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NEW INEQUALITIES FOR LOCAL FRACTIONAL INTEGRALS

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Abstract. In this paper, we establish double inequalities for twice local fractional differentiable mappings on fractal sets R^{α} $(0<\alpha\leq 1)$ of real line numbers. We also give some generalized Hermite-Hadamard like inequalities for local fractional integrals.

1. Introduction

In [10], Ujević established the following interesting integral inequalities for twice differentiable mappings:

Theorem 1. *Let $f:[a,b]\rightarrow R$ be a twice differentiable mapping on (a,b) and suppose that $\gamma\leq f''(t)\leq\Gamma$ for all $t\in(a,b)$. Then we have the double inequalities*

$$\frac{3M-2\Gamma}{24}(b-a)^2\leq\frac{1}{b-a}\int_a^b f(t)dt-f\left(\frac{a+b}{2}\right)\leq\frac{3M-2\gamma}{24}(b-a)^2 \tag{1.1}$$

and

$$\frac{3M-\Gamma}{24}(b-a)^2\leq\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\leq\frac{3M-\gamma}{24}(b-a)^2 \tag{1.2}$$

where

$$M=\frac{f'(b)-f'(a)}{b-a}.$$

The purpose of this paper is to obtain a certain local fractional integral inequality on fractal space. This paper is divided into the following four sections. In Section 2, we give the definitions of the local fractional derivatives and local fractional integral and introduce several useful notations on fractal sapace used our main results. In Section 3, the main result is presented. Some double inequalities for twice local fractional differentiable mappings are given. In section 4, we establish some generalized Hermite-Hadamard like inequalities using our main results.

2. Preliminaries

Recall the set R^{α} of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [11, 12] and so on.

Recently, the theory of Yang's fractional sets [11] was introduced as follows.

For $0<\alpha\leq 1$, we have the following α-type set of element sets:

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Z^{α} : The α-type set of integer is defined as the set $\{0^{\alpha}, \pm 1^{\alpha}, \pm 2^{\alpha}, ..., \pm n^{\alpha}, ...\}$.

Q^{α} : The α-type set of the rational numbers is defined as the set $\{m^{\alpha} = \left(\frac{p}{q}\right)^{\alpha} : p, q \in Z, \, q \neq 0\}$.

J^{α} : The α-type set of the irrational numbers is defined as the set $\{m^{\alpha} \neq \left(\frac{p}{q}\right)^{\alpha} : p, q \in Z, \, q \neq 0\}$.

R^{α} : The α-type set of the real line numbers is defined as the set $R^{\alpha} = Q^{\alpha} \cup J^{\alpha}$.

If a^{α}, b^{α} and c^{α} belongs the set R^{α} of real line numbers, then

(1) $a^{\alpha} + b^{\alpha}$ and $a^{\alpha}b^{\alpha}$ belongs the set R^{α};
(2) $a^{\alpha} + b^{\alpha} = b^{\alpha} + a^{\alpha} = (a+b)^{\alpha} = (b+a)^{\alpha}$;
(3) $a^{\alpha} + (b^{\alpha} + c^{\alpha}) = (a+b)^{\alpha} + c^{\alpha}$;
(4) $a^{\alpha}b^{\alpha} = b^{\alpha}a^{\alpha} = (ab)^{\alpha} = (ba)^{\alpha}$;
(5) $a^{\alpha}(b^{\alpha}c^{\alpha}) = (a^{\alpha}b^{\alpha})c^{\alpha}$;
(6) $a^{\alpha}(b^{\alpha} + c^{\alpha}) = a^{\alpha}b^{\alpha} + a^{\alpha}c^{\alpha}$;
(7) $a^{\alpha} + 0^{\alpha} = 0^{\alpha} + a^{\alpha} = a^{\alpha}$ and $a^{\alpha}1^{\alpha} = 1^{\alpha}a^{\alpha} = a^{\alpha}$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 1. [11] *A non-differentiable function $f : R \to R^{\alpha}$, $x \to f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that*

$$|f(x) - f(x_0)| < \varepsilon^{\alpha}$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in R$. If $f(x)$ is local continuous on the interval (a, b), we denote $f(x) \in C_{\alpha}(a, b)$.

Definition 2. [11]*The local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by*

$$f^{(\alpha)}(x_0) = \left.\frac{d^{\alpha}f(x)}{dx^{\alpha}}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha}(f(x) - f(x_0))}{(x - x_0)^{\alpha}},$$

where $\Delta^{\alpha}(f(x) - f(x_0)) \cong \Gamma(\alpha + 1)(f(x) - f(x_0))$.

