x) \in D_\alpha [a,b]$ and $f^{(\alpha)}(x)$, $g^{(\alpha)}(x) \in C_\alpha [a,b]$, then we have

$${}_aI_b^\alpha f(x) g^{(\alpha)}(x) = f(x)g(x)|_a^b - {}_aI_b^\alpha f^{(\alpha)}(x) g(x).$$

Lemma 2. [26] *We have*

i) $\dfrac{d^\alpha x^{k\alpha}}{dx^\alpha} = \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha};$

ii) $\dfrac{1}{\Gamma(\alpha+1)} \displaystyle\int_a^b x^{k\alpha} (dx)^\alpha = \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha} \right), \ k \in R.$

Lemma 3. [26] *Suppose that $f(x) \in C_\alpha [a,b]$, then*

$$\frac{d^\alpha ({}_aI_x^\alpha f(t))}{dx^\alpha} = f(x) \quad a < x < b.$$

Lemma 4 (Generalized Hölder's inequality)**.** [26] *Let $f, g \in C_\alpha [a,b]$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)g(x)| (dx)^\alpha \leq \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)|^p (dx)^\alpha \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |g(x)|^q (dx)^\alpha \right)^{\frac{1}{q}} .$$

In [21], Sarikaya and Budak proved the following generalized Ostrowski inequality:

Theorem 3 (Generalized Ostrowski inequality)**.** *Let $I \subseteq \mathbb{R}$ be an interval, $f : I^0 \subseteq \mathbb{R} \to \mathbb{R}^\alpha$ (I^0 is the interior of I) such that $f \in D_\alpha(I^0)$ and $f^{(\alpha)} \in C_\alpha [a,b]$ for $a, b \in I^0$ with $a < b$ Then. for all $x \in [a,b]$, we have the inequality*

(2.1)

$$\left| f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_aI_b^\alpha f(t) \right| \leq 2^\alpha \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[\frac{1}{4^\alpha} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^{2\alpha} \right] (b-a)^\alpha \left\| f^{(\alpha)} \right\|_\infty .$$

For more information and recent developments on local fractional theory, please refer to [6],[12],[13],[21]-[30].

The aim of the this paper is to obtain some generalized weighted Ostrowski inequality for local fractional integrals.

3. Main Results

We will give a identity for local fractional integrals as follow:

Theorem 4. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I^0 \subseteq \mathbb{R} \rightarrow \mathbb{R}^\alpha$ (I^0 is the interior of I) such that $f \in D_\alpha(I^0)$ and $f^{(\alpha)} \in C_\alpha[a,b]$ for $a, b \in I^0$ with $a < b$ and $w : [a,b] \rightarrow \mathbb{R}^\alpha$, non-negative and $w(x) \in I_x^\alpha[a,b]$. Then, for all $x \in [a,b]$, we have the identity*

$$[{}_aI_b^\alpha w(t)]\, f(x) - {}_aI_b^\alpha w(t) f(t) = \frac{1}{\Gamma(1+\alpha)} \int_a^b p_w(x,t) f^{(\alpha)}(t)\,(dt)^\alpha \tag{3.1}$$

where

$$p(x,t) = \begin{cases} \frac{1}{\Gamma(1+\alpha)} \int_a^t w(u)\,(du)^\alpha, & t \in [a,x] \\ \frac{1}{\Gamma(1+\alpha)} \int_b^t w(u)\,(du)^\alpha, & t \in (x,b]. \end{cases}$$

Proof. We have

$$\begin{aligned} K &= \frac{1}{\Gamma(1+\alpha)} \int_a^b p_w(x,t) f^{(\alpha)}(t)\,(dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \int_a^x \left(\frac{1}{\Gamma(1+\alpha)} \int_a^t w(u)\,(du)^\alpha \right) f^{(\alpha)}(t)\,(dt)^\alpha \\ &\quad + \frac{1}{\Gamma(1+\alpha)} \int_x^b \left(\frac{1}{\Gamma(1+\alpha)} \int_b^t w(u)\,(du)^\alpha \right) f^{(\alpha)}(t)\,(dt)^\alpha \\ &= K_1 + K_2. \end{aligned}$$

