

GENERALIZED STEFFENSEN INEQUALITIES FOR LOCAL FRACTIONAL INTEGRALS

MEHMET ZEKI SARIKAYA, TUBA TUNC, AND SAMET ERDEN

ABSTRACT. Firstly we give a important integral inequality which is generalized Steffensen's inequality. Then, we establish weighted version of generalized Steffensen's inequality for local fractional integrals. Finally, we obtain several inequalities related these inequalities using the local fractional integral.

1. INTRODUCTION

In [17], J. S. Steffensen established the following result which is known as Steffensen's inequality in the literature.

Theorem 1. *Let a and b be real numbers such that $a < b$, $f, g : [a, b] \rightarrow \mathbb{R}$ be integrable functions such that f is nonincreasing and for every $x \in [a, b]$, $0 \leq g(x) \leq 1$. Then*

$$(1.1) \quad \int_{b-\lambda}^b f(x)dx \leq \int_a^b f(x)g(x)dx \leq \int_a^{a+\lambda} f(x)dx$$

where

$$\lambda = \int_a^b g(x)dx.$$

The most basic inequality which deals with the comparison between integrals over a whole interval $[a, b]$ and integrals over a subset of $[a, b]$ is the following inequality. The inequality (1.1) has attracted considerable attention and interest from mathematicans and researchers. Due to this, over the years, the interested reader is also refered to ([1], [2], [4]-[7], [10], [11] and [18]) for integral inequalities.

In [19], Wu and Srivastava proved the following inequality which is weighted version of the inequality (1.1).

Theorem 2. *Let f , g and h be integrable functions defined on $[a, b]$ with f nonincreasin. Also let $0 \leq g(x) \leq h(x)$ for all $x \in [a, b]$. Then, the following inequalities*

2000 *Mathematics Subject Classification.* 26D15, 26A33.

Key words and phrases. Steffensen's inequality, local fractional integral, fractal space, generalized convex function.

hold:

$$\begin{aligned}
& \int_{b-\lambda}^b f(x)h(x)dx \\
& \leq \int_{b-\lambda}^b (f(x)h(x) - [f(x) - f(b-\lambda)][h(x) - g(x)]) dx \leq \int_a^b f(x)g(x)dx \\
& \leq \int_a^{a+\lambda} (f(x)h(x) - [f(x) - f(a+\lambda)][h(x) - g(x)]) dx \leq \int_a^{a+\lambda} f(x)h(x)dx
\end{aligned}$$

where λ is given by

$$\int_a^{a+\lambda} h(x)dx = \int_a^b g(x)dx = \int_{b-\lambda}^b h(x)dx.$$

2. PRELIMINARIES

Recall the set R^α of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [20, 21] and so on.

Recently, the theory of Yang's fractional sets [20] was introduced as follows.

For $0 < \alpha \leq 1$, we have the following α -type set of element sets:

Z^α : The α -type set of integer is defined as the set $\{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, \dots, \pm n^\alpha, \dots\}$.

Q^α : The α -type set of the rational numbers is defined as the set $\{m^\alpha = \left(\frac{p}{q}\right)^\alpha : p, q \in Z, q \neq 0\}$.

J^α : The α -type set of the irrational numbers is defined as the set $\{m^\alpha \neq \left(\frac{p}{q}\right)^\alpha : p, q \in Z, q \neq 0\}$.

R^α : The α -type set of the real line numbers is defined as the set $R^\alpha = Q^\alpha \cup J^\alpha$.

If a^α, b^α and c^α belongs the set R^α of real line numbers, then

- (1) $a^\alpha + b^\alpha$ and $a^\alpha b^\alpha$ belongs the set R^α ;
- (2) $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a+b)^\alpha = (b+a)^\alpha$;
- (3) $a^\alpha + (b^\alpha + c^\alpha) = (a+b)^\alpha + c^\alpha$;
- (4) $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$;
- (5) $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$;
- (6) $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$;
- (7) $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 1. [20] *A non-differentiable function $f : R \rightarrow R^\alpha$, $x \rightarrow f(x)$ is called to be local fractional continuous at x_0 , if for any $\varepsilon > 0$, there exists $\delta > 0$, such that*

$$|f(x) - f(x_0)| < \varepsilon^\alpha$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in R$. If $f(x)$ is local continuous on the interval (a, b) , we denote $f(x) \in C_\alpha(a, b)$.