If there exists $f^{(k+1)\alpha}(x) = \overbrace{D_x^{\alpha}...D_x^{\alpha}}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq R$, then we denoted $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, ...$

Remark 1. [16] *It is found that, in this expression, α is precisely the Hölder exponent of function defined cantor's set. That is to say, $[d(x - x_0)]^{\alpha}$, which is a fractal span, is a fractal geometrical meaning. K-G local fractional derivative of a constant is zero.*

Definition 3. [11] *Let $f(x) \in C_{\alpha}[a, b]$. Then the local fractional integral is defined by,*

$${}_aI_b^{\alpha}f(x) = \frac{1}{\Gamma(\alpha+1)} \int_a^b f(t)(dt)^{\alpha} = \frac{1}{\Gamma(\alpha+1)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^{\alpha},$$

with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_1, \Delta t_2, ..., \Delta t_{N-1}\}$, where $[t_j, t_{j+1}]$, $j = 0, ..., N-1$ and $a = t_0 < t_1 < ... < t_{N-1} < t_N = b$ is partition of interval $[a, b]$.

Here, it follows that ${}_aI_b^{\alpha}f(x) = 0$ if $a = b$ and ${}_aI_b^{\alpha}f(x) = -{}_bI_a^{\alpha}f(x)$ if $a < b$. If for any $x \in [a, b]$, there exists ${}_aI_x^{\alpha}f(x)$, then we denoted by $f(x) \in I_x^{\alpha}[a, b]$.

Theorem 2. [17]*Constant function $f(x) = c$ is Local fractional integrable from a to b and*

$$ {}_aI_b^{\alpha} f(x) = \frac{c\,(b-a)^{\alpha}}{\Gamma(\alpha+1)}. $$

Definition 4 (Generalized convex function)**.** [11] *Let $f : I \subseteq R \to R^{\alpha}$. For any $x_1, x_2 \in I$ and $\lambda \in [0,1]$, if the following inequality*

$$ f(\lambda x_1 + (1-\lambda)x_2) \le \lambda^{\alpha} f(x_1) + (1-\lambda)^{\alpha} f(x_2) $$

holds, then f is called a generalized convex function on I.

Here are two basic examples of generalized convex functions:
(1) $f(x) = x^{\alpha p}$, $x \ge 0$, $p > 1$;
(2) $f(x) = E_{\alpha}(x^{\alpha})$, $x \in R$ where $E_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Leffer function.

Theorem 3. *Let $f \in D_{\alpha}(I)$, then the following conditions are equivalent*
a) f is a generalized convex function on I
b) $f^{(\alpha)}$ is an increasing function on I
c) for any $x_1, x_2 \in I$,

$$ f(x_2) - f(x_1) \ge \frac{f^{(\alpha)}(x_1)}{\Gamma(1+\alpha)} (x_2 - x_1)^{\alpha}. $$

Corollary 1. *Let $f \in D_{2\alpha}(a,b)$. Then f is a generalized convex function (or a generalized concave function) if and only if*

$$ f^{(2\alpha)}(x) \ge 0 \left(or\ f^{(2\alpha)}(x) \le 0 \right) $$

for all $x \in (a,b)$.

Lemma 1. [11]
(1) (Local fractional integration is anti-differentiation) Suppose that $f(x) = g^{(\alpha)}(x) \in C_{\alpha}[a,b]$, then we have

$$ {}_aI_b^{\alpha} f(x) = g(b) - g(a). $$

(2) (Local fractional integration by parts) Suppose that $f(x), g(x) \in D_{\alpha}[a,b]$ and $f^{(\alpha)}(x)$, $g^{(\alpha)}(x) \in C_{\alpha}[a,b]$, then we have

$$ {}_aI_b^{\alpha} f(x) g^{(\alpha)}(x) = f(x)g(x)\big|_a^b - {}_aI_b^{\alpha} f^{(\alpha)}(x) g(x). $$

Lemma 2. [11]

$$ \frac{d^{\alpha} x^{k\alpha}}{dx^{\alpha}} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha}; $$

$$ \frac{1}{\Gamma(\alpha+1)} \int_a^b x^{k\alpha} (dx)^{\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha} \right),\ k \in R. $$

