Received 17/06/15

ON GENERALIZED SOME INTEGRAL INEQUALITIES FOR LOCAL FRACTIONAL INTEGRALS

MEHMET ZEKI SARIKAYA, TUBA TUNC, AND HÜSEYIN BUDAK

Abstract. In this study, we establish generaized Grüss type inequality and some generaized Cebysev type inequalities for local fractional integrals on fractal sets R^{α} $(0<\alpha\leq 1)$ of real line numbers.

1. Introduction

In 1935, G. Grüss [8] proved the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx\right|\leq\frac{1}{4}(M-m)(N-n), \tag{1.1}$$

provided that f and g are two integrable function on $[a,b]$ satisfying the condition

$$m\leq f(x)\leq M \text{ and } n\leq g(x)\leq N \text{ for all } x\in[a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible.

In 1882, P. L. Čebyšev [3] gave the following inequality:

$$|T(f,g)|\leq\frac{1}{12}(b-a)^2\left\|f'\right\|_{\infty}\left\|g'\right\|_{\infty}, \tag{1.3}$$

where $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous function, whose first derivatives f' and g' are bounded,

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \tag{1.4}$$

and $\|.\|_{\infty}$ denotes the norm in $L_{\infty}[a,b]$ defined as $\|p\|_{\infty}=ess\sup_{t\in[a,b]}|p(t)|$.

The following result of Grüss type was proved by Dragomir and Fedotov [5]:

Theorem 1. *Let $f,u:[a,b]\to\mathbb{R}$ be such that u is L-Lipshitzian on $[a,b]$, i.e,*

$$|u(x)-u(y)|\leq L|x-y| \quad \text{for all } x\in[a,b], \tag{1.5}$$

f is Riemann integrable on $[a,b]$ and there exist the real numbers m,M so that

$$m\leq f(x)\leq M \quad \text{for all } x\in[a,b]. \tag{1.6}$$

2000 *Mathematics Subject Classification.* 26D07, 26D10, 26D15, 26A33.

Key words and phrases. Grüss inequality, Cebysev inequality, local fractional integrals.

Then we have the inequality,

$$\left|\int_a^b f(x)du(x)-\frac{u(b)-u(a)}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{2}L(M-m)(b-a).$$

From [10], if $f:[a,b]\to\mathbb{R}$ is differentiable on (a,b) with the first derivative f' integrable on $[a,b]$, then Montgomery identity holds:

$$f(x)=\frac{1}{b-a}\int_a^b f(t)dt+\int_a^b P(x,t)f'(t)dt, \tag{1.7}$$

where $P(x,t)$ is the Peano kernel defined by

$$P(x,t)=\begin{cases}\frac{t-a}{b-a}, & a\leq t\leq x\\ \frac{t-b}{b-a}, & x<t\leq b.\end{cases}$$

In [13], Pachpatte established new inequalities of the Cebysev type by using Pecaris's extension of the Montgomery identity [14].

In the last years, many papers were deveted to the generalization of Cebysev-Grüss type inequalities, we can mention the works [1], [5]-[7], [9], [13]-[17].

2. Preliminaries

Recall the set R^α of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [23, 24] and so on.

Recently, the theory of Yang's fractional sets [23] was introduced as follows.

For $0<\alpha\leq 1$, we have the following α-type set of element sets:

Z^α : The α-type set of integer is defined as the set $\{0^\alpha,\pm 1^\alpha,\pm 2^\alpha,...,\pm n^\alpha,...\}$.

Q^α : The α-type set of the rational numbers is defined as the set $\{m^\alpha=\left(\frac{p}{q}\right)^\alpha : p,q\in Z,\ q\neq 0\}$.

J^α : The α-type set of the irrational numbers is defined as the set $\{m^\alpha\neq\left(\frac{p}{q}\right)^\alpha : p,q\in Z,\ q\neq 0\}$.

R^α : The α-type set of the real line numbers is defined as the set $R^\alpha=Q^\alpha\cup J^\alpha$.

If a^α,b^α and c^α belongs the set R^α of real line numbers, then

(1) $a^\alpha+b^\alpha$ and $a^\alpha b^\alpha$ belongs the set R^α;
(2) $a^\alpha+b^\alpha=b^\alpha+a^\alpha=(a+b)^\alpha=(b+a)^\alpha$;
(3) $a^\alpha+(b^\alpha+c^\alpha)=(a+b)^\alpha+c^\alpha$;
(4) $a^\alpha b^\alpha=b^\alpha a^\alpha=(ab)^\alpha=(ba)^\alpha$;
(5) $a^\alpha(b^\alpha c^\alpha)=(a^\alpha b^\alpha)c^\alpha$;
(6) $a^\alpha(b^\alpha+c^\alpha)=a^\alpha b^\alpha+a^\alpha c^\alpha$;
(7) $a^\alpha+0^\alpha=0^\alpha+a^\alpha=a^\alpha$ and $a^\alpha 1^\alpha=1^\alpha a^\alpha=a^\alpha$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 1. [23] *A non-differentiable function $f:R\to R^\alpha$, $x\to f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon>0$, there exists $\delta>0$, such that*

$$|f(x)-f(x_0)|<\varepsilon^\alpha$$

holds for $|x-x_0|<\delta$, where $\varepsilon,\delta\in R$. If $f(x)$ is local continuous on the interval (a,b), we denote $f(x)\in C_\alpha(a,b)$.

