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SOME GENERALIZED STEFFENSEN TYPE INEQUALITIES VIA LOCAL FRACTIONAL INTEGRALS

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Abstract. The purpose of this study is to give new inequalities concerning Steffensen's inequality for local fractional integral.

1. Introduction

In [12], J. S. Steffensen established following result which is known as Steffensen's inequality.

Theorem 1. *Let a and b be real numbers such that $a<b$, $f,g:[a,b]\to\mathbb{R}$ be integrable functions such that f is nonincreasing and for every $x\in[a,b]$, $0\leq g(x)\leq 1$. Then*

$$\int_{b-\lambda}^{b} f(x)dx \leq \int_{a}^{b} f(x)g(x)dx \leq \int_{a}^{a+\lambda} f(x)dx \tag{1.1}$$

where

$$\lambda=\int_{a}^{b} g(x)dx.$$

The most basic inequality which deals with the comparison between integrals over a whole interval $[a,b]$ and integrals over a subset of $[a,b]$ is the following inequality. The inequality (1.1) has attracted considerable attention and interest from mathematicans and researchers. Due to this, over the years, the interested reader is also refered to ([1],[2],[4]-[7],[10] [11],[19]) for integral inequalities.

2. Preliminaries

Recall the set R^{α} of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [20, 21] and so on.

Recently, the theory of Yang's fractional sets [20] was introduced as follows.

For $0<\alpha\leq 1$, we have the following α-type set of element sets:

Z^{α} : The α-type set of integer is defined as the set $\{0^{\alpha},\pm 1^{\alpha},\pm 2^{\alpha},...,\pm n^{\alpha},...\}$.

Q^{α} : The α-type set of the rational numbers is defined as the set $\{m^{\alpha}=\left(\frac{p}{q}\right)^{\alpha}: p,q\in Z,\ q\neq 0\}$.

J^{α} : The α-type set of the irrational numbers is defined as the set $\{m^{\alpha}\neq\left(\frac{p}{q}\right)^{\alpha}: p,q\in Z,\ q\neq 0\}$.

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R^{α} : The α-type set of the real line numbers is defined as the set $R^{\alpha}=Q^{\alpha}\cup J^{\alpha}$.

If a^{α},b^{α} and c^{α} belongs the set R^{α} of real line numbers, then

(1) $a^{\alpha}+b^{\alpha}$ and $a^{\alpha}b^{\alpha}$ belongs the set R^{α};

(2) $a^{\alpha}+b^{\alpha}=b^{\alpha}+a^{\alpha}=(a+b)^{\alpha}=(b+a)^{\alpha}$;

(3) $a^{\alpha}+(b^{\alpha}+c^{\alpha})=(a+b)^{\alpha}+c^{\alpha}$;

(4) $a^{\alpha}b^{\alpha}=b^{\alpha}a^{\alpha}=(ab)^{\alpha}=(ba)^{\alpha}$;

(5) $a^{\alpha}(b^{\alpha}c^{\alpha})=(a^{\alpha}b^{\alpha})c^{\alpha}$;

(6) $a^{\alpha}(b^{\alpha}+c^{\alpha})=a^{\alpha}b^{\alpha}+a^{\alpha}c^{\alpha}$;

(7) $a^{\alpha}+0^{\alpha}=0^{\alpha}+a^{\alpha}=a^{\alpha}$ and $a^{\alpha}1^{\alpha}=1^{\alpha}a^{\alpha}=a^{\alpha}$.

If $a^{\alpha}-b^{\alpha}$ is non-negative number, we say that a^{α}greater than or equal to b^{α} or b^{α} is less than or equal to a^{α}, and write, respectively, $a^{\alpha}\geq b^{\alpha}$ or $b^{\alpha}\leq a^{\alpha}$. If there is no possibility that $a^{\alpha}=b^{\alpha}$, we write $a^{\alpha}>b^{\alpha}$ or $b^{\alpha}<a^{\alpha}$.