Using the local fractional integration by parts, we have

$$\begin{aligned} K_1 &= \frac{1}{\Gamma(1+\alpha)} \int_a^x \left(\frac{1}{\Gamma(1+\alpha)} \int_a^t w(u)\,(du)^\alpha \right) f^{(\alpha)}(t)\,(dt)^\alpha \\ &= \left(\frac{1}{\Gamma(1+\alpha)} \int_a^t w(u)\,(du)^\alpha \right) f(t) \Bigg|_a^x - \frac{1}{\Gamma(1+\alpha)} \int_a^x w(t) f(t)\,(dt)^\alpha \\ &= \left(\frac{1}{\Gamma(1+\alpha)} \int_a^x w(u)\,(du)^\alpha \right) f(x) - \frac{1}{\Gamma(1+\alpha)} \int_a^x w(t) f(t)\,(dt)^\alpha \end{aligned} \tag{3.2}$$

and similarly,

$$K_2 = \left(\frac{1}{\Gamma(1+\alpha)} \int_x^b w(u)\,(du)^\alpha \right) f(x) - \frac{1}{\Gamma(1+\alpha)} \int_x^b w(t) f(t)\,(dt)^\alpha . \tag{3.3}$$

Adding (3.2) and (3.3), we obtain

$$\begin{aligned} K &= \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b w(u)\,(du)^\alpha \right) f(x) - \frac{1}{\Gamma(1+\alpha)} \int_a^b w(t) f(t)\,(dt)^\alpha \\ &= \left[{}_aI_b^\alpha w(t) \right] f(x) - {}_aI_b^\alpha w(t) f(t) \end{aligned}$$

which completes the proof. □

Remark 1. *If we take $w \equiv 1$ in Theorem 4, then the Theorem 4 reduces the Theorem 3 in* [21].

Theorem 5 (Generalized weighted Ostrowski inequality)**.** *Suppose that the assumptions of Theorem 4 are satisfied, $\left\| f^{(\alpha)} \right\|_\infty = \sup_{x\in[a,b]} \left| f^{(\alpha)}(x) \right|$, then we have the following generalized weighted Ostrowski inequality*

$$\begin{aligned} &\left| \left[{}_aI_b^\alpha w(t) \right] f(x) - {}_aI_b^\alpha w(t) f(t) \right| \\ &\leq \frac{2^\alpha (b-a)^{2\alpha}}{\Gamma(1+2\alpha)} \left[\frac{1}{4^\alpha} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^{2\alpha} \right] \|w\|_\infty \left\| f^{(\alpha)} \right\|_\infty . \end{aligned} \tag{3.4}$$

Proof. Taking modulus in Theorem 4, we have

$$\begin{aligned} &\left| \left[{}_aI_b^\alpha w(t) \right] f(x) - {}_aI_b^\alpha w(t) f(t) \right| \\ &\leq \frac{1}{\Gamma(1+\alpha)} \int_a^b \left| p_w(x,t) \right| \left| f^{(\alpha)}(t) \right| (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \int_a^x \left(\frac{1}{\Gamma(1+\alpha)} \int_a^t w(u)\,(du)^\alpha \right) \left| f^{(\alpha)}(t) \right| (dt)^\alpha \\ &\quad + \frac{1}{\Gamma(1+\alpha)} \int_x^b \left(\frac{1}{\Gamma(1+\alpha)} \int_t^b w(u)\,(du)^\alpha \right) \left| f^{(\alpha)}(t) \right| (dt)^\alpha . \end{aligned}$$