Definition 2. [20] *The local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by*

$$f^{(\alpha)}(x_0) = \left. \frac{d^\alpha f(x)}{dx^\alpha} \right|_{x=x_0} = \lim_{x \rightarrow x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha},$$

where $\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(\alpha + 1) (f(x) - f(x_0))$.

If there exists $f^{(k+1)\alpha}(x) = \overbrace{D_x^\alpha \dots D_x^\alpha}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq \mathbb{R}$, then we denoted $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, \dots$

Definition 3. [20] *Let $f(x) \in C_\alpha [a, b]$. Then the local fractional integral is defined by,*

$${}_a I_b^\alpha f(x) = \frac{1}{\Gamma(\alpha + 1)} \int_a^b f(t) (dt)^\alpha = \frac{1}{\Gamma(\alpha + 1)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} f(t_j) (\Delta t_j)^\alpha,$$

with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max \{\Delta t_1, \Delta t_2, \dots, \Delta t_{N-1}\}$, where $[t_j, t_{j+1}]$, $j = 0, \dots, N-1$ and $a = t_0 < t_1 < \dots < t_{N-1} < t_N = b$ is partition of interval $[a, b]$.

Here, it follows that ${}_a I_b^\alpha f(x) = 0$ if $a = b$ and ${}_a I_b^\alpha f(x) = -{}_b I_a^\alpha f(x)$ if $a < b$. If for any $x \in [a, b]$, there exists ${}_a I_x^\alpha f(x)$, then we denoted by $f(x) \in I_x^\alpha [a, b]$.

Lemma 1. [20] *We have*

$$\begin{aligned} i) \quad & \frac{d^\alpha x^{k\alpha}}{dx^\alpha} = \frac{\Gamma(1 + k\alpha)}{\Gamma(1 + (k-1)\alpha)} x^{(k-1)\alpha}; \\ ii) \quad & \frac{1}{\Gamma(\alpha + 1)} \int_a^b x^{k\alpha} (dx)^\alpha = \frac{\Gamma(1 + k\alpha)}{\Gamma(1 + (k+1)\alpha)} (b^{(k+1)\alpha} - a^{(k+1)\alpha}), \quad k \in \mathbb{R}. \end{aligned}$$

The interested reader is able to look over the references [3],[8],[9],[12]-[16], [20]-[24] for local freactional theory.

In this study, generalized Steffensen's inequality is established. Then, some inequalities related generalized this inequality are given by using local fractional integrals.

3. MAIN RESULTS

We start the following important inequality for local fractional integrals:

Theorem 3 (Generalized Steffensen's Inequality). *Let $f(x), g(x) \in I_x^\alpha [a, b]$ such that f never increases and $0 \leq g(x) \leq 1$ on $[a, b]$ with $a < b$. Then*

$$(3.1) \quad {}_{b-\lambda} I_b^\alpha f(x) \leq {}_a I_b^\alpha f(x) g(x) \leq {}_a I_{a+\lambda}^\alpha f(x)$$

where

$$(3.2) \quad \lambda^\alpha = \Gamma(\alpha + 1) {}_a I_b^\alpha g(x).$$

Proof. For the proof of theorem, we give two different methods:

First method: By direct computation, we get

$$\begin{aligned}
 (3.3) \quad & \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} f(x)(dx)^\alpha - {}_a I_b^\alpha f(x)g(x) \\
 &= \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} [f(x) - f(a+\lambda)] [1-g(x)] (dx)^\alpha \\
 & \quad + \frac{1}{\Gamma(\alpha+1)} \int_{a+\lambda}^b [f(a+\lambda) - f(x)] g(x)(dx)^\alpha.
 \end{aligned}$$

Using the equality (3.3), because f is nonincreasing, we obtain the second inequality of (3.1).