Suppose that a^{α},b^{α} and c^{α} are any given real line numbers, then we have the following relations:

(1) Either $a^{\alpha}>b^{\alpha}$, $a^{\alpha}=b^{\alpha}$ or $a^{\alpha}<b^{\alpha}$ (Law of trichotomy);

(2) If $a^{\alpha}>b^{\alpha}$ and $b^{\alpha}>c^{\alpha}$, then $a^{\alpha}>c^{\alpha}$(Law of Transitivity);

(3) If $a^{\alpha}>b^{\alpha}$, then $a^{\alpha}+c^{\alpha}>b^{\alpha}+c^{\alpha}$;

(4) If $a^{\alpha}>b^{\alpha}$ and $c^{\alpha}>0^{\alpha}$, then $a^{\alpha}c^{\alpha}>b^{\alpha}c^{\alpha}$;

(5) If $a^{\alpha}>b^{\alpha}$ and $c^{\alpha}<0^{\alpha}$, then $a^{\alpha}c^{\alpha}<b^{\alpha}c^{\alpha}$.

The formula is similar to classical one in case of $\alpha=1$. As direct results, the following inequalies are valid:

If $a^{\alpha}>b^{\alpha}$, then $a>b$

If $a^{\alpha}=b^{\alpha}$, then $a=b$

If $a^{\alpha}<b^{\alpha}$, then $a<b$

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 1. [20] *A non-differentiable function $f:R\rightarrow R^{\alpha}$, $x\rightarrow f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon>0$, there exists $\delta>0$, such that*

$$|f(x)-f(x_0)|<\varepsilon^{\alpha}$$

holds for $|x-x_0|<\delta$, where $\varepsilon,\delta\in R$. If $f(x)$ is local continuous on the interval (a,b), we denote $f(x)\in C_{\alpha}(a,b)$.

Definition 2. [20] *The local fractional derivative of $f(x)$ of order α at $x=x_0$ is defined by*

$$f^{(\alpha)}(x_0)=\left.\frac{d^{\alpha}f(x)}{dx^{\alpha}}\right|_{x=x_0}=\lim_{x\rightarrow x_0}\frac{\Delta^{\alpha}\left(f(x)-f(x_0)\right)}{(x-x_0)^{\alpha}},$$

where $\Delta^{\alpha}\left(f(x)-f(x_0)\right)\cong\Gamma(\alpha+1)\left(f(x)-f(x_0)\right)$.

If there exists $f^{(k+1)\alpha}(x)=\overbrace{D_x^{\alpha}...D_x^{\alpha}}^{k+1\text{ times}}f(x)$ for any $x\in I\subseteq R$, then we denoted $f\in D_{(k+1)\alpha}(I)$, where $k=0,1,2,...$

Definition 3. [20] *Let $f(x)\in C_{\alpha}[a,b]$. Then the local fractional integral is defined by,*

$${}_aI_b^{\alpha}f(x)=\frac{1}{\Gamma(\alpha+1)}\int_a^b f(t)(dt)^{\alpha}=\frac{1}{\Gamma(\alpha+1)}\lim_{\Delta t\rightarrow 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^{\alpha},$$

with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\left\{\Delta t_1,\Delta t_2,...,\Delta t_{N-1}\right\}$, where $[t_j,t_{j+1}]$, $j=0,...,N-1$ and $a=t_0<t_1<...<t_{N-1}<t_N=b$ is partition of interval $[a,b]$.

Here, it follows that ${}_aI_b^{\alpha}f(x)=0$ *if* $a=b$ *and* ${}_aI_b^{\alpha}f(x)=-{}_bI_a^{\alpha}f(x)$ *if* $a<b$*. If for any* $x\in[a,b]$ *, there exists* ${}_aI_x^{\alpha}f(x)$*, then we denoted by* $f(x)\in I_x^{\alpha}[a,b]$ *.*