Definition 2. [23] *The local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by*

$$f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x\to x_0} \frac{\Delta^\alpha \left(f(x) - f(x_0)\right)}{(x - x_0)^\alpha},$$

where $\Delta^\alpha \left(f(x) - f(x_0)\right) \cong \Gamma(\alpha + 1) \left(f(x) - f(x_0)\right).$

If there exists $f^{(k+1)\alpha}(x) = \overbrace{D_x^\alpha ... D_x^\alpha}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq R$, then we denoted $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, ...$

Definition 3. [23] *Let $f(x) \in C_\alpha [a, b]$. Then the local fractional integral is defined by,*

$${}_aI_b^\alpha f(x) = \frac{1}{\Gamma(\alpha + 1)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(\alpha + 1)} \lim_{\Delta t\to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha,$$

with $\Delta t_j = t_{j+1} - t_j$ *and* $\Delta t = \max\{\Delta t_1, \Delta t_2, ..., \Delta t_{N-1}\}$, *where* $[t_j, t_{j+1}]$, $j = 0, ..., N - 1$ *and* $a = t_0 < t_1 < ... < t_{N-1} < t_N = b$ *is partition of interval* $[a, b]$.

Here, it follows that ${}_aI_b^\alpha f(x) = 0$ *if* $a = b$ *and* ${}_aI_b^\alpha f(x) = -{}_bI_a^\alpha f(x)$ *if* $a < b$. *If for any* $x \in [a, b]$, *there exists* ${}_aI_x^\alpha f(x)$, *then we denoted by* $f(x) \in I_x^\alpha [a, b]$.

Definition 4 (Generalized convex function)**.** [23] *Let $f : I \subseteq R \to R^\alpha$. For any $x_1, x_2 \in I$ and $\lambda \in [0, 1]$, if the following inequality*

$$f(\lambda x_1 + (1 - \lambda)x_2) \le \lambda^\alpha f(x_1) + (1 - \lambda)^\alpha f(x_2)$$

holds, then f is called a generalized convex function on I.

Here are two basic examples of generalized convex functions:

(1) $f(x) = x^{\alpha p}$, $x \ge 0$, $p > 1$;

(2) $f(x) = E_\alpha(x^\alpha)$, $x \in R$ where $E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Lrffer function.

Lemma 1. [23]

(1) (Local fractional integration is anti-differentiation) Suppose that $f(x) = g^{(\alpha)}(x) \in C_\alpha [a, b]$, then we have

$${}_aI_b^\alpha f(x) = g(b) - g(a).$$

(2) (Local fractional integration by parts) Suppose that $f(x), g(x) \in D_\alpha [a, b]$ and $f^{(\alpha)}(x)$, $g^{(\alpha)}(x) \in C_\alpha [a, b]$, then we have

$${}_aI_b^\alpha f(x)g^{(\alpha)}(x) = \left. f(x)g(x)\right|_a^b - {}_a I_b^\alpha f^{(\alpha)}(x)g(x).$$

Lemma 2. [23] *We have*

i) $\dfrac{d^\alpha x^{k\alpha}}{dx^\alpha} = \dfrac{\Gamma(1 + k\alpha)}{\Gamma(1 + (k - 1)\alpha)} x^{(k-1)\alpha}$;

ii) $\dfrac{1}{\Gamma(\alpha + 1)} \displaystyle\int_a^b x^{k\alpha}(dx)^\alpha = \dfrac{\Gamma(1 + k\alpha)}{\Gamma(1 + (k + 1)\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right)$, $k \in R$.

For more information and recent developments on local freactional theory, please refer to [2],[4],[11],[12],[18]-[27].

The aim of the this paper is to obtain generaized Grüss inequality and some generaized Cebysev type inequalities for local fractional integrals.