Similarly, we have

$$\begin{aligned}
 (3.4) \quad & {}_a I_b^\alpha f(x)g(x) - \frac{1}{\Gamma(\alpha+1)} \int_{b-\lambda}^b f(x)(dx)^\alpha \\
 &= \frac{1}{\Gamma(\alpha+1)} \int_a^{b-\lambda} [f(x) - f(b-\lambda)] g(x)(dx)^\alpha \\
 & \quad + \frac{1}{\Gamma(\alpha+1)} \int_{b-\lambda}^b [f(b-\lambda) - f(x)] [1-g(x)] (dx)^\alpha.
 \end{aligned}$$

Using the equality (3.4), because f is nonincreasing, we obtain the first inequality of (3.1). Thus, the proof is completed.

Second method: Now, we prove the same of above Theorem in a deffrent way.

Because f is nonincreasing, the second inequality of (3.1) may be derived as follows:

$$\begin{aligned}
 & \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} f(x)(dx)^\alpha - {}_a I_b^\alpha f(x)g(x) \\
 &= \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} f(x) [1-g(x)] (dx)^\alpha - \frac{1}{\Gamma(\alpha+1)} \int_{a+\lambda}^b f(x)g(x)(dx)^\alpha \\
 &\geq \frac{f(a+\lambda)}{\Gamma(\alpha+1)} \int_a^{a+\lambda} [1-g(x)] (dx)^\alpha - \frac{1}{\Gamma(\alpha+1)} \int_{a+\lambda}^b f(x)g(x)(dx)^\alpha
 \end{aligned}$$

$$\begin{aligned}
&= \frac{f(a+\lambda)}{\Gamma(\alpha+1)} \left[\lambda^\alpha - \int_a^{a+\lambda} g(x)(dx)^\alpha \right] - \frac{1}{\Gamma(\alpha+1)} \int_{a+\lambda}^b f(x)g(x)(dx)^\alpha \\
&= \frac{1}{\Gamma(\alpha+1)} \int_{a+\lambda}^b [f(a+\lambda) - f(x)]g(x)(dx)^\alpha \geq 0.
\end{aligned}$$

The first inequality of (3.1) can be proved in a similar way. However, the second inequality implies the first.

Indeed, let

$$G(x) = 1 - g(x)$$

and

$$\Lambda^\alpha = \Gamma(\alpha+1) {}_a I_b^\alpha G(x).$$

Note that $0 \leq G(x) \leq 1$ if $0 \leq g(x) \leq 1$ in (a, b) .

Suppose the second inequality of (3.1) holds. Then, we obtain

$$\begin{aligned}
\frac{1}{\Gamma(\alpha+1)} \int_a^b f(x)G(x)(dx)^\alpha &\leq \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\Lambda} f(x)(dx)^\alpha \\
\frac{1}{\Gamma(\alpha+1)} \int_a^b f(x)(dx)^\alpha - \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\Lambda} f(x)(dx)^\alpha &\leq \frac{1}{\Gamma(\alpha+1)} \int_a^b f(x)g(x)(dx)^\alpha \\
\frac{1}{\Gamma(\alpha+1)} \int_{a+\Lambda}^b f(x)(dx)^\alpha &\leq \frac{1}{\Gamma(\alpha+1)} \int_a^b f(x)g(x)(dx)^\alpha.
\end{aligned}$$

Because of

$$\Lambda^\alpha = \Gamma(\alpha+1) {}_a I_b^\alpha G(x) = b^\alpha - a^\alpha - \lambda^\alpha$$

we have the identity

$$(3.5) \quad \Lambda + a = b - \lambda.$$

From (3.5), we get the inequality

$$\frac{1}{\Gamma(\alpha+1)} \int_{b-\lambda}^b f(x)(dx)^\alpha \leq {}_a I_b^\alpha f(x)g(x)$$

which is the first inequality of (3.1). The proof is thus completed. \square

In order to prove weighted version of generalized Steffensen's inequality we need the following lemma:

Lemma 2. *Let f , g and h belong to $I_x^\alpha[a, b]$. Suppose also that λ is a real number such that*

$${}_a I_{a+\lambda}^\alpha h(x) = {}_a I_b^\alpha g(x) = {}_{b-\lambda} I_b^\alpha h(x).$$