Lemma 3. [11] *Suppose that $f(x) \in C_{\alpha}[a,b]$, then*

$$ \frac{d^{\alpha} \left({}_aI_x^{\alpha} f(t) \right)}{dx^{\alpha}} = f(x) \quad a < x < b. $$

In [2], Mo et al. proved the following generalized Hermite-Hadamard inequality for generalized convex function:

Theorem 4 (Gereneralized Hermite-Hadamard's inequality)**.** *Let $f(x) \in I_x^{\alpha}[a,b]$ be generalized convex function on $[a,b]$ with $a<b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha} f(x) \leq \frac{f(a)+f(b)}{2^{\alpha}}.$$

In [9], authors gave the following generalized Grüss inequality:

Theorem 5 (Geneneralized Grüss inequality)**.** *Let $f,g:[a,b]\to\mathbb{R}^{\alpha}$ are two functions defined and local fractional integrable on $[a,b]$. Then, $\varphi \leq f(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$, for all $x\in[a,b], \varphi, \Phi, \gamma$ and $\Gamma \in \mathbb{R}^{\alpha}$ we have*

$$|T_{\alpha}(f,g)| \leq \frac{(b-a)^{2\alpha}}{4^{\alpha}\Gamma^2(1+\alpha)}(\Phi-\varphi)(\Gamma-\gamma) \tag{2.1}$$

where

$$T_{\alpha}(f,g) = \frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}\ {}_aI_b^{\alpha} f(x)g(x) - \left[\ {}_aI_b^{\alpha} f(x)\right]\left[\ {}_aI_b^{\alpha} g(x)\right].$$

For more information and recent developments on local fractional theory, please refer to [1]-[9],[11]-[17].

3. Main Results

We will give our first main inequality as follow:

Theorem 6. *Let $I\subseteq R$ be an interval, $f: I^0 \subseteq R \to R^{\alpha}$ (I^0 is the interior of I) such that $f\in D_{2\alpha}(I^0)$ and $f^{(2\alpha)} \in C_{2\alpha}(I)$ for $a,b\in I^0$ with $a<b$. Then, $\varphi \leq f^{(2\alpha)}(x) \leq \Phi$ for all $x\in[a,b]$, $\varphi,\Phi\in\mathbb{R}^{\alpha}$, we have following the inequality*

$$\begin{aligned} &\frac{(b-a)^{2\alpha}}{4^{\alpha}}\left[\Phi\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}-\frac{1}{\Gamma(1+2\alpha)}\right)+S\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right] \\ \leq\ & \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha} f(t) - f\left(\frac{a+b}{2}\right) \\ \leq\ & \frac{(b-a)^{2\alpha}}{4^{\alpha}}\left[\varphi\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}-\frac{1}{\Gamma(1+2\alpha)}\right)+S\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right], \end{aligned} \tag{3.1}$$

where

$$S = \frac{f^{(\alpha)}(b)-f^{(\alpha)}(a)}{(b-a)^{\alpha}}.$$

Proof. We define

$$p(t) = \begin{cases} (t-a)^{2\alpha}, & t\in\left[a,\frac{a+b}{2}\right] \\ (t-b)^{2\alpha}, & t\in\left(\frac{a+b}{2},b\right]. \end{cases}$$

Then we have

$$K = \frac{1}{\Gamma(1+\alpha)}\int_a^b p(t)f^{(2\alpha)}(t)\,(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)}\int_a^{\frac{a+b}{2}} (t-a)^{2\alpha} f^{(2\alpha)}(t)\,(dt)^\alpha + \frac{1}{\Gamma(1+\alpha)}\int_{\frac{a+b}{2}}^b (t-b)^{2\alpha} f^{(2\alpha)}(t)\,(dt)^\alpha = K_1 + K_2. \tag{3.2}$$