Lemma 1. [20] *We have*
i) $\dfrac{d^{\alpha}x^{k\alpha}}{dx^{\alpha}}=\dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)}x^{(k-1)\alpha}$;
ii) $\dfrac{1}{\Gamma(\alpha+1)}\displaystyle\int_a^b x^{k\alpha}(dx)^{\alpha}=\dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}\left(b^{(k+1)\alpha}-a^{(k+1)\alpha}\right),\ k\in R.$

In [18], Sarikaya et al. proved the following generalized Steffensen inequality for local fractional integrals:

Theorem 2 (Generalized Steffensen's Inequality)**.** *Let* $f(x),g(x)\in I_x^{\alpha}[a,b]$ *such that* f *never increases and* $0\le g(x)\le 1$ *on* $[a,b]$ *with* $a<b$*. Then*

$${}_{b-\lambda}I_b^{\alpha}f(x)\le\ {}_aI_b^{\alpha}f(x)g(x)\le\ {}_aI_{a+\lambda}^{\alpha}f(x)$$

where

$$\lambda^{\alpha}=\Gamma(1+\alpha)\ {}_aI_b^{\alpha}g(x).$$

The interested reader is able to look over the references [3],[8], [9],[13]-[18], [20]-[24] for local freactional theory.

In this study,we give the new inequalities of Steffensen type inequality for local fractional integrals.

3. **Main Results**

Theorem 3. *Let* $g,\phi\in I_x^{\alpha}[0,1]$ *be nonnegative functions. Assume that* $f:R\to[0,1]$ *non-increasing,* $0^{\alpha}\le g\le 1^{\alpha}$ *and* $p\ge 1$*,*

$$\phi(\lambda^{\frac{1}{p}})\le\lambda^{\alpha},$$

then, we have the following inequality

$${}_0I_{\lambda}^{\alpha}\phi(f(x))\ge\lambda^{\frac{-\alpha}{p}}\phi(\lambda^{\frac{1}{p}})\ {}_0I_1^{\alpha}\phi(f(x))g(x)$$

where $\lambda^{\frac{\alpha}{p}}=\Gamma(1+\alpha)\ {}_0I_1^{\alpha}g(x)$*.*

Proof. As $\phi\ge 0^{\alpha}$ and f is non-increasing, then $\phi(f)$ is non-increasing. Also,

$$\lambda^{\frac{-\alpha}{p}}\phi(\lambda^{\frac{1}{p}})g(x)\le\lambda^{\frac{-\alpha}{p}}\phi(\lambda^{\frac{1}{p}})\le\lambda^{\alpha-\frac{\alpha}{p}}\le 1^{\alpha},$$

then, we have

$$
\begin{aligned}
& \frac{1}{\Gamma(1+\alpha)}\int_0^\lambda \phi(f(x))(dx)^\alpha - \lambda^{\frac{-\alpha}{p}}\phi(\lambda^{\frac{1}{p}})\frac{1}{\Gamma(1+\alpha)}\int_0^1 \phi(f(x))g(x)(dx)^\alpha \\
= & \frac{1}{\Gamma(1+\alpha)}\int_0^\lambda \phi(f(x))(dx)^\alpha - \lambda^{\frac{-\alpha}{p}}\phi(\lambda^{\frac{1}{p}})\frac{1}{\Gamma(1+\alpha)}\left(\int_0^\lambda + \int_\lambda^1\right)\phi(f(x))g(x)(dx)^\alpha \\
= & \frac{1}{\Gamma(1+\alpha)}\int_0^\lambda \phi(f(x))\left[1^\alpha - \lambda^{\frac{-\alpha}{p}}\phi(\lambda^{\frac{1}{p}})g(x)\right](dx)^\alpha \\
& -\lambda^{\frac{-\alpha}{p}}\phi(\lambda^{\frac{1}{p}})\frac{1}{\Gamma(1+\alpha)}\int_\lambda^1 \phi(f(x))g(x)(dx)^\alpha \\
\geq & \phi(f(\lambda))\frac{1}{\Gamma(1+\alpha)}\left[\int_0^\lambda \left[1^\alpha - \lambda^{\frac{-\alpha}{p}}\phi(\lambda^{\frac{1}{p}})g(x)\right](dx)^\alpha - \lambda^{\frac{-\alpha}{p}}\phi(\lambda^{\frac{1}{p}})\int_\lambda^1 g(x)(dx)^\alpha\right] \\
= & \phi(f(\lambda))\frac{1}{\Gamma(1+\alpha)}\left[\lambda^\alpha - \lambda^{\frac{-\alpha}{p}}\phi(\lambda^{\frac{1}{p}})\int_0^1 g(x)(dx)^\alpha\right] \\
= & \phi(f(\lambda))\frac{1}{\Gamma(1+\alpha)}\left[\lambda^\alpha - \phi(\lambda^{\frac{1}{p}})\right] \geq 0^\alpha
\end{aligned}
$$