Then, it follows that

$$\begin{aligned}
&\left|\left[{}_aI_b^{\alpha}w(t)\right]f(x)-{}_aI_b^{\alpha}w(t)f(t)\right| \\
&\leq \frac{\left\|f^{(\alpha)}\right\|_{\infty}\|w\|_{\infty}}{\Gamma(1+\alpha)}\left[\frac{1}{\Gamma(1+\alpha)}\int_a^x (t-a)^{\alpha}(dt)^{\alpha}+\frac{1}{\Gamma(1+\alpha)}\int_a^x (b-t)^{\alpha}(dt)^{\alpha}\right] \\
&= \frac{\left\|f^{(\alpha)}\right\|_{\infty}\|w\|_{\infty}}{\Gamma(1+\alpha)}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[(x-a)^{2\alpha}+(b-x)^{2\alpha}\right] \\
&= \frac{2^{\alpha}(b-a)^{2\alpha}}{\Gamma(1+2\alpha)}\left[\frac{1}{4^{\alpha}}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^{2\alpha}\right]\left\|f^{(\alpha)}\right\|_{\infty}\|w\|_{\infty}.
\end{aligned}$$

which completes the proof. □

Remark 2. *If we take $w\equiv 1$ in Theorem 5, then the inequality (3.4) reduces the inequality (2.1).*

Theorem 6. *Suppose that the assumptions of Theorem 4 are satisfied, then we have the inequality*

$$\begin{aligned}
&\left|\left[{}_aI_b^{\alpha}w(t)\right]f(x)-{}_aI_b^{\alpha}w(t)f(t)\right| \\
&\leq \frac{\left\|f^{(\alpha)}\right\|_{q}\|w\|_{p}}{\Gamma(1+\alpha)}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\left[(x-a)^{(p+1)\alpha}+(b-x)^{(p+1)\alpha}\right]^{\frac{1}{p}}
\end{aligned}$$

where $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\left\|f^{(\alpha)}\right\|_{q}$ is defined by

$$\left\|f^{(\alpha)}\right\|_{q}=\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|f^{(\alpha)}(t)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}.$$

Proof. Taking modulus in Theorem 4 and using the generalized Hölder's inequality (Lemma 4), we obtain

$$
\begin{aligned}
& \left|\left[{}_aI_b^{\alpha}w(t)\right]f(x)-{}_aI_b^{\alpha}w(t)f(t)\right| \\
\leq\ & \frac{1}{\Gamma(1+\alpha)}\int_a^b |p_w(x,t)|\left|f^{(\alpha)}(t)\right|(dt)^{\alpha} \\
\leq\ & \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |p(x,t)|^p (dt)^{\alpha}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|f^{(\alpha)}(t)\right|^q (dt)^{\alpha}\right)^{\frac{1}{q}} \\
=\ & \left\|f^{(\alpha)}\right\|_q\left[\frac{1}{\Gamma(1+\alpha)}\int_a^x\left(\frac{1}{\Gamma(1+\alpha)}\int_a^t w(u)\,(du)^{\alpha}\right)^p (dt)^{\alpha}\right. \\
& \left.+\frac{1}{\Gamma(1+\alpha)}\int_x^b\left(\frac{1}{\Gamma(1+\alpha)}\int_t^b w(u)\,(du)^{\alpha}\right)^p (dt)^{\alpha}\right]^{\frac{1}{p}} \\
\leq\ & \frac{\left\|f^{(\alpha)}\right\|_q\|w\|_p}{\Gamma(1+\alpha)}\left[\frac{1}{\Gamma(1+\alpha)}\int_a^x (t-a)^{p\alpha}(dt)^{\alpha}+\frac{1}{\Gamma(1+\alpha)}\int_x^b (b-t)^{p\alpha}(dt)^{\alpha}\right] \\
=\ & \frac{\left\|f^{(\alpha)}\right\|_q\|w\|_p}{\Gamma(1+\alpha)}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\left[(x-a)^{(p+1)\alpha}+(b-x)^{(p+1)\alpha}\right]\right)^{\frac{1}{p}}
\end{aligned}
$$

which completes the proof. □

Remark 3. *If we take $w \equiv 1$ in Theorem 6, then we have the inequality*

$$
\begin{aligned}
& \left|f(x)-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(t)\right| \\
\leq\ & \frac{\left\|f^{(\alpha)}\right\|_q}{(b-a)^{\alpha}}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\left[(x-a)^{(p+1)\alpha}+(b-x)^{(p+1)\alpha}\right]^{\frac{1}{p}}
\end{aligned}
$$

which is proved by Sarikaya and Budak in [21].