Then, we have

$$\begin{aligned}
(3.6) \quad & {}_a I_b^\alpha f(x)g(x) \\
&= \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} (f(x)h(x) - [f(x) - f(a+\lambda)][h(x) - g(x)]) (dx)^\alpha \\
&\quad + \frac{1}{\Gamma(\alpha+1)} \int_{a+\lambda}^b [f(x) - f(a+\lambda)] g(x)(dx)^\alpha
\end{aligned}$$

and

$$\begin{aligned}
(3.7) \quad & {}_a I_b^\alpha f(x)g(x) \\
&= \frac{1}{\Gamma(\alpha+1)} \int_{b-\lambda}^b (f(x)h(x) - [f(x) - f(b-\lambda)][h(x) - g(x)]) (dx)^\alpha \\
&\quad + \frac{1}{\Gamma(\alpha+1)} \int_a^{b-\lambda} [f(x) - f(b-\lambda)] g(x)(dx)^\alpha.
\end{aligned}$$

Proof. The assumptions of the Lemma imply that

$$a \leq a + \lambda \leq b \text{ and } a \leq b - \lambda \leq b.$$

Firstly, we prove the validity of the equality (3.6). Indeed, by direct computation, we find that

$$\begin{aligned}
(3.8) \quad & {}_a I_b^\alpha f(x)g(x) \\
&= \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} f(x)g(x)(dx)^\alpha + \frac{1}{\Gamma(\alpha+1)} \int_{a+\lambda}^b f(x)g(x)(dx)^\alpha \\
&\quad + \frac{f(a+\lambda)}{\Gamma(\alpha+1)} \left(\int_a^b g(x)(dx)^\alpha - \int_a^{a+\lambda} g(x)(dx)^\alpha - \int_{a+\lambda}^b g(x)(dx)^\alpha \right)
\end{aligned}$$

Now, if we apply the following assumption of the Lemma:

$${}_a I_{a+\lambda}^\alpha h(x) = {}_a I_b^\alpha g(x)$$

to (3.8), we obtain

$$\begin{aligned}
(3.9) \quad & {}_a I_b^\alpha f(x)g(x) \\
&= \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} (f(x)g(x) + f(a+\lambda)[h(x) - g(x)]) (dx)^\alpha \\
&\quad + \frac{1}{\Gamma(\alpha+1)} \int_{a+\lambda}^b [f(x) - f(a+\lambda)] g(x)(dx)^\alpha.
\end{aligned}$$

If we add

$$\frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} f(x)h(x)(dx)^\alpha - \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} f(x)h(x)(dx)^\alpha$$

to right side of (3.9) and also we use elementary analysis, then we easily get the equality (3.6).

Secondly, if we apply above the operations for the following assumption of the Lemma

$${}_a I_b^\alpha g(x) = {}_{b-\lambda} I_b^\alpha h(x)$$

and also we consider the case $a \leq b - \lambda \leq b$, then we obtain the equality (3.7). Thus, the proof is completed. \square

Now, we prove weighted version generalized Steffensen's inequality using local fractional integrals.

Theorem 4. *Let f, g and h belong to $I_x^\alpha [a, b]$ with f nonincreasing. Suppose also that $0 \leq g(x) \leq h(x)$ for all $x \in [a, b]$. Then, we have the following inequalities*

$$\begin{aligned} (3.10) \quad & {}_{b-\lambda} I_b^\alpha f(x)h(x) \\ & \leq \frac{1}{\Gamma(\alpha+1)} \int_{b-\lambda}^b (f(x)h(x) - [f(x) - f(b-\lambda)] [h(x) - g(x)]) (dx)^\alpha \\ & \leq {}_a I_b^\alpha f(x)g(x) \\ & \leq \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} (f(x)h(x) - [f(x) - f(a+\lambda)] [h(x) - g(x)]) (dx)^\alpha \\ & \leq {}_a I_{a+\lambda}^\alpha f(x)h(x) \end{aligned}$$

where λ is given by

$${}_a I_{a+\lambda}^\alpha h(x) = {}_a I_b^\alpha g(x) = {}_{b-\lambda} I_b^\alpha h(x).$$