3. Main Results

Theorem 2 (Genenneralized Grüss inequality)**.** *Let* $f,g\in I_x^{\alpha}[a,b]$. *Then,* $\varphi\leq f(x)\leq\Phi$ *and* $\gamma\leq g(x)\leq\Gamma$, *for all* $x\in[a,b],\varphi,\Phi,\gamma$ *and* $\Gamma\in\mathbb{R}^{\alpha}$, *we have*

$$|T_{\alpha}(f,g)|\leq\frac{(b-a)^{2\alpha}}{4^{\alpha}\Gamma^2(1+\alpha)}(\Phi-\varphi)(\Gamma-\gamma)\tag{3.1}$$

where

$$T_{\alpha}(f,g)=\frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}\ {}_aI_b^{\alpha}f(x)g(x)-[\ {}_aI_b^{\alpha}f(x)]\,[\ {}_aI_b^{\alpha}g(x)]\,.\tag{3.2}$$

Proof. By using the local fractional integrals for mappings $f,g\in I_x^{\alpha}[a,b]$, we have

$$\begin{aligned}
&\frac{1}{\Gamma^2(1+\alpha)}\int_a^b\int_a^b[f(x)-f(y)]\,[g(x)-g(y)]\,(dy)^{\alpha}(dx)^{\alpha} \qquad (3.3)\\
=\;&\frac{1}{\Gamma^2(1+\alpha)}\int_a^b\int_a^b[f(x)g(x)-f(x)g(y)-f(y)g(x)+f(y)g(y)]\,(dy)^{\alpha}(dx)^{\alpha}\\
=\;&\frac{2^{\alpha}(b-a)^{\alpha}}{\Gamma^2(1+\alpha)}\int_a^b f(x)g(x)\,(dx)^{\alpha}\\
&-2^{\alpha}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)\,(dx)^{\alpha}\right)\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b g(x)\,(dx)^{\alpha}\right)\\
=\;&\frac{2^{\alpha}(b-a)^{\alpha}}{\Gamma(1+\alpha)}\ {}_aI_b^{\alpha}f(x)g(x)-2^{\alpha}[\ {}_aI_b^{\alpha}f(x)]\,[\ {}_aI_b^{\alpha}g(x)]\\
=\;&2^{\alpha}T_{\alpha}(f,g).
\end{aligned}$$

Appling generalized Hölder's integral inequality for $p=q=2$, we obtain

$$\begin{aligned}
&\left[\frac{1}{2^{\alpha}(b-a)^{2\alpha}\Gamma^2(1+\alpha)}\right. \qquad (3.4)\\
&\left.\times\int_a^b\int_a^b[f(x)-f(y)]\,[g(x)-g(y)]\,(dy)^{\alpha}(dx)^{\alpha}\right]^2\\
\leq\;&\left(\frac{1}{2^{\alpha}(b-a)^{2\alpha}\Gamma^2(1+\alpha)}\int_a^b\int_a^b[f(x)-f(y)]^2(dy)^{\alpha}(dx)^{\alpha}\right)\\
&\times\left(\frac{1}{2^{\alpha}(b-a)^{2\alpha}\Gamma^2(1+\alpha)}\int_a^b\int_a^b[g(x)-g(y)]^2(dy)^{\alpha}(dx)^{\alpha}\right)
\end{aligned}$$

$$= \left[\frac{1}{(b-a)^{\alpha}\Gamma^{2}(1+\alpha)}\int_{a}^{b}f^{2}(x)(dx)^{\alpha}-\left(\frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_{a}^{b}f(x)(dx)^{\alpha}\right)^{2}\right]$$

$$\times\left[\frac{1}{(b-a)^{\alpha}\Gamma^{2}(1+\alpha)}\int_{a}^{b}g^{2}(x)(dx)^{\alpha}-\left(\frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_{a}^{b}g(x)(dx)^{\alpha}\right)^{2}\right].$$

It is easy to observe that

$$\frac{1}{(b-a)^{\alpha}\Gamma^{2}(1+\alpha)}\int_{a}^{b}f^{2}(x)(dx)^{\alpha}-\left(\frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_{a}^{b}f(x)(dx)^{\alpha}\right)^{2}$$

$$= \left(\frac{\Phi}{\Gamma(1+\alpha)}-\frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_{a}^{b}f(x)(dx)^{\alpha}\right)$$

$$\times\left(\frac{1}{\Gamma(1+\alpha)(b-a)^{\alpha}}\int_{a}^{b}f(x)(dx)^{\alpha}-\frac{\varphi}{\Gamma(1+\alpha)}\right)$$

$$-\frac{1}{(b-a)^{\alpha}\Gamma^{2}(1+\alpha)}\int_{a}^{b}[\Phi-f(x)][f(x)-\varphi](dx)^{\alpha}.$$

Since $[\Phi-f(x)][f(x)-\varphi]\geq 0$ for each $x\in[a,b]$, then we get

$$\frac{1}{(b-a)^{\alpha}\Gamma^{2}(1+\alpha)}\int_{a}^{b}f^{2}(x)(dx)^{\alpha}-\left(\frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_{a}^{b}f(x)(dx)^{\alpha}\right)^{2} \tag{3.5}$$

$$= \left(\frac{\Phi}{\Gamma(1+\alpha)}-\frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_{a}^{b}f(x)(dx)^{\alpha}\right)$$

$$\times\left(\frac{1}{\Gamma(1+\alpha)(b-a)^{\alpha}}\int_{a}^{b}f(x)(dx)^{\alpha}-\frac{\varphi}{\Gamma(1+\alpha)}\right).$$