which completes the proof. □

Theorem 4. *Let $g, \phi, h \in I_x^\alpha[0,1]$ be nonnegative functions. Assume that $f : R \to [0,1]$ non-increasing, $0^\alpha \leq g \leq 1^\alpha$ and $p \geq 1$,*

$$\phi(\lambda^{\frac{1}{p}}) \leq \Gamma(1+\alpha)\ {}_0I_\lambda^\alpha h(x),\ \ \lambda^{\frac{-\alpha}{p}}\phi(\lambda^{\frac{1}{p}}) \leq h(x).$$

Then, the following inequality holds

$${}_0I_\lambda^\alpha \phi(f(x))h(x) \geq \lambda^{\frac{-\alpha}{p}}\phi(\lambda^{\frac{1}{p}})\ {}_0I_1^\alpha \phi(f(x))g(x)$$

where $\lambda^{\frac{\alpha}{p}} = \Gamma(1+\alpha)\ {}_0I_1^\alpha g(x)$.

Proof. As $\phi \geq 0^\alpha$ and f is non-increasing, then $\phi(f)$ is non-increasing. Therefore, we obtain

$$
\begin{aligned}
& \frac{1}{\Gamma(1+\alpha)}\int_0^\lambda \phi(f(x))h(x)(dx)^\alpha - \lambda^{\frac{-\alpha}{p}}\phi(\lambda^{\frac{1}{p}})\frac{1}{\Gamma(1+\alpha)}\int_0^1 \phi(f(x))g(x)(dx)^\alpha \\
= & \frac{1}{\Gamma(1+\alpha)}\int_0^\lambda \phi(f(x))h(x)(dx)^\alpha - \lambda^{\frac{-\alpha}{p}}\phi(\lambda^{\frac{1}{p}})\frac{1}{\Gamma(1+\alpha)}\left(\int_0^\lambda + \int_\lambda^1\right)\phi(f(x))g(x)(dx)^\alpha
\end{aligned}
$$

$$= \frac{1}{\Gamma(1+\alpha)}\int_0^\lambda \phi(f(x))\left[h(x)-\lambda^{\frac{-\alpha}{p}}\phi(\lambda^{\frac{1}{p}})g(x)\right](dx)^\alpha$$

$$-\lambda^{\frac{-\alpha}{p}}\phi(\lambda^{\frac{1}{p}})\frac{1}{\Gamma(1+\alpha)}\int_\lambda^1 \phi(f(x))g(x)(dx)^\alpha$$

$$\geq \phi(f(\lambda))\frac{1}{\Gamma(1+\alpha)}\left[\int_0^\lambda [h(x)-\lambda^{\frac{-\alpha}{p}}\phi(\lambda^{\frac{1}{p}})g(x)](dx)^\alpha - \lambda^{\frac{-\alpha}{p}}\phi(\lambda^{\frac{1}{p}})\int_\lambda^1 g(x)(dx)^\alpha\right]$$

$$= \phi(f(\lambda))\frac{1}{\Gamma(1+\alpha)}\left[\int_0^\lambda h(x)(dx)^\alpha - \lambda^{\frac{-\alpha}{p}}\phi(\lambda^{\frac{1}{p}})\int_0^1 g(x)(dx)^\alpha\right]$$

$$= \phi(f(\lambda))\frac{1}{\Gamma(1+\alpha)}\left[\int_0^\lambda h(x)(dx)^\alpha - \phi(\lambda^{\frac{1}{p}})\right] \geq 0^\alpha.$$