Proof. In view of the assumptions that the function f is nonincreasing on $[a, b]$ and that $0 \leq g(x) \leq h(x)$ for all $x \in [a, b]$, we find that

$$(3.11) \quad \frac{1}{\Gamma(\alpha+1)} \int_a^{b-\lambda} [f(x) - f(b-\lambda)] g(x)(dx)^\alpha \geq 0,$$

$$(3.12) \quad \frac{1}{\Gamma(\alpha+1)} \int_{b-\lambda}^b [f(b-\lambda) - f(x)] [h(x) - g(x)] (dx)^\alpha \geq 0,$$

$$(3.13) \quad \frac{1}{\Gamma(\alpha+1)} \int_{a+\lambda}^b [f(x) - f(a+\lambda)] g(x)(dx)^\alpha \leq 0,$$

and

$$(3.14) \quad \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} [f(a+\lambda) - f(x)] [h(x) - g(x)] (dx)^\alpha \leq 0.$$

Using the equality (3.7) together with the inequalities (3.11) and (3.12), we obtain that

$$(3.15) \quad \begin{aligned} & {}_{b-\lambda}I_b^\alpha f(x)h(x) \\ & \leq \frac{1}{\Gamma(\alpha+1)} \int_{b-\lambda}^b (f(x)h(x) - [f(x) - f(b-\lambda)] [h(x) - g(x)]) (dx)^\alpha \\ & \leq {}_aI_b^\alpha f(x)g(x). \end{aligned}$$

Using the equality (3.6) together with the inequalities (3.13) and (3.14) either, we get that

$$(3.16) \quad \begin{aligned} & {}_aI_b^\alpha f(x)g(x) \\ & \leq \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} (f(x)h(x) - [f(x) - f(a+\lambda)] [h(x) - g(x)]) (dx)^\alpha \\ & \leq {}_aI_{a+\lambda}^\alpha f(x)h(x). \end{aligned}$$

Combining the inequalities (3.15) and (3.16), we easily deduce required inequalities. \square

In particular, if we chose $h(t) = 1$ in (3.10), we obtain the following refinement of generalized Steffensen's inequality.

Corollary 1. *Let $f(x), g(x) \in I_x^\alpha [a, b]$ such that f never increases and $0 \leq g(x) \leq 1$ on $[a, b]$ with $a < b$. Then*

$$\begin{aligned} & {}_{b-\lambda}I_b^\alpha f(x) \\ & \leq \frac{1}{\Gamma(\alpha+1)} \int_{b-\lambda}^b (f(x) - [f(x) - f(b-\lambda)] [1 - g(x)]) (dx)^\alpha \\ & \leq {}_aI_b^\alpha f(x)g(x) \\ & \leq \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} (f(x) - [f(x) - f(a+\lambda)] [1 - g(x)]) (dx)^\alpha \\ & \leq {}_aI_{a+\lambda}^\alpha f(x) \end{aligned}$$

where

$$\lambda^\alpha = \Gamma(\alpha+1) {}_aI_b^\alpha g(x).$$

Theorem 5. *Let f, g and h belong to $I_x^\alpha [a, b]$ with f nonincreasing. Also let*

$$0 \leq \psi(x) \leq g(x) \leq h(x) - \psi(x)$$

for all $x \in [a, b]$. Then we have the inequalities

$$\begin{aligned}
& {}_{b-\lambda}I_b^\alpha f(x)h(x) + \frac{1}{\Gamma(\alpha+1)} \int_a^b |[f(x) - f(b-\lambda)]\psi(x)| (dx)^\alpha \\
& \leq {}_aI_b^\alpha f(x)g(x) \\
& \leq {}_aI_{a+\lambda}^\alpha f(x)h(x) - \frac{1}{\Gamma(\alpha+1)} \int_a^b |[f(x) - f(a+\lambda)]\psi(x)| (dx)^\alpha
\end{aligned}$$

where λ is given by

$${}_aI_{a+\lambda}^\alpha h(x) = {}_aI_b^\alpha g(x) = {}_{b-\lambda}I_b^\alpha h(x).$$