Using the local fractional integration by parts, we have

$$\begin{aligned} K_1 &= (t-a)^{2\alpha} f^{(\alpha)}(t)\Big|_a^{\frac{a+b}{2}} - \frac{1}{\Gamma(1+\alpha)}\int_a^{\frac{a+b}{2}} \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}(t-a)^{\alpha} f^{(\alpha)}(t)\,(dt)^\alpha \\ &= \frac{(b-a)^{2\alpha}}{2^\alpha} f^{(\alpha)}\left(\frac{a+b}{2}\right) - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}(t-a)^\alpha f(t)\Big|_a^{\frac{a+b}{2}} + \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\int_a^{\frac{a+b}{2}} f(t)\,(dt)^\alpha \\ &= \frac{(b-a)^{2\alpha}}{2^\alpha} f^{(\alpha)}\left(\frac{a+b}{2}\right) - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\frac{(b-a)^\alpha}{2^\alpha} f\left(\frac{a+b}{2}\right) + \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\int_a^{\frac{a+b}{2}} f(t)\,(dt)^\alpha \end{aligned} \tag{3.3}$$

and similarly,

$$K_2 = -\frac{(b-a)^{2\alpha}}{2^\alpha} f^{(\alpha)}\left(\frac{a+b}{2}\right) - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\frac{(b-a)^\alpha}{2^\alpha} f\left(\frac{a+b}{2}\right) + \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\int_{\frac{a+b}{2}}^b f(t)\,(dt)^\alpha. \tag{3.4}$$

Adding (3.3) and (3.4), we obtain

$$K = -\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}(b-a)^\alpha f\left(\frac{a+b}{2}\right) + \Gamma(1+2\alpha)\ {}_aI_b^\alpha f(t). \tag{3.5}$$

Moreover, we have

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b p(t)\,(dt)^\alpha = \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{(b-a)^{3\alpha}}{4^\alpha}. \tag{3.6}$$

From (3.5) and (3.6) we have

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)}\int_a^b p(t)\left[f^{(2\alpha)}(t)-\varphi\right](dt)^\alpha \\ = {}& \Gamma(1+2\alpha)\ {}_aI_b^\alpha f(t) - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}(b-a)^\alpha f\left(\frac{a+b}{2}\right) \\ &-\varphi\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{(b-a)^{3\alpha}}{4^\alpha}. \end{aligned} \tag{3.7}$$

We also get

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)}\int_a^b p(t)\left[f^{(2\alpha)}(t)-\varphi\right](dt)^\alpha \\ \le {}& \sup_{t\in[a,b]}|p(t)|\,\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|f^{(2\alpha)}(t)-\varphi\right|(dt)^\alpha \\ = {}& \frac{(b-a)^{2\alpha}}{4^\alpha}\left[f^{(\alpha)}(b)-f^{(\alpha)}(a)-\frac{\varphi\,(b-a)^\alpha}{\Gamma(1+\alpha)}\right] \\ = {}& \frac{(b-a)^{3\alpha}}{4^\alpha}\left[S-\frac{\varphi}{\Gamma(1+\alpha)}\right] \end{aligned} \tag{3.8}$$

since $f^{(2\alpha)}(t)\ge\varphi$, $t\in[a,b]$.

From (3.7) and (3.8) we get

$$\begin{aligned} &\Gamma(1+2\alpha)\ {}_aI_b^\alpha f(t) - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}(b-a)^\alpha f\left(\frac{a+b}{2}\right) \\ \le {}& \frac{(b-a)^{3\alpha}}{4^\alpha}\left[S+\varphi\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\frac{1}{\Gamma(1+\alpha)}\right)\right] \end{aligned}$$

which completes the proof of second inequality in (3.1).

On the other hand, we have

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)}\int_a^b p(t)\left[\Phi-f^{(2\alpha)}(t)\right](dt)^\alpha \\ = {}& \Phi\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{(b-a)^{3\alpha}}{4^\alpha} \\ &-\Gamma(1+2\alpha)\ {}_aI_b^\alpha f(t) + \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}(b-a)^\alpha f\left(\frac{a+b}{2}\right) \end{aligned} \tag{3.9}$$

and

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b p(t)\left[\Phi - f^{(2\alpha)}(t)\right](dt)^{\alpha} \tag{3.10}$$

$$\leq \sup_{t\in[a,b]}|p(t)|\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|\Phi - f^{(2\alpha)}(t)\right|(dt)^{\alpha}$$

$$= \frac{(b-a)^{2\alpha}}{4^{\alpha}}\left[\frac{\Phi(b-a)^{\alpha}}{\Gamma(1+\alpha)} - \left(f^{(\alpha)}(b) - f^{(\alpha)}(a)\right)\right]$$

$$= \frac{(b-a)^{3\alpha}}{4^{\alpha}}\left[\frac{\Phi}{\Gamma(1+\alpha)} - S\right]$$

since $f^{(2\alpha)}(t) \leq \Phi$, $t \in [a,b]$.