Theorem 7. *The assumptions of Theorem 4 are satisfied. If* $\left|f^{(\alpha)}\right|^q$ *is the generalized convex, then we have the following inequality*

$$(3.5)\quad \left|\left[{}_aI_b^{\alpha}w(t)\right]f(x)-{}_aI_b^{\alpha}w(t)f(t)\right|$$

$$\leq \frac{\|w\|_{[a,b],p}}{(b-a)^{\frac{\alpha}{q}}\Gamma(1+\alpha)}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{1}{q}}$$

$$\times\left[(x-a)^{\left(\frac{p+1}{p}\right)\alpha}\left(\left[(b-a)^{2\alpha}-(b-x)^{2\alpha}\right]\left|f^{(\alpha)}(a)\right|^q+(x-a)^{2\alpha}\left|f^{(\alpha)}(b)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+(b-x)^{\left(\frac{p+1}{p}\right)\alpha}\left((b-x)^{2\alpha}\left|f^{(\alpha)}(a)\right|^q+\left[(b-a)^{2\alpha}-(x-a)^{2\alpha}\right]\left|f^{(\alpha)}(b)\right|^q\right)^{\frac{1}{q}}\right]$$

where $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $\|w\|_{[a,b],p}$ *is defined by*

$$\|w\|_{[a,b],p}=\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b|w(t)|^p(dt)^{\alpha}\right)^{\frac{1}{p}}.$$

Proof. Taking modulus in Theorem 4

$$(3.6)\quad \left|\left[{}_aI_b^{\alpha}w(t)\right]f(x)-{}_aI_b^{\alpha}w(t)f(t)\right|$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(1+\alpha)}\int_a^b|p_w(x,t)|\left|f^{(\alpha)}(t)\right|(dt)^{\alpha}\\
&= \frac{1}{\Gamma(1+\alpha)}\int_a^x\left(\frac{1}{\Gamma(1+\alpha)}\int_a^t w(u)(du)^{\alpha}\right)\left|f^{(\alpha)}(t)\right|(dt)^{\alpha}\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_x^b\left(\frac{1}{\Gamma(1+\alpha)}\int_t^b w(u)(du)^{\alpha}\right)\left|f^{(\alpha)}(t)\right|(dt)^{\alpha}.\\
&= K_3+K_4.
\end{aligned}$$

Using the generalized Hölder's inequality, we obtain

$$\begin{aligned}
K_3 &\leq \left(\frac{1}{\Gamma(1+\alpha)}\int_a^x\left(\frac{1}{\Gamma(1+\alpha)}\int_a^t w(u)(du)^{\alpha}\right)^p(dt)^{\alpha}\right)^{\frac{1}{p}}\\
&\quad\times\left(\frac{1}{\Gamma(1+\alpha)}\int_a^x\left|f^{(\alpha)}(t)\right|^q(dt)^{\alpha}\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|f^{(\alpha)}\right|^{q}$ is the generalized convex, we have

$$\begin{aligned}\left|f^{(\alpha)}(t)\right|^{q} &= \left|f^{(\alpha)}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|^{q} \\ &\leq \left(\frac{b-t}{b-a}\right)^{\alpha}\left|f^{(\alpha)}(a)\right|^{q}+\left(\frac{t-a}{b-a}\right)^{\alpha}\left|f^{(\alpha)}(b)\right|^{q}.\end{aligned}$$

