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GENERALIZED POMPEIU TYPE INEQUALITIES FOR LOCAL FRACTIONAL INTEGRALS AND ITS APPLICATIONS

SAMET ERDEN AND MEHMET ZEKI SARIKAYA

ABSTRACT. First of all, the generalized Pompeiu's mean value theorem is established. Then, some generalized Pompeiu type inequalities are obtained. Finally, some applications of these inequalities in numerical integration and for special means are given.

1. INTRODUCTION

In 1938, the classical integral inequality established by Ostrowski [9] as follows:

Theorem 1. *Let $f : [a,b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f' : (a,b) \rightarrow \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, the inequality holds:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2} \right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Inequality (1.1) has wide applications in numerical analysis and in the theory of some special means; estimating error bounds for some special means, some mid-point, trapezoid and Simpson rules and quadrature rules, etc. Hence inequality (1.1) has attracted considerable attention and interest from mathematicans and researchers.

In 1946, Pompeiu [11] derived a variant of Lagrange's mean value theorem, now known as *Pompeiu's mean value theorem.*

Theorem 2. *For every real valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a,b]$, there exist a point ξ between x_1 and x_2 such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi).$$

The following Pompeiu type inequality is proved by Dragomir in [4].

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Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $[a,b]$ not containing 0. Then for any $x\in[a,b]$, we have the inequality*

$$\begin{aligned} &\left|\frac{a+b}{2}\frac{f(x)}{x}+\frac{1}{b-a}\int_a^b f(t)dt\right| \\ &\leq \frac{b-a}{|x|}\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right]\left\|f-lf'\right\|_\infty . \end{aligned}$$

where $l(t)=t$ for all $t\in[a,b]$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

In [10], the authors have proved the following Ostrowski type inequality:

Theorem 4. *Let $f:[a,b]\longrightarrow\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0<a<b$. Then for $1\leq p,\ q\leq\infty$ with $\frac{1}{p}+\frac{1}{q}=1$ we have the inequality*

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}+\frac{1}{b-a}\int_a^b f(t)dt\right|\leq PU(x,p)\left\|f-lf'\right\|_p ,$$

for $x\in[a,b]$, where

$$\begin{aligned} PU(x,p) \quad : &= (b-a)^{\frac{1}{p}-1}\left[\left(\frac{a^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-a^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{\frac{1}{p}}\right. \\ &\left.+\left(\frac{b^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-b^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{\frac{1}{q}}\right]. \end{aligned}$$

In the cases $(p,q)=(1,\infty),(\infty,1)$ and $(2,2)$ the quantity $PU(x,p)$ has to be taken as the limit as $p\longrightarrow 1,\infty$ and 2, respectively.

The interested reader is also referred to ([1], [2], [4], [5], [8], [10], [12]-[14]) for integral inequalities by using Pompeiu's mean value theorem.

2. Preliminaries

Recall the set R^α of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [20, 22] and so on.

Recently, the theory of Yang's fractional sets [20] was introduced as follows.

For $0<\alpha\leq 1$, we have the following α-type set of element sets:

Z^α : The α-type set of integer is defined as the set $\{0^\alpha,\pm1^\alpha,\pm2^\alpha,...,\pm n^\alpha,...\}$.

Q^α : The α-type set of the rational numbers is defined as the set $\{m^\alpha=\left(\frac{p}{q}\right)^\alpha : p,q\in Z,\ q\neq 0\}$.

J^α : The α-type set of the irrational numbers is defined as the set $\{m^\alpha\neq\left(\frac{p}{q}\right)^\alpha : p,q\in Z,\ q\neq 0\}$.

R^α : The α-type set of the real line numbers is defined as the set $R^\alpha=Q^\alpha\cup J^\alpha$.

If a^α, b^α and c^α belongs the set R^α of real line numbers, then

(1) $a^\alpha+b^\alpha$ and $a^\alpha b^\alpha$ belongs the set R^α;

(2) $a^\alpha+b^\alpha=b^\alpha+a^\alpha=(a+b)^\alpha=(b+a)^\alpha$;

(3) $a^\alpha+(b^\alpha+c^\alpha)=(a+b)^\alpha+c^\alpha$;

(4) $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$;
(5) $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$;
(6) $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$;
(7) $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 1. [20] *A non-differentiable function $f : R \to R^\alpha$, $x \to f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that*

$$|f(x) - f(x_0)| < \varepsilon^\alpha$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in R$. If $f(x)$ is local continuous on the interval (a, b), we denote $f(x) \in C_\alpha(a, b)$.