Similarly, we have

$$\frac{1}{(b-a)^{\alpha}\Gamma^{2}(1+\alpha)}\int_{a}^{b}g^{2}(x)(dx)^{\alpha}-\left(\frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_{a}^{b}g(x)(dx)^{\alpha}\right)^{2} \tag{3.6}$$

$$= \left(\frac{\Gamma}{\Gamma(1+\alpha)}-\frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_{a}^{b}g(x)(dx)^{\alpha}\right)$$

$$\times\left(\frac{1}{\Gamma(1+\alpha)(b-a)^{\alpha}}\int_{a}^{b}g(x)(dx)^{\alpha}-\frac{\gamma}{\Gamma(1+\alpha)}\right).$$

Using (3.5) and (3.6) in (3.4), we get the following inequality

$$\left[\frac{1}{2^{\alpha}(b-a)^{2\alpha}\Gamma^{2}(1+\alpha)}\int_{a}^{b}\int_{a}^{b}[f(x)-f(y)][g(x)-g(y)](dy)^{\alpha}(dx)^{\alpha}\right]^{2}$$
$$\leq \left(\frac{\Phi}{\Gamma(1+\alpha)}-\frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_{a}^{b}f(x)(dx)^{\alpha}\right)$$
$$\times\left(\frac{1}{\Gamma(1+\alpha)(b-a)^{\alpha}}\int_{a}^{b}f(x)(dx)^{\alpha}-\frac{\varphi}{\Gamma(1+\alpha)}\right)$$
$$\times\left(\frac{\Gamma}{\Gamma(1+\alpha)}-\frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_{a}^{b}g(x)(dx)^{\alpha}\right)$$
$$\times\left(\frac{1}{\Gamma(1+\alpha)(b-a)^{\alpha}}\int_{a}^{b}g(x)(dx)^{\alpha}-\frac{\gamma}{\Gamma(1+\alpha)}\right).$$

Now, using the elementary inequality for $\alpha-$type set of the real line numbers

$$4^{\alpha}pq\leq(p+q)^{2},\quad p,q\in\mathbb{R}^{\alpha}$$

we get

$$\left[\frac{1}{2^{\alpha}(b-a)^{2\alpha}\Gamma^{2}(1+\alpha)}\int_{a}^{b}\int_{a}^{b}[f(x)-f(y)][g(x)-g(y)](dy)^{\alpha}(dx)^{\alpha}\right]^{2}$$
$$\leq \frac{1}{16^{\alpha}\Gamma^{4}(1+\alpha)}(\Phi-\varphi)^{2}(\Gamma-\gamma)^{2}$$

which completes the proof. To prove the sharpness of (3.1), let choose

$$f(x)=g(x)=\begin{cases}-1^{\alpha}, & a\leq x<\frac{a+b}{2}\\ 1^{\alpha}, & \frac{a+b}{2}\leq x\leq b\end{cases}$$

then

$${}_{a}I_{b}^{\alpha}f(x)g(x)=\frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)},$$
$${}_{a}I_{b}^{\alpha}f(x)={}_{a}I_{b}^{\alpha}g(x)=0^{\alpha}$$

and

$$(\Phi-\varphi)=(\Gamma-\gamma)=2^{\alpha}$$

which the equality (3.1) is realized. □

Theorem 3 (Geneneralized Cebysev inequality)**.** *Let* $f(x),g(x)\in D_{\alpha}[a,b]$ *and* $f^{(\alpha)}(x)$, $g^{(\alpha)}(x)$ *are bounded, i.e*

$$\left\|f^{(\alpha)}\right\|_{\infty}=\sup_{t\in[a,b]}\left|f^{(\alpha)}(t)\right|<\infty$$

$$\left\|g^{(\alpha)}\right\|_{\infty}=\sup_{t\in[a,b]}\left|g^{(\alpha)}(t)\right|<\infty$$

then, we have

$$|T_{\alpha}(f,g)|\leq\frac{\left\|f^{(\alpha)}\right\|_{\infty}\left\|g^{(\alpha)}\right\|_{\infty}}{\Gamma^{2}(1+\alpha)}\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}(b-a)^{4\alpha}\tag{3.7}$$

where $T_{\alpha}(f,g)$ is defined (3.2).