Then, it follows that

$$\begin{aligned}K_3 &\leq \|w\|_{[a,x],p}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^x\frac{(t-a)^{p\alpha}}{[\Gamma(1+\alpha)]^{p}}(dt)^{\alpha}\right)^{\frac{1}{p}} \\ &\quad\times\left(\frac{\left|f^{(\alpha)}(a)\right|^{q}}{\Gamma(1+\alpha)}\int_a^x\left(\frac{b-t}{b-a}\right)^{\alpha}(dt)^{\alpha}+\frac{\left|f^{(\alpha)}(b)\right|^{q}}{\Gamma(1+\alpha)}\int_a^x\left(\frac{t-a}{b-a}\right)^{\alpha}(dt)^{\alpha}\right)^{\frac{1}{q}} \\ &= \frac{\|w\|_{[a,x],p}}{(b-a)^{\frac{\alpha}{q}}\Gamma(1+\alpha)}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}(x-a)^{(p+1)\alpha}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{1}{q}} \\ &\quad\times\left(\left[(b-a)^{2\alpha}-(b-x)^{2\alpha}\right]\left|f^{(\alpha)}(a)\right|^{q}+(x-a)^{2\alpha}\left|f^{(\alpha)}(b)\right|^{q}\right)^{\frac{1}{q}}.\end{aligned}$$

Using the similar way, we have

$$\begin{aligned}K_4 &\leq \frac{\|w\|_{[x,b],p}}{(b-a)^{\frac{\alpha}{q}}\Gamma(1+\alpha)}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}(b-x)^{(p+1)\alpha}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{1}{q}} \\ &\quad\times\left((b-x)^{2\alpha}\left|f^{(\alpha)}(a)\right|^{q}+\left[(b-a)^{2\alpha}-(x-a)^{2\alpha}\right]\left|f^{(\alpha)}(b)\right|^{q}\right)^{\frac{1}{q}}.\end{aligned}$$

Using the fact that $\|w\|_{[a,x],p}\leq\|w\|_{[a,b],p}$ and $\|w\|_{[x,b],p}\leq\|w\|_{[a,b],p}$, then we obtain required result. □

Corollary 2. *Under assumptions of Theorem 7 with $w\equiv 1$, then we have the inequality*

$$\begin{aligned}&\left|f(x)-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}f(t)\right| \quad (3.7)\\ &\leq \frac{\Gamma(1+\alpha)}{(b-a)^{\left(1+\frac{1}{q}\right)\alpha}}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{1}{q}} \\ &\quad\times\Bigg[(x-a)^{\left(\frac{p+1}{p}\right)\alpha}\left(\left[(b-a)^{2\alpha}-(b-x)^{2\alpha}\right]\left|f^{(\alpha)}(a)\right|^{q}+(x-a)^{2\alpha}\left|f^{(\alpha)}(b)\right|^{q}\right)^{\frac{1}{q}} \\ &\quad+(b-x)^{\left(\frac{p+1}{p}\right)\alpha}\left((b-x)^{2\alpha}\left|f^{(\alpha)}(a)\right|^{q}+\left[(b-a)^{2\alpha}-(x-a)^{2\alpha}\right]\left|f^{(\alpha)}(b)\right|^{q}\right)^{\frac{1}{q}}\Bigg].\end{aligned}$$

Corollary 3. *If we choose $x=\frac{a+b}{2}$ in inequality (3.7), then we obtain the following midpoint inequality*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma\left(1+\alpha\right)}{\left(b-a\right)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(t)\right| \\
\leq\ & \frac{\Gamma\left(1+\alpha\right)\left(b-a\right)^{\alpha}}{4^{\alpha}}\left(\frac{\Gamma\left(1+p\alpha\right)}{\Gamma\left(1+\left(p+1\right)\alpha\right)}\right)^{\frac{1}{p}}\left(\frac{\Gamma\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)}\right)^{\frac{1}{q}} \\
&\times\left[\left(\frac{3^{\alpha}\left|f^{(\alpha)}(a)\right|^{q}+\left|f^{(\alpha)}(b)\right|^{q}}{4^{\alpha}}\right)^{\frac{1}{q}}+\left(\frac{\left|f^{(\alpha)}(a)\right|^{q}+3^{\alpha}\left|f^{(\alpha)}(b)\right|^{q}}{4^{\alpha}}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

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Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY

E-mail address: hsyn.budak@gmail.com

E-mail address: sarikayamz@gmail.com