Definition 2. [20] *The local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by*

$$f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x\to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha},$$

where $\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(\alpha + 1) (f(x) - f(x_0))$.

If there exists $f^{(k+1)\alpha}(x) = \overbrace{D_x^\alpha ... D_x^\alpha}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq R$, then we denoted $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, ...$

Lemma 1. [21] *Suppose that $f(x) \in C_\alpha [a, b]$ and $f(x) \in D_\alpha(a, b)$, then for $0 < \alpha \leq 1$ we have a $\alpha-$differential form*

$$d^\alpha f(x) = f^{(\alpha)}(x) dx^\alpha.$$

Lemma 2. [21] *Let I be an interval, $f, g : I \subset R \to R^\alpha$ (I° is the interior of I) such that $f, g \in D_\alpha(I^\circ)$. Then, the following differentiation rules are valid.*
(1) $\frac{d^\alpha [f(x) \pm g(x)]}{dx^\alpha} = f^{(\alpha)}(x) \pm g^{(\alpha)}(x)$;
(2) $\frac{d^\alpha f(x)g(x)}{dx^\alpha} = f^{(\alpha)}(x)g(x) + f(x)g^{(\alpha)}(x)$;
(3) $\frac{d^\alpha \left(\frac{f(x)}{g(x)}\right)}{dx^\alpha} = \frac{f^{(\alpha)}(x)g(x) - f(x)g^{(\alpha)}(x)}{[g(x)]^2}$ *where* $g(x) \neq 0$;
(4) $\frac{d^\alpha [cf(x)]}{dx^\alpha} = cf^{(\alpha)}(x)$ *where c is a constant;*
(5) If $y(x) = (f \circ g)(x)$, then

$$\frac{d^\alpha y(x)}{dx^\alpha} = f^{(\alpha)}(g(x)) \left(g^{(1)}(x)\right)^\alpha.$$

Theorem 5 (Generalized mean value theorem)**.** [24] *Suppose that $f(x) \in C_\alpha [a, b]$, $f^{(\alpha)}(x) \in C (a, b)$, we have*

$$\frac{f(x) - f(x_0)}{(x - x_0)^\alpha} = \frac{f^{(\alpha)}(\xi)}{\Gamma(\alpha + 1)}$$

where $a < x_0 < \xi < x < b$.

Definition 3. [20] *Let $f(x) \in C_\alpha [a, b]$. Then the local fractional integral is defined by,*

$${}_aI_b^\alpha f(x) = \frac{1}{\Gamma(\alpha + 1)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(\alpha + 1)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha,$$

with $\Delta t_j = t_{j+1} - t_j$ *and* $\Delta t = \max\{\Delta t_1, \Delta t_2, ..., \Delta t_{N-1}\}$, *where* $[t_j, t_{j+1}]$, $j = 0, ..., N-1$ *and* $a = t_0 < t_1 < ... < t_{N-1} < t_N = b$ *is partition of interval* $[a,b]$.

Here, it follows that ${}_aI_b^{\alpha} f(x) = 0$ *if* $a = b$ *and* ${}_aI_b^{\alpha} f(x) = -{}_bI_a^{\alpha} f(x)$ *if* $a < b$. *If for any* $x \in [a,b]$, *there exists* ${}_aI_x^{\alpha} f(x)$, *then we denoted by* $f(x) \in I_x^{\alpha}[a,b]$.

Lemma 3. [20]

(1) (Local fractional integration is anti-differentiation) Suppose that $f(x) = g^{(\alpha)}(x) \in C_{\alpha}[a,b]$, *then we have*

$$ {}_aI_b^{\alpha} f(x) = g(b) - g(a). $$

(2) (Local fractional integration by parts) Suppose that $f(x), g(x) \in D_{\alpha}[a,b]$ *and* $f^{(\alpha)}(x)$, $g^{(\alpha)}(x) \in C_{\alpha}[a,b]$, *then we have*

$$ {}_aI_b^{\alpha} f(x) g^{(\alpha)}(x) = f(x)g(x)|_a^b - {}_aI_b^{\alpha} f^{(\alpha)}(x) g(x). $$

Lemma 4. [20] *We have*

i) $\dfrac{d^{\alpha} x^{k\alpha}}{dx^{\alpha}} = \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha}$;

ii) $\dfrac{1}{\Gamma(\alpha+1)} \displaystyle\int_a^b x^{k\alpha} (dx)^{\alpha} = \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right)$, $k \in R$.