Proof. By the assumptions that the function f is nonincreasing on $[a, b]$ and that

$$0 \leq \psi(x) \leq g(x) \leq h(x) - \psi(x)$$

for all $x \in [a, b]$, it follows that

$$\begin{aligned}
(3.17) \quad & \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} [f(x) - f(a+\lambda)] [h(x) - g(x)] (dx)^\alpha \\
& + \frac{1}{\Gamma(\alpha+1)} \int_{a+\lambda}^b [f(a+\lambda) - f(x)] g(x) (dx)^\alpha \\
& = \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} |f(x) - f(a+\lambda)| [h(x) - g(x)] (dx)^\alpha \\
& + \frac{1}{\Gamma(\alpha+1)} \int_{a+\lambda}^b |f(a+\lambda) - f(x)| g(x) (dx)^\alpha \\
& \geq \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} |f(x) - f(a+\lambda)| \psi(x) (dx)^\alpha \\
& + \frac{1}{\Gamma(\alpha+1)} \int_{a+\lambda}^b |f(a+\lambda) - f(x)| \psi(x) (dx)^\alpha \\
& = \frac{1}{\Gamma(\alpha+1)} \int_a^b |[f(x) - f(a+\lambda)]\psi(x)| (dx)^\alpha.
\end{aligned}$$

Similarly, we find that

$$\begin{aligned}
(3.18) \quad & \frac{1}{\Gamma(\alpha+1)} \int_{b-\lambda}^b [f(b-\lambda) - f(x)] [h(x) - g(x)] (dx)^\alpha \\
& + \frac{1}{\Gamma(\alpha+1)} \int_a^{b-\lambda} [f(x) - f(b-\lambda)] g(x) (dx)^\alpha \\
& \geq \frac{1}{\Gamma(\alpha+1)} \int_a^b |[f(x) - f(b-\lambda)] \psi(x)| (dx)^\alpha.
\end{aligned}$$

If we use the equalities (3.6) and (3.7) and the inequalities (3.17) and (3.18), we obtain required inequalities. \square

Corollary 2. *Under the same assumptions of Theorem 5 with $h(x) = 1$ and $\psi(x) = M^\alpha$, then the following inequalities hold:*

$$\begin{aligned}
& {}_{b-\lambda}I_b^\alpha f(x) + \frac{M^\alpha}{\Gamma(\alpha+1)} \int_a^b |[f(x) - f(b-\lambda)]| (dx)^\alpha \\
& \leq {}_aI_b^\alpha f(x)g(x) \\
& \leq {}_{a+\lambda}I_b^\alpha f(x) - \frac{M^\alpha}{\Gamma(\alpha+1)} \int_a^b |[f(x) - f(a+\lambda)]| (dx)^\alpha
\end{aligned}$$

where $M^\alpha \in \mathbb{R}_+^\alpha \cup \{0^\alpha\}$ and

$$\lambda^\alpha = \Gamma(\alpha+1) {}_aI_b^\alpha g(x).$$

Finally, we give a general result on a considerably improved version of generalized Steffensen's inequality by introducing the additional paramaters λ_1 and λ_2 .

Theorem 6. *Let $f(x), g(x) \in I_x^\alpha [a, b]$ such that f never increases on $[a, b]$. Also let*

$$0^\alpha \leq \lambda_1^\alpha \leq \lambda^\alpha = \Gamma(\alpha+1) {}_aI_b^\alpha g(x) \leq \lambda_2^\alpha \leq (b-a)^\alpha$$

and

$$0 \leq M^\alpha \leq g(x) \leq (1-M)^\alpha$$

for all $x \in [a, b]$. Then, we have the inequalities

$$\begin{aligned}
(3.19) \quad & {}_{b-\lambda_1}I_b^\alpha f(x) + \frac{f(b)}{\Gamma(\alpha+1)} (\lambda - \lambda_1)^\alpha + \frac{M^\alpha}{\Gamma(\alpha+1)} \int_a^b |[f(x) - f(b-\lambda)]| (dx)^\alpha \\
& \leq {}_aI_b^\alpha f(x)g(x) \\
& \leq {}_aI_{a+\lambda_2}^\alpha f(x) + \frac{f(b)}{\Gamma(\alpha+1)} (\lambda_2 - \lambda)^\alpha - \frac{M^\alpha}{\Gamma(\alpha+1)} \int_a^b |[f(x) - f(a+\lambda)]| (dx)^\alpha.
\end{aligned}$$