Using (3.9) and (3.10) we obtain

$$\frac{(b-a)^{3\alpha}}{4^{\alpha}}\left[\Phi\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} - \frac{1}{\Gamma(1+\alpha)}\right) + S\right]$$

$$\leq \Gamma(1+2\alpha)\ {}_aI_b^{\alpha} f(t) - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}(b-a)^{\alpha} f\left(\frac{a+b}{2}\right)$$

which completely completes the proof. □

Theorem 7. *Under the assumptions of Theorem 6, we have*

$$\frac{(b-a)^{2\alpha}}{4^{\alpha}}\left[S\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - \Phi\frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}\right] \tag{3.11}$$

$$\leq \frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha} f(t)$$

$$\leq \frac{(b-a)^{2\alpha}}{4^{\alpha}}\left[S\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - \varphi\frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}\right],$$

where

$$S = \frac{f^{(\alpha)}(b) - f^{(\alpha)}(a)}{(b-a)^{\alpha}}.$$

Proof. We define

$$q(t) = \left(t - \frac{a+b}{2}\right)^{2\alpha} - \frac{(b-a)^{2\alpha}}{4^{\alpha}}.$$

Then we have

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b q(t)(dt)^{\alpha} = \frac{(b-a)^{3\alpha}}{4^{\alpha}}\left[\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} - \frac{1}{\Gamma(1+\alpha)}\right]. \tag{3.12}$$

Using the local fractional integration by parts, we have

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b q(t)f^{(2\alpha)}(t)\,(dt)^\alpha \tag{3.13}$$

$$= \left[\left(t-\frac{a+b}{2}\right)^{2\alpha}-\frac{(b-a)^{2\alpha}}{4^\alpha}\right]f^{(\alpha)}(t)\Big|_a^b$$

$$-\frac{1}{\Gamma(1+\alpha)}\int_a^b \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\left(t-\frac{a+b}{2}\right)^{\alpha}f^{(\alpha)}(t)\,(dt)^\alpha$$

$$= -\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\left(t-\frac{a+b}{2}\right)^{\alpha}f(t)\Big|_a^b+\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\int_a^b f(t)\,(dt)^\alpha$$

$$= \Gamma(1+2\alpha)\ {}_aI_b^{\alpha}f(t)-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}(b-a)^\alpha\,\frac{f(a)+f(b)}{2^\alpha}.$$

From (3.12) and (3.13), we have

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b q(t)\left[\varphi-f^{(2\alpha)}(t)\right](dt)^\alpha \tag{3.14}$$

$$= \varphi\frac{(b-a)^{3\alpha}}{4^\alpha}\left[\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\frac{1}{\Gamma(1+\alpha)}\right]$$

$$-\Gamma(1+2\alpha)\ {}_aI_b^{\alpha}f(t)+\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}(b-a)^\alpha\,\frac{f(a)+f(b)}{2^\alpha}.$$

We also get

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b q(t)\left[\varphi-f^{(2\alpha)}(t)\right](dt)^\alpha \tag{3.15}$$

$$\leq \sup_{t\in[a,b]}|q(t)|\,\frac{1}{\Gamma(1+\alpha)}\int_a^b\left|\varphi-f^{(2\alpha)}(t)\right|(dt)^\alpha$$

$$= \frac{(b-a)^{2\alpha}}{4^\alpha}\left[f^{(\alpha)}(b)-f^{(\alpha)}(a)-\frac{\varphi(b-a)^\alpha}{\Gamma(1+\alpha)}\right]$$

$$= \frac{(b-a)^{3\alpha}}{4^\alpha}\left[S-\frac{\varphi}{\Gamma(1+\alpha)}\right]$$

since $f^{(2\alpha)}(t)\geq\varphi$, $t\in[a,b]$.

Using (3.14) and (3.15) we have

$$\begin{aligned}&\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}(b-a)^{\alpha}\frac{f(a)+f(b)}{2^{\alpha}}-\Gamma(1+2\alpha)\ {}_{a}I_{b}^{\alpha}f(t)\\ \leq\ &\frac{(b-a)^{3\alpha}}{4^{\alpha}}\left[S-\varphi\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]\end{aligned}$$

which completes the proof of second inequality in (3.11).