Lemma 5 (Generalized Hölder's inequality)**.** [20] *Let* $f, g \in C_{\alpha}[a,b]$, $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$ \frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)g(x)| (dx)^{\alpha} \leq \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)|^p (dx)^{\alpha} \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |g(x)|^q (dx)^{\alpha} \right)^{\frac{1}{q}}. $$

The interested reader is able to look over the references [3], [6], [7], [15]-[25] for local fractional theory.

The main aim of this paper is to establish some generalized Pompeiu's type inequlities involving local fractional integrals and obtain some applications of these inequalities in numerical integration and for special means.

3. Main Results

We prove generalized Pompeiu's mean value theorem for local fractional derivative.

Theorem 6 (Generalized Pompeiu's mean value theorem)**.** *Let* $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}^{\alpha}$ *be a mapping such that* $f \in D_{\alpha}(a,b)$, *with* $[a,b]$ *not containing* 0 *and for all pairs* $x_1 \neq x_2$ *in* $[a,b]$, *there exist a point* ξ *in* (x_1, x_2) *such that the following equality holds:*

$$ \frac{x_1^{\alpha} f(x_2) - x_2^{\alpha} f(x_1)}{(x_1 - x_2)^{\alpha}} = f(\xi) - \frac{\xi^{\alpha}}{\Gamma(1+\alpha)} f^{(\alpha)}(\xi). $$

Proof. Define function F on $\left[\frac{1}{b}, \frac{1}{a}\right]$ by

$$ F(t) = t^{\alpha} f(\frac{1}{t}). \tag{3.1} $$

Using the fifth item of Theorem 2, we have

$$ \frac{d^{\alpha} f(\frac{1}{t})}{dx^{\alpha}} = -\frac{1}{t^{2\alpha}} f^{(\alpha)}(\frac{1}{t}). \tag{3.2} $$

Because of $f \in D_\alpha(\frac{1}{b}, \frac{1}{a})$ and using the second item of Theorem 2, the identity (3.2) and Lemma 4, we get

$$F^{(\alpha)}(t) = \Gamma(1+\alpha) f(\frac{1}{t}) - \frac{1}{t^\alpha} f^{(\alpha)}(\frac{1}{t}). \tag{3.3}$$

In addition, applying the generalized mean value theorem to F on the interval $[x, y] \subset \left[\frac{1}{b}, \frac{1}{a}\right]$, we obtain

$$\frac{F(x) - F(y)}{(x-y)^\alpha} = \frac{F^{(\alpha)}(\varphi)}{\Gamma(1+\alpha)} \tag{3.4}$$

for all $\varphi \in (x, y)$.

Now, using (3.1) and (3.3) on (3.4), we obtain

$$\frac{x^\alpha f(\frac{1}{x}) - y^\alpha f(\frac{1}{y})}{(x-y)^\alpha} = \frac{\Gamma(1+\alpha) f(\frac{1}{\varphi}) - \frac{1}{\varphi^\alpha} f^{(\alpha)}(\frac{1}{\varphi})}{\Gamma(1+\alpha)}.$$

Let $x_2 = \frac{1}{x}$, $x_1 = \frac{1}{y}$ and $\xi = \frac{1}{\varphi}$. Then, since $\varphi \in (x, y)$, we have

$$x_1 < \xi < x_2$$

and we write

$$\frac{x_1^\alpha f(x_2) - x_2^\alpha f(x_1)}{(x_1 - x_2)^\alpha} = f(\xi) - \frac{\xi^\alpha}{\Gamma(1+\alpha)} f^{(\alpha)}(\xi).$$

The proof is thus completed. □

Remark 1. *Let $f \in D_\alpha(a, b)$, $[a, b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a, b]$, there exist a point ξ in (x_1, x_2) such that*

$$\frac{x_1^\alpha f(x_2) - x_2^\alpha f(x_1)}{(x_1 - x_2)^\alpha} = f(\xi) - \frac{\xi^\alpha}{\Gamma(1+\alpha)} f^{(\alpha)}(\xi).$$