Proof. After dividing (3.3) by 2^{α}, if we take modulus in (3.3), we have

$$\begin{aligned}
&|T_{\alpha}(f,g)|\\
\leq\ & \frac{1}{2^{\alpha}\Gamma^{2}(1+\alpha)}\int_{a}^{b}\int_{a}^{b}|f(x)-f(y)|\,|g(x)-g(y)|\,(dy)^{\alpha}(dx)^{\alpha}\\
=\ & \frac{1}{2^{\alpha}\Gamma^{2}(1+\alpha)}\int_{a}^{b}\int_{a}^{b}\left|\frac{1}{\Gamma(1+\alpha)}\int_{y}^{x}f^{(\alpha)}(u)\,(du)^{\alpha}\right|\left|\frac{1}{\Gamma(1+\alpha)}\int_{y}^{x}g^{(\alpha)}(u)\,(du)^{\alpha}\right|(dy)^{\alpha}(dx)^{\alpha}\\
\leq\ & \frac{\left\|f^{(\alpha)}\right\|_{\infty}\left\|g^{(\alpha)}\right\|_{\infty}}{2^{\alpha}\Gamma^{2}(1+\alpha)}\int_{a}^{b}\int_{a}^{b}\frac{|x-y|^{2\alpha}}{\Gamma^{2}(1+\alpha)}(dy)^{\alpha}(dx)^{\alpha}\\
=\ & \frac{\left\|f^{(\alpha)}\right\|_{\infty}\left\|g^{(\alpha)}\right\|_{\infty}}{2^{\alpha}\Gamma^{3}(1+\alpha)}\int_{a}^{b}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left[(x-a)^{3\alpha}+(b-x)^{3\alpha}\right](dx)^{\alpha}\\
=\ & \frac{\left\|f^{(\alpha)}\right\|_{\infty}\left\|g^{(\alpha)}\right\|_{\infty}}{\Gamma^{2}(1+\alpha)}\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}(b-a)^{4\alpha}
\end{aligned}$$

which completes the proof. □

Theorem 4. *Let f be as in Theorem 2. Then, we have the following inequality*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_{a}I_{b}^{\alpha}f(x)\right.\\
&\left.-\frac{(b-a)^{\alpha}}{2^{\alpha}}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]\left[f^{(\alpha)}(b)-f^{(\alpha)}(a)\right]\right|\\
\leq\ & \frac{(b-a)^{2\alpha}}{32^{\alpha}\Gamma^{2}(1+\alpha)}(\Phi-\varphi).
\end{aligned}$$

Proof. By generalized Grüss inequality, we have that

$$\begin{aligned}
&\left|\frac{(b-a)^{\alpha}}{\left[\Gamma(1+\alpha)\right]^{2}}\int_{a}^{b}(x-a)^{\alpha}(b-x)^{\alpha}f^{(2\alpha)}(x)\,(dx)^{\alpha}\right. \\
&\left.-\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}(x-a)^{\alpha}(b-x)^{\alpha}(x)\,(dx)^{\alpha}\right)\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}f^{(2\alpha)}(x)\,(dx)^{\alpha}\right)\right| \\
\leq\; &\frac{(b-a)^{2\alpha}}{4^{\alpha}\Gamma^{2}(1+\alpha)}(\Phi-\varphi)(L-l),
\end{aligned}$$

where

$$L=\sup_{x\in[a,b]}\left\{(x-a)^{\alpha}(b-x)^{\alpha}\right\}=\frac{(b-a)^{2\alpha}}{4^{\alpha}}$$

and

$$l=\inf_{x\in[a,b]}\left\{(x-a)^{\alpha}(b-x)^{\alpha}\right\}=0^{\alpha}.$$

As a simple calculation shows us that

$$\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}(x-a)^{\alpha}(b-x)^{\alpha}(x)\,(dx)^{\alpha}=(b-a)^{3\alpha}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right],$$

and one has the identity [2]

$$\begin{aligned}
&\frac{(b-a)^{\alpha}}{\left[\Gamma(1+\alpha)\right]^{2}}\int_{a}^{b}(x-a)^{\alpha}(b-x)^{\alpha}f^{(2\alpha)}(x)\,(dx)^{\alpha} \\
=\;& 2^{\alpha}(b-a)^{2\alpha}\left[\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(x)\right].
\end{aligned}$$

Then, it follows that,

$$\begin{aligned}
&\left|2^{\alpha}(b-a)^{2\alpha}\left[\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(x)\right]\right. \\
&\left.-(b-a)^{3\alpha}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]\left[f^{(\alpha)}(b)-f^{(\alpha)}(a)\right]\right| \\
\leq\; &\frac{(b-a)^{4\alpha}}{16^{\alpha}\Gamma^{2}(1+\alpha)}(\Phi-\varphi).
\end{aligned}$$