Proof. By direct computation, we obtain

$$\begin{aligned}
(3.20) \quad & {}_a I_b^\alpha f(x)g(x) - {}_a I_{a+\lambda_2}^\alpha f(x) + \frac{f(b)}{\Gamma(\alpha+1)} \left(\lambda_2^\alpha - \int_a^b g(x)(dx)^\alpha \right) \\
&= \frac{1}{\Gamma(\alpha+1)} \left(\int_a^b f(x)g(x)(dx)^\alpha - \int_a^{a+\lambda_2} f(x)(dx)^\alpha \right) \\
&\quad + \frac{1}{\Gamma(\alpha+1)} \left(\int_a^{a+\lambda_2} f(b)(dx)^\alpha - \int_a^b f(b)g(x)(dx)^\alpha \right) \\
&\leq \frac{1}{\Gamma(\alpha+1)} \int_a^b [f(x) - f(b)]g(x)(dx)^\alpha - \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} [f(x) - f(b)](dx)^\alpha.
\end{aligned}$$

Because of the following assumption of the Theorem

$$0^\alpha \leq \lambda_1^\alpha \leq \lambda^\alpha \leq \lambda_2^\alpha \leq (b-a)^\alpha,$$

we find that

$$a^\alpha \leq a^\alpha + \lambda^\alpha \leq a^\alpha + \lambda_2^\alpha \leq b^\alpha$$

that is

$$a \leq a + \lambda \leq a + \lambda_2 \leq b.$$

Also, since f is nonincreasing, we have

$$f(x) - f(b) \geq 0$$

for all $x \in [a, b]$.

On the other hand, since the hypothesis of the Theorem, we conclude that the function $f(x) - f(b)$ belong to $I_x^\alpha [a, b]$ and nonincreasing on $[a, b]$. Thus, substituting $f(x) - f(b)$ instead of $f(x)$ in Corollary 2, we find that

$$\begin{aligned}
(3.21) \quad & \frac{1}{\Gamma(\alpha+1)} \int_a^b [f(x) - f(b)]g(x)(dx)^\alpha - \frac{1}{\Gamma(\alpha+1)} \int_{a+\lambda}^b [f(x) - f(b)](dx)^\alpha \\
&\leq -\frac{M}{\Gamma(\alpha+1)} \int_a^b |[f(x) - f(a+\lambda)]|(dx)^\alpha.
\end{aligned}$$

Combining the inequalities (3.20) and (3.21), we obtain

$$\begin{aligned}
& {}_a I_b^\alpha f(x)g(x) - {}_a I_{a+\lambda_2}^\alpha f(x) + \frac{f(b)}{\Gamma(\alpha+1)} (\lambda_2^\alpha - \lambda^\alpha) \\
&\leq -\frac{M^\alpha}{\Gamma(\alpha+1)} \int_a^b |[f(x) - f(a+\lambda)]|(dx)^\alpha.
\end{aligned}$$

which is the second inequality of (3.19).

In a similar way, we can prove that

$$\begin{aligned}
& {}_a I_b^\alpha f(x)g(x) - {}_{b-\lambda_1} I_b^\alpha f(x) - \frac{f(b)}{\Gamma(\alpha+1)} \left(\int_a^b g(x)(dx)^\alpha - \lambda_1^\alpha \right) \\
& \geq \frac{1}{\Gamma(\alpha+1)} \int_a^b [f(x) - f(b)] g(x)(dx)^\alpha + \frac{1}{\Gamma(\alpha+1)} \int_{b-\lambda}^b [f(b) - f(x)] (dx)^\alpha \\
& \geq \frac{M^\alpha}{\Gamma(\alpha+1)} \int_a^b |[f(x) - f(b-\lambda)]| (dx)^\alpha
\end{aligned}$$