On the other hand, we have

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}q(t)\left[f^{(2\alpha)}(t)-\Phi\right](dt)^{\alpha} \qquad (3.16)\\ =\ &\Gamma(1+2\alpha)\ {}_{a}I_{b}^{\alpha}f(t)-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}(b-a)^{\alpha}\frac{f(a)+f(b)}{2^{\alpha}}\\ &-\Phi\frac{(b-a)^{3\alpha}}{4^{\alpha}}\left[\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\frac{1}{\Gamma(1+\alpha)}\right]\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}q(t)\left[f^{(2\alpha)}(t)-\Phi\right](dt)^{\alpha} \qquad (3.17)\\ \leq\ &\sup_{t\in[a,b]}|q(t)|\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}\left|f^{(2\alpha)}(t)-\Phi\right|(dt)^{\alpha}\\ =\ &\frac{(b-a)^{2\alpha}}{4^{\alpha}}\left[\frac{\Phi(b-a)^{\alpha}}{\Gamma(1+\alpha)}-\left(f^{(\alpha)}(b)-f^{(\alpha)}(a)\right)\right]\\ =\ &\frac{(b-a)^{3\alpha}}{4^{\alpha}}\left[\frac{\Phi}{\Gamma(1+\alpha)}-S\right]\end{aligned}$$

since $f^{(2\alpha)}(t)\leq\Phi$, $t\in[a,b]$.

Using (3.16) and (3.17) we have

$$\begin{aligned}&\frac{(b-a)^{3\alpha}}{4^{\alpha}}\left[S-\varphi\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]\\ \leq\ &\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}(b-a)^{\alpha}\frac{f(a)+f(b)}{2^{\alpha}}-\Gamma(1+2\alpha)\ {}_{a}I_{b}^{\alpha}f(t).\end{aligned}$$

This completes the proof. □

Theorem 8. *Under the assumptions of Theorem 6, we have the inequality*

$$\begin{aligned}&\left|\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(t)-f\left(\frac{a+b}{2}\right)-\frac{L}{4^{\alpha}}\frac{\Gamma^{2}(1+\alpha)}{\Gamma(1+3\alpha)}\right|\\ \leq\ &\frac{(b-a)^{2\alpha}}{16^{\alpha}\Gamma(1+2\alpha)}(\Phi-\varphi)\end{aligned}$$

where $L=\left(f^{(\alpha)}(b)-f^{(\alpha)}(a)\right)(b-a)^{\alpha}$.

Proof. By generalized Grüss inequality, we have that

$$\begin{aligned}&\left|\frac{(b-a)^{\alpha}}{\Gamma^{2}(1+\alpha)}\int_{a}^{b}p(t)f^{(2\alpha)}(t)\,(dt)^{\alpha}\right.\\&\left.-\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}p(t)\,(dt)^{\alpha}\right)\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}f^{(2\alpha)}(t)\,(dt)^{\alpha}\right)\right|\\\leq\;&\frac{(b-a)^{2\alpha}}{4^{\alpha}\Gamma^{2}(1+\alpha)}(\Phi-\varphi)(N-n)\end{aligned}$$

where

$$N=\sup_{t\in[a,b]}p(t)=\frac{(b-a)^{2\alpha}}{4^{\alpha}}$$

and

$$n=\inf_{t\in[a,b]}p(t)=0.$$

From (3.5) and (3.6) we have

$$\begin{aligned}&\left|\frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}\left[\Gamma(1+2\alpha)\ {}_{a}I_{b}^{\alpha}f(t)-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}(b-a)^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right.\\&\left.-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{(b-a)^{3\alpha}}{4^{\alpha}}\left(f^{(\alpha)}(b)-f^{(\alpha)}(a)\right)\right|\\\leq\;&\frac{(b-a)^{4\alpha}}{16^{\alpha}\Gamma^{2}(1+\alpha)}(\Phi-\varphi).\end{aligned}$$

That is,

$$\begin{aligned}&\left|\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(t)-f\left(\frac{a+b}{2}\right)\right.\\&\left.-\frac{\Gamma^{2}(1+\alpha)}{\Gamma(1+3\alpha)}\frac{(b-a)^{\alpha}}{4^{\alpha}}\left(f^{(\alpha)}(b)-f^{(\alpha)}(a)\right)\right|\\\leq\;&\frac{(b-a)^{2\alpha}}{16^{\alpha}\Gamma(1+2\alpha)}(\Phi-\varphi).\end{aligned}$$

This completes the proof. □

Theorem 9. *Under the assumptions of Theorem 6, we have the inequality*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(t)-\frac{L}{4^{\alpha}}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma^{2}(1+\alpha)}{\Gamma(1+3\alpha)}\right]\right|\\\leq\;&\frac{(b-a)^{2\alpha}}{16^{\alpha}\Gamma(1+2\alpha)}(\Phi-\varphi).\end{aligned}$$

where $L=\left(f^{(\alpha)}(b)-f^{(\alpha)}(a)\right)(b-a)^{\alpha}$.