That is,

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(x)\right. \\
&\left.-\frac{(b-a)^{\alpha}}{2^{\alpha}}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]\left[f^{(\alpha)}(b)-f^{(\alpha)}(a)\right]\right| \\
\leq\; &\frac{(b-a)^{2\alpha}}{32^{\alpha}\Gamma^{2}(1+\alpha)}(\Phi-\varphi),
\end{aligned}$$

and the theorem is thus proved. □

Theorem 5. *Let $I\subseteq\mathbb{R}$ be an interval, $f,g:I^0\subseteq\mathbb{R}\to\mathbb{R}^\alpha$ (I^0 is the interior of I) such that $f,g\in D_\alpha(I^0)$ and $f^{(\alpha)},g^{(\alpha)}\in C_\alpha[a,b]$ for $a,b\in I^0$ with $a<b$ Then. we have the inequality*

$$|T_\alpha(f,g)| \le \left[2^\alpha\frac{\Gamma(1+4\alpha)}{\Gamma(1+5\alpha)}-2^\alpha\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right] \times\frac{2^\alpha\Gamma(1+\alpha)}{\Gamma^2(1+2\alpha)}(b-a)^{4\alpha}\left\|f^{(\alpha)}\right\|_\infty\left\|g^{(\alpha)}\right\|_\infty. \tag{3.8}$$

Proof. From the hypotheses the following identities hold [18]:

$$f(x)-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(t)=\frac{1}{\Gamma(1+\alpha)(b-a)^\alpha}\int_a^b p(x,t)f^{(\alpha)}(t)\,(dt)^\alpha, \tag{3.9}$$

$$g(x)-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha g(t)=\frac{1}{\Gamma(1+\alpha)(b-a)^\alpha}\int_a^b p(x,t)g^{(\alpha)}(t)\,(dt)^\alpha \tag{3.10}$$

where

$$p(x,t)=\begin{cases}(t-a)^\alpha, & t\in[a,x]\\ (t-b)^\alpha, & t\in(x,b].\end{cases}$$

From (3.9) and (3.10), we observe that

$$\left[f(x)-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(t)\right]\left[g(x)-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha g(t)\right]$$

$$=\frac{1}{(b-a)^{2\alpha}}\left[\frac{1}{\Gamma(1+\alpha)}\int_a^b p(x,t)f^{(\alpha)}(t)\,(dt)^\alpha\right]\times\left[\frac{1}{\Gamma(1+\alpha)}\int_a^b p(x,t)g^{(\alpha)}(t)\,(dt)^\alpha\right],$$

that is,

$$f(x)g(x)-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}g(x)\,{}_aI_b^\alpha f(t) -\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}f(x)\,{}_aI_b^\alpha g(t)+\frac{\Gamma^2(1+\alpha)}{(b-a)^{2\alpha}}\left[{}_aI_b^\alpha f(x)\right]\left[{}_aI_b^\alpha g(x)\right] \tag{3.11}$$

$$=\frac{1}{(b-a)^{2\alpha}}\left[\frac{1}{\Gamma(1+\alpha)}\int_a^b p(x,t)f^{(\alpha)}(t)\,(dt)^\alpha\right]\times\left[\frac{1}{\Gamma(1+\alpha)}\int_a^b p(x,t)g^{(\alpha)}(t)\,(dt)^\alpha\right].$$

Integrating both sides of the resulting identity with respect to x from a to b, we obtain

$$\begin{aligned} &\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}T_{\alpha}(f,g) \\ =\ & \frac{1}{(b-a)^{2\alpha}\Gamma(1+\alpha)}\int_a^b\left[\frac{1}{\Gamma(1+\alpha)}\int_a^b p(x,t)f^{(\alpha)}(t)\,(dt)^{\alpha}\right] \\ & \times\left[\frac{1}{\Gamma(1+\alpha)}\int_a^b p(x,t)g^{(\alpha)}(t)\,(dt)^{\alpha}\right](dx)^{\alpha}. \end{aligned} \tag{3.12}$$

From (3.12) and using the properties of modulus, we observe that

$$\begin{aligned} &\left|\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}T_{\alpha}(f,g)\right| \\ \leq\ & \frac{1}{(b-a)^{2\alpha}\Gamma(1+\alpha)}\int_a^b\left[\frac{1}{\Gamma(1+\alpha)}\int_a^b |p(x,t)|\left|f^{(\alpha)}(t)\right|(dt)^{\alpha}\right] \\ & \times\left[\frac{1}{\Gamma(1+\alpha)}\int_a^b |p(x,t)|\left|g^{(\alpha)}(t)\right|(dt)^{\alpha}\right](dx)^{\alpha} \\ \leq\ & \frac{\left\|f^{(\alpha)}\right\|_{\infty}\left\|g^{(\alpha)}\right\|_{\infty}}{(b-a)^{2\alpha}\Gamma(1+\alpha)}\int_a^b\left[\frac{1}{\Gamma(1+\alpha)}\int_a^b |p(x,t)|\,(dt)^{\alpha}\right]^2(dx)^{\alpha} \\ =\ & (b-a)^{3\alpha}\left\|f^{(\alpha)}\right\|_{\infty}\left\|g^{(\alpha)}\right\|_{\infty}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^2 \\ & \times\left[4^{\alpha}\frac{\Gamma(1+4\alpha)}{\Gamma(1+5\alpha)}-4^{\alpha}\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}+2^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]. \end{aligned}$$