This completes the proof. □

Theorem 5. *Let $g, \phi \in I_x^\alpha[0,1]$ be nonnegative functions. Assume that $f : R \to [0,1]$ non-increasing and $\varphi(p) > 0$, $\lambda^{\alpha\varphi(p)} g \leq 1^\alpha$, $\lambda^{\alpha(1-\varphi(p))} = \Gamma(1+\alpha)\ {}_0I_1^\alpha g(x)$, then we have the following inequality*

$${}_0I_\lambda^\alpha \phi(f(x)) \geq \lambda^{\alpha\varphi(p)}\ {}_0I_1^\alpha \phi(f(x))g(x).$$

Proof. As $\phi \geq 0^\alpha$ and f is non-increasing, then $\phi(f)$ is non-increasing. Also

$$\lambda^{\alpha\varphi(p)} g \leq 1^\alpha \Longrightarrow \lambda^{\alpha\varphi(p)}\int_0^1 g(x)(dx)^\alpha \leq 1^\alpha \Longrightarrow \lambda^\alpha \leq 1^\alpha \Longrightarrow \lambda \leq 1.$$

Then, we get

$$\frac{1}{\Gamma(1+\alpha)}\int_0^\lambda \phi(f(x))(dx)^\alpha - \lambda^{\alpha\varphi(p)}\frac{1}{\Gamma(1+\alpha)}\int_0^1 \phi(f(x))g(x)(dx)^\alpha$$

$$= \frac{1}{\Gamma(1+\alpha)}\int_0^\lambda \phi(f(x))(dx)^\alpha - \lambda^{\alpha\varphi(p)}\frac{1}{\Gamma(1+\alpha)}\left(\int_0^\lambda + \int_\lambda^1\right)\phi(f(x))g(x)(dx)^\alpha$$

$$= \frac{1}{\Gamma(1+\alpha)}\int_0^\lambda \phi(f(x))\left[1^\alpha - \lambda^{\alpha\varphi(p)}g(x)\right](dx)^\alpha - \lambda^{\alpha\varphi(p)}\frac{1}{\Gamma(1+\alpha)}\int_\lambda^1 \phi(f(x))g(x)(dx)^\alpha$$

$$\geq \phi(f(\lambda))\frac{1}{\Gamma(1+\alpha)}\left[\int_0^\lambda \left[1^\alpha - \lambda^{\alpha\varphi(p)}g(x)\right](dx)^\alpha - \lambda^{\alpha\varphi(p)}\int_\lambda^1 g(x)(dx)^\alpha\right]$$

$$= \phi(f(\lambda))\frac{1}{\Gamma(1+\alpha)}\left[\lambda^\alpha - \lambda^{\alpha\varphi(p)}\int_0^1 g(x)(dx)^\alpha\right]$$

$$= \phi(f(\lambda))\frac{1}{\Gamma(1+\alpha)}\left[\lambda^\alpha - \lambda^\alpha\right] = 0^\alpha$$

which completes the proof. □

Corollary 1. *Let $g,\phi \in I_x^\alpha[0,1]$ be nonnegative functions. Assume that $f : R \to [0,1]$ non-increasing, $0^\alpha \leq g \leq 1^\alpha$, $p > 0$ where $\lambda^\alpha = \left(\Gamma(1+\alpha)\,{}_0I_1^\alpha g(x)\right)^{\frac{p}{2p-1}}$. Then*

$${}_0I_\lambda^\alpha \phi(f(x)) \geq \lambda^{\frac{\alpha}{(\frac{1}{p}-1)}}\ {}_0I_1^\alpha \phi(f(x))g(x)$$

Proof. The proof obtain from Theorem 5, by putting $\varphi(p) = \frac{1}{p} - 1,\ 0 < p < 1.$ □

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