which is the first inequality of (3.19). The proof is thus completed. \square

REFERENCES

- [1] S. Abramovich, M.K. Bakula, M. Matic', J.E. Pečarić, *A variant of Jensen–Steffensen's inequality and quasi-arithmetic means*, J. Math. Anal. Appl., 307 (2005), 370–386.
- [2] J. Bergh, *A generalization of Steffensen's inequality*, J. Math. Anal. Appl., 41 (1973), 187–191.
- [3] G-S. Chen, *Generalizations of Hölder's and some related integral inequalities on fractal space*, Journal of Function Spaces and Applications Volume 2013, Article ID 198405.
- [4] H. Gauchman, *On a further generalization of Steffensen's inequality*, J. Inequal. Appl., 5 (2000), 505–513.
- [5] F. Qi, B.-N. Guo, *On Steffensen pairs*, J. Math. Anal. Appl. 271 (2002) 534–541.
- [6] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlang New-York, Heidelberg, Berlin, 1970, pp. 107-118.
- [7] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and new inequalities in analysis*, ser. Math. Appl. (East European Ser.). Dordrecht: Kluwer Academic Publishers Group, 1993, vol. 61, pp. 311-331.
- [8] H. Mo, X Sui and D Yu, *Generalized convex functions on fractal sets and two related inequalities*, Abstract and Applied Analysis, Volume 2014, Article ID 636751, 7 pages.
- [9] H. Mo, *Generalized Hermite-Hadamard inequalities involving local fractional integral*, arXiv:1410.1062.
- [10] J. E. Pečarić, *On the Bellman generalization of Steffensen's inequality. II*, J. Math. Anal. Appl. 104 (1984) 432–434.
- [11] J. E. Pečarić, *On the Bellman generalization of Steffensen's inequality*, J. Math. Anal. Appl. 88 (1982) 505–507.
- [12] M. Z. Sarikaya and H Budak, *Generalized Ostrowski type inequalities for local fractional integrals*, RGMIA Research Report Collection, 18(2015), Article 62, 11 pp.
- [13] M. Z. Sarikaya, S.Erden and H. Budak, *Some generalized Ostrowski type inequalities involving local fractional integrals and applications*, RGMIA Research Report Collection, 18(2015), Article 63, 12 pp.
- [14] M. Z. Sarikaya H. Budak, *On generalized Hermite-Hadamard inequality for generalized convex function*, RGMIA Research Report Collection, 18(2015), Article 64, 15 pp.
- [15] M. Z. Sarikaya, S.Erden and H. Budak, *Some integral inequalities for local fractional integrals*, RGMIA Research Report Collection, 18(2015), Article 65, 12 pp.
- [16] M. Z. Sarikaya, H. Budak and S.Erden, *On new inequalities of Simpson's type for generalized convex functions*, RGMIA Research Report Collection, 18(2015), Article 66, 13 pp.
- [17] J. F. Steffensen, *On certain inequalities and methods of approximation*, J. Inst. Actuaries, 51 (1919), 274–297.
- [18] U. M. Ozkan and H. Yildirim, *Steffensen's integral inequality on time scales*, Journal of Inequalities and Applications, Article ID 46524, 10 pages, 2007.
- [19] S.-H. Wu and H.M. Srivastava, *Some improvements and generalizations of Steffensen's integral inequality*, Applied Mathematics and Computaion, 192 (2007) 422-428.

- [20] X. J. Yang, *Advanced Local Fractional Calculus and Its Applications*, World Science Publisher, New York, 2012.
- [21] J. Yang, D. Baleanu and X. J. Yang, *Analysis of fractal wave equations by local fractional Fourier series method*, Adv. Math. Phys. , 2013 (2013), Article ID 632309.
- [22] X. J. Yang, *Local fractional integral equations and their applications*, Advances in Computer Science and its Applications (ACSA) 1(4), 2012.
- [23] X. J. Yang, *Generalized local fractional Taylor's formula with local fractional derivative*, Journal of Expert Systems, 1(1) (2012) 26-30.
- [24] X. J. Yang, *Local fractional Fourier analysis*, Advances in Mechanical Engineering and its Applications 1(1), 2012 12-16.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, DÜZCE UNIVERSITY, KONURALP CAMPUS, DÜZCE-TURKEY
E-mail address: sarikayamz@gmail.com

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, DÜZCE UNIVERSITY, KONURALP CAMPUS, DÜZCE-TURKEY
E-mail address: tubatunc@duzce.edu.tr

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, BARTIN UNIVERSITY, BARTIN-TURKEY
E-mail address: erdensmt@gmail.com