Here, we used the fact that

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)}\int_a^b\left[\frac{1}{\Gamma(1+\alpha)}\int_a^b |p(x,t)|\,(dt)^{\alpha}\right]^2(dx)^{\alpha} \\ =\ & \left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^2\frac{1}{\Gamma(1+\alpha)}\int_a^b\left[(x-a)^{2\alpha}+(b-x)^{2\alpha}\right]^2(dx)^{\alpha} \\ =\ & (b-a)^{5\alpha}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^2\left[4^{\alpha}\frac{\Gamma(1+4\alpha)}{\Gamma(1+5\alpha)}-4^{\alpha}\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}+2^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]. \end{aligned}$$

This completes the proof. □

Theorem 6. *Let f, g be as in Theorem 5. Then, we have the inequality*

$$\begin{aligned} & \left|T_{\alpha}(f,g)\right| \\ \leq & \frac{1}{2^{\alpha}\Gamma^{2}\left(1+\alpha\right)}\int_{a}^{b}\left[\left\|f^{(\alpha)}\right\|_{\infty}\left|g(x)\right|+\left\|g^{(\alpha)}\right\|_{\infty}\left|f(x)\right|\right]H(x)\left(dx\right)^{\alpha} \end{aligned}$$

where

$$H(x)=\frac{1}{\Gamma\left(1+\alpha\right)}\int_{a}^{b}\left|p(x,t)\right|\left(dt\right)^{\alpha}.$$

Proof. Multiplying both sides of (3.9) and (3.10) by $g(x)$ and $f(x)$, respectively, and adding the resulting identities and rewriting, we have

$$\begin{aligned} & f(x)g(x)-\frac{\Gamma\left(1+\alpha\right)}{2^{\alpha}\left(b-a\right)^{\alpha}}g(x)\ {}_{a}I_{b}^{\alpha}f(t)-\frac{\Gamma\left(1+\alpha\right)}{2^{\alpha}\left(b-a\right)^{\alpha}}f(x)\ {}_{a}I_{b}^{\alpha}g(t) \\ = & \frac{1}{2^{\alpha}\left(b-a\right)^{\alpha}}\left[\frac{g(x)}{\Gamma\left(1+\alpha\right)}\int_{a}^{b}p(x,t)f^{(\alpha)}(t)\left(dt\right)^{\alpha}\right. \\ & \left.+\frac{f(x)}{\Gamma\left(1+\alpha\right)}\int_{a}^{b}p(x,t)g^{(\alpha)}(t)\left(dt\right)^{\alpha}\right]. \end{aligned} \tag{3.13}$$

Integrating both sides of (3.13) with respect to x from a to b and rewriting, we have

$$\begin{aligned} & \frac{\Gamma\left(1+\alpha\right)}{\left(b-a\right)^{\alpha}}T_{\alpha}(f,g) \\ = & \frac{1}{2^{\alpha}\left(b-a\right)^{\alpha}\Gamma\left(1+\alpha\right)}\int_{a}^{b}\left[\frac{g(x)}{\Gamma\left(1+\alpha\right)}\int_{a}^{b}p(x,t)f^{(\alpha)}(t)\left(dt\right)^{\alpha}\right. \\ & \left.+\frac{1}{\Gamma\left(1+\alpha\right)}\int_{a}^{b}f(x)p(x,t)g^{(\alpha)}(t)\right]\left(dx\right)^{\alpha}. \end{aligned} \tag{3.14}$$

From (3.6) and using the properties of modulus we observe that

$$\begin{aligned}
&\left|\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}T_{\alpha}(f,g)\right| \\
\leq\ & \frac{1}{2^{\alpha}(b-a)^{\alpha}\Gamma(1+\alpha)}\int_{a}^{b}\left[\frac{|g(x)|}{\Gamma(1+\alpha)}\int_{a}^{b}|p(x,t)|\left|f^{(\alpha)}(t)\right|(dt)^{\alpha}\right. \\
&\left.+\frac{|f(x)|}{\Gamma(1+\alpha)}\int_{a}^{b}\int_{a}^{b}|p(x,t)|\,g^{(\alpha)}(t)\right](dx)^{\alpha}. \\
\leq\ & \frac{1}{2^{\alpha}(b-a)^{\alpha}\Gamma(1+\alpha)}\int_{a}^{b}\left[\left\|f^{(\alpha)}\right\|_{\infty}|g(x)|+\left\|g^{(\alpha)}\right\|_{\infty}|f(x)|\right]H(x)\,(dx)^{\alpha}
\end{aligned}$$

which completes the proof. □

References

[1] K. Boukerrioua and A.G. Lakoud, *On generalization of Čebyšev type inequalities,* J. Inequal. Pure and Appl. Math. 8(2), Art 55, 2007.