Proof. Defining $r(t) = -q(t)$ and using generalized Grüss inequality, we have that

$$\begin{aligned} & \left| \frac{(b-a)^{\alpha}}{\Gamma^2(1+\alpha)} \int_a^b r(t) f^{(2\alpha)}(t) (dt)^{\alpha} \right. \\ & \left. - \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b r(t) (dt)^{\alpha} \right) \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b f^{(2\alpha)}(t) (dt)^{\alpha} \right) \right| \\ \leq & \frac{(b-a)^{2\alpha}}{4^{\alpha}\Gamma^2(1+\alpha)} (\Phi - \varphi)(\Omega - \omega) \end{aligned}$$

where

$$\Omega = \sup_{t \in [a,b]} r(t) = \frac{(b-a)^{2\alpha}}{4^{\alpha}}$$

and

$$\omega = \inf_{t \in [a,b]} r(t) = 0.$$

Using (3.12) and (3.13) we have

$$\begin{aligned} & \left| \frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)} \left[\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} (b-a)^{\alpha} \frac{f(a)+f(b)}{2^{\alpha}} - \Gamma(1+2\alpha)\ {}_aI_b^{\alpha} f(t) \right] \right. \\ & \left. - \frac{(b-a)^{3\alpha}}{4^{\alpha}} \left[\frac{1}{\Gamma(1+\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \right] \left(f^{(\alpha)}(b) - f^{(\alpha)}(a) \right) \right| \\ \leq & \frac{(b-a)^{4\alpha}}{16^{\alpha}\Gamma^2(1+\alpha)} (\Phi - \varphi). \end{aligned}$$

i.e.,

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha} f(t) \right. \\ & \left. - \frac{(b-a)^{\alpha}}{4^{\alpha}} \left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma^2(1+\alpha)}{\Gamma(1+3\alpha)} \right] \left(f^{(\alpha)}(b) - f^{(\alpha)}(a) \right) \right| \\ \leq & \frac{(b-a)^{2\alpha}}{16^{\alpha}\Gamma(1+2\alpha)} (\Phi - \varphi). \end{aligned}$$

The proof is completed. □

4. Results for generalized convex and generalized concave functions

From Theorem 4, one has the following the generalized Hermite-Hadamard's inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha} f(x) \leq \frac{f(a)+f(b)}{2^{\alpha}}. \tag{4.1}$$

If $f^{(2\alpha)}(x) \geq 0$, $x \in (a,b)$, i.e. f is a generalized convex function (Corollary 1), then we can set $\varphi = 0$ in (3.1). Thus,

$$\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha} f(t) \leq f\left(\frac{a+b}{2}\right) + \frac{L}{4^{\alpha}} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \tag{4.2}$$

where $L=\left(f^{(\alpha)}(b)-f^{(\alpha)}(a)\right)(b-a)^{\alpha}$. On the other hand, from (3.11) we get

$$\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(t)\geq\frac{f(a)+f(b)}{2^{\alpha}}-\frac{L}{4^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}. \tag{4.3}$$

Therefore, the following result holds.

Theorem 10. *Let the assumptions of Theorem 6 hold. If* $f^{(2\alpha)}(x)\geq 0,\ x\in(a,b)$ *, then we have*

$$\frac{f(a)+f(b)}{2^{\alpha}}-\frac{L}{4^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\leq\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(x)\leq f\left(\frac{a+b}{2}\right)+\frac{L}{4^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}$$

where $L=\left(f^{(\alpha)}(b)-f^{(\alpha)}(a)\right)(b-a)^{\alpha}$.