[2] H. Budak and M. Z. Sarikaya, *Some new generalized Hermite-Hadamard inequality for generalized convex function and application,* Submitted.

[3] P. L. Čebyšev, *Sur less expressions approximatives des integrales definies par les autres prises entre les memes limites,* Proc. Math. Soc. Charkov, 2, 93-98, 1882.

[4] G-S. Chen, *Generalizations of Hölder's and some related integral inequalities on fractal space,* Journal of Function Spaces and Applications Volume 2013, Article ID 198405.

[5] S. S. Dragomir and I. Fedotov, *An inequality of Grüss type for Riemann-Stieltjes integral and applications for special means,* Tamkang J. of Math., 29(4), 287-292, 1998.

[6] S.S. Dragomir, *Some integral inequalities of Grüss type,* Indian J. Pure Appl. Math., 31(4), 397-415, 2000.

[7] A. G. Lakoud and F. Aissaouinew, *Čebyšev type inequalities for double integrals*, Journal Math. Inequali. Volume 5, Number 4 (2011), 453–462.

[8] G. Grüss, *Über das maximum des absoluten Betrages von* $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$, Math. Z., 39, 215-226, 1935.

[9] S. Hussain and M. W. Alomari, *Weighted Osrowski and Čebyšev type inequalities with applications*, Konuralp Journal of Mathematics, Volume 1 No. 2 pp. 1–16 (2013).

[10] D. S. Mitrinovic, J. E. Pecaric and A. M. Fink, *Inequalities involving functions and their integrals and derivatives,* Kluwer Academic Publishers, Dordrecht, 1991.

[11] H. Mo, X Sui and D Yu, *Generalized convex functions on fractal sets and two related inequalities*, Abstract and Applied Analysis, Volume 2014, Article ID 636751, 7 pages.

[12] H. Mo, *Generalized Hermite-Hadamard inequalities involving local fractional integral,* arXiv:1410.1062.

[13] B. G. Pachpatte, *On Čebyšev-Grüss type inequalities via Pecaric's extention of the Montgomery identity,* J. Inequal. Pure and Appl. Math. 7(1), Art 108, 2006.

[14] J. E. Pecaric, *On the Čebyšev inequality,* Bul. Sti. Tehn. Inst. Politehn "Tralan Vuia" Timişora(Romania), 25(39)(1), 5-9, 1980.

[15] J. E. Pecaric and B. Tebes, *On Grüss type inequalities Dragomir and Fedotov,* J. Inequal. Pure and Appl. Math. 4(5), Art 91, 2003.

[16] M. Z. Sarikaya, N. Aktan, H. Yıldırım, *Weighted Čebyšev-Grüss type inequalities on time scales*, J. Math. Inequal. 2, 2 (2008), 185–195.

[17] Sarikaya, M. Z., *A Note on Grüss type inequalities on time scales*, Dynamic Systems and Applications, 17 (2008), 663-666.

[18] M. Z. Sarikaya and H Budak, *Generalized Ostrowski type inequalities for local fractional integrals*, RGMIA Research Report Collection, 18(2015), Article 62, 11 pp.
[19] M. Z. Sarikaya, S.Erden and H. Budak, *Some generalized Ostrowski type inequalities involving local fractional integrals and applications*, RGMIA Research Report Collection, 18(2015), Article 63, 12 pp.
[20] M. Z. Sarikaya H. Budak, *On generalized Hermite-Hadamard inequality for generalized convex function*, RGMIA Research Report Collection, 18(2015), Article 64, 15 pp.
[21] M. Z. Sarikaya, S.Erden and H. Budak, *Some integral inequalities for local fractional integrals*, RGMIA Research Report Collection, 18(2015), Article 65, 12 pp.
[22] M. Z. Sarikaya, H. Budak and S.Erden, *On new inequalities of Simpson's type for generalized convex functions*, RGMIA Research Report Collection, 18(2015), Article 66, 13 pp.
[23] X. J. Yang, *Advanced Local Fractional Calculus and Its Applications*, World Science Publisher, New York, 2012.
[24] J. Yang, D. Baleanu and X. J. Yang, *Analysis of fractal wave equations by local fractional Fourier series method*, Adv. Math. Phys. , 2013 (2013), Article ID 632309.
[25] X. J. Yang, *Local fractional integral equations and their applications*, Advances in Computer Science and its Applications (ACSA) 1(4), 2012.
[26] X. J. Yang, *Generalized local fractional Taylor's formula with local fractional derivative*, Journal of Expert Systems, 1(1) (2012) 26-30.
[27] X. J. Yang, *Local fractional Fourier analysis,* Advances in Mechanical Engineering and its Applications 1(1), 2012 12-16.

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY

E-mail address: sarikayamz@gmail.com

E-mail address: tubatunc03@gmail.com

E-mail address: hsyn.budak@gmail.coms