Corollary 2. *Under the assumptions of Theorem 10, we have*

$$\frac{f(a)+f(b)}{2^{\alpha}}-\frac{L}{4^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\leq\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(x)\leq\frac{f(a)+f(b)}{2^{\alpha}}$$

and

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(x)\leq f\left(\frac{a+b}{2}\right)+\frac{L}{4^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}.$$

Proof. From (4.1)-(4.3), th proof is obvious. □

If $f^{(2\alpha)}(x)\leq 0,\ x\in(a,b)$, i.e. f is a generalized concave function (Corollary 1), then we can set $\Phi=0$ in (3.1). Thus,

$$\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(t)\geq f\left(\frac{a+b}{2}\right)+\frac{L}{4^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \tag{4.4}$$

where $L=\left(f^{(\alpha)}(b)-f^{(\alpha)}(a)\right)(b-a)^{\alpha}$. On the other hand, from (3.11) we get

$$\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(t)\leq\frac{f(a)+f(b)}{2^{\alpha}}-\frac{L}{4^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}. \tag{4.5}$$

Hence, the following result is valid.

Theorem 11. *Let the assumptions of Theorem 6 hold. If* $f^{(2\alpha)}(x)\leq 0,\ x\in(a,b)$ *, then we have*

$$f\left(\frac{a+b}{2}\right)+\frac{L}{4^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\leq\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(x)\leq\frac{f(a)+f(b)}{2^{\alpha}}-\frac{L}{4^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}$$

where $L=\left(f^{(\alpha)}(b)-f^{(\alpha)}(a)\right)(b-a)^{\alpha}$.

Corollary 3. *Under the assumptions of Theorem 11, we have*

$$\frac{f(a)+f(b)}{2^{\alpha}}\leq\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(x)\leq\frac{f(a)+f(b)}{2^{\alpha}}-\frac{L}{4^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}$$

and

$$f\left(\frac{a+b}{2}\right)+\frac{L}{4^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\leq\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(x)\leq f\left(\frac{a+b}{2}\right).$$

References

[1] G-S. Chen, *Generalizations of Hölder's and some related integral inequalities on fractal space,* Journal of Function Spaces and Applications Volume 2013, Article ID 198405.

[2] H. Mo, X Sui and D Yu, *Generalized convex functions on fractal sets and two related inequalities*, Abstract and Applied Analysis, Volume 2014, Article ID 636751, 7 pages.

[3] H. Mo, *Generalized Hermite-Hadamard inequalities involving local fractional integral,* arXiv:1410.1062.

[4] M. Z. Sarikaya and H Budak, *Generalized Ostrowski type inequalities for local fractional integrals*, RGMIA Research Report Collection, 18(2015), Article 62, 11 pp.

[5] M. Z. Sarikaya, S.Erden and H. Budak, *Some generalized Ostrowski type inequalities involving local fractional integrals and applications*, RGMIA Research Report Collection, 18(2015), Article 63, 12 pp.

[6] M. Z. Sarikaya and H. Budak, *On generalized Hermite-Hadamard inequality for generalized convex function*, RGMIA Research Report Collection, 18(2015), Article 64, 15 pp.

[7] M. Z. Sarikaya, S.Erden and H. Budak, *Some integral inequalities for local fractional integrals*, RGMIA Research Report Collection, 18(2015), Article 65, 12 pp.

[8] M. Z. Sarikaya, H. Budak and S.Erden, *On new inequalities of Simpson's type for generalized convex functions*, RGMIA Research Report Collection, 18(2015), Article 66, 13 pp.

[9] M. Z. Sarikaya and H Budak, *On generalized some integral inequalities for local fractional integrals*, Submitted.

[10] Ujević N., *Some double integral inequalities and applications*, Acta. Math. Univ. Comenianae, 71(2) (2002), 189–199.

[11] X. J. Yang, *Advanced Local Fractional Calculus and Its Applications*, World Science Publisher, New York, 2012.

[12] J. Yang, D. Baleanu and X. J. Yang, *Analysis of fractal wave equations by local fractional Fourier series method*, Adv. Math. Phys. , 2013 (2013), Article ID 632309.

[13] X. J. Yang, *Local fractional integral equations and their applications*, Advances in Computer Science and its Applications (ACSA) 1(4), 2012.

[14] X. J. Yang, *Generalized local fractional Taylor's formula with local fractional derivative*, Journal of Expert Systems, 1(1) (2012) 26-30.

[15] X. J. Yang, *Local fractional Fourier analysis,* Advances in Mechanical Engineering and its Applications 1(1), 2012 12-16.

[16] X. J. Yang, *A Short note on local fractional calculus of function of one variable*, Journal of Applied Library and Information Science (JALIS) 1(1), 2012.

[17] X.J. Yang. *Local fractional functional analysis and its applications.* Asian Academic Publisher Limited. 2011.

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