Now, we will mention here a geometrical interpretation of generalized Pompeiu's theorem. The generalized secant line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is given by

$$y = f(x_1) + \frac{f(x_2) - f(x_1)}{(x_1 - x_2)^\alpha}(x - x_1)^\alpha.$$

This line intersects the $y-$axis at the point $(0, y)$, where y is

$$\begin{aligned} y &= f(x_1) + \frac{f(x_2) - f(x_1)}{(x_1 - x_2)^\alpha}(0 - x_1)^\alpha \\ &= \frac{x_1^\alpha f(x_2) - x_2^\alpha f(x_1)}{(x_1 - x_2)^\alpha}. \end{aligned}$$

On the other hand, the equation of the generalized tangent line at the point $(\xi, f(\xi))$ is

$$y = (x - \xi)^\alpha \frac{f^{(\alpha)}(\xi)}{\Gamma(1+\alpha)} + f(\xi).$$

The deneralized tangent line intersects the $y-$axis at the point $(0, y)$, where y is

$$y = f(\xi) - \frac{\xi^\alpha}{\Gamma(1+\alpha)} f^{(\alpha)}(\xi).$$

Hence, the geometric meaning of generalized Pompeiu's mean value theorem is that the generalized tangent of the point $(\xi, f(\xi))$ intersects on the $y-$axis at the *same point as the generalized secant line connecting the points* $(x_1, f(x_1))$ *and* $(x_2, f(x_2))$.

Now, we give a generalized Pompeiu type ineqaulity using generalized Pompeiu's mean value theorem.

Theorem 7. *Let* $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}^\alpha$ *be a mapping such that* $f \in C_\alpha[a,b]$ *and* $f \in D_\alpha(a,b)$, *with* $[a,b]$ *not containing* 0. *Then for any* $x \in [a,b]$, *we have the inequality*

$$\left| \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \frac{f(x)}{x^\alpha} (a+b)^\alpha - \frac{1}{(b-a)^\alpha} \, {}_aI_b^\alpha f(t) \right| \tag{3.5}$$
$$\leq \frac{2^\alpha \Gamma(1+\alpha)(b-a)^\alpha}{\Gamma(1+2\alpha) |x|^\alpha} \left[\frac{1}{4^\alpha} + \frac{\left(x - \frac{a+b}{2}\right)^{2\alpha}}{(b-a)^{2\alpha}} \right] \left\| f - lf^{(\alpha)} \right\|_\infty$$

where $l(t) = \frac{t^\alpha}{\Gamma(1+\alpha)}$, $t \in [a,b]$, *and* $\left\| f - lf^{(\alpha)} \right\|_\infty = \sup_{\xi \in (a,b)} \left| f(\xi) - lf^{(\alpha)}(\xi) \right| < \infty$. *The constant* $\frac{1}{4^\alpha}$ *is best possible in the sense that it cannot be replaced by a smaller constant.*

Proof. Applying Generalized Pompeiu's mean value theorem, for any $x, t \in [a,b]$, there is ξ beyween x and t such that

$$t^\alpha f(x) - x^\alpha f(t) = \left[f(\xi) - \frac{\xi^\alpha}{\Gamma(1+\alpha)} f^{(\alpha)}(\xi) \right] (t-x)^\alpha. \tag{3.6}$$

Because of the equality (3.6) and the inequality

$$\begin{aligned} \left| f(\xi) - \frac{\xi^\alpha}{\Gamma(1+\alpha)} f^{(\alpha)}(\xi) \right| &\leq \sup_{\xi \in (a,b)} \left| f(\xi) - \frac{\xi^\alpha}{\Gamma(1+\alpha)} f^{(\alpha)}(\xi) \right| \\ &= \left\| f - lf^{(\alpha)} \right\|_\infty, \end{aligned}$$

we have the inequality

$$\left| t^\alpha f(x) - x^\alpha f(t) \right| \leq \left\| f - lf^{(\alpha)} \right\|_\infty |x-t|^\alpha. \tag{3.7}$$

Integrating both sides of (3.7) with respect to t from a to b for local fractional integrals, we get

$$\left| \frac{f(x)}{\Gamma(\alpha+1)} \int_a^b t^\alpha (dt)^\alpha - \frac{x^\alpha}{\Gamma(\alpha+1)} \int_a^b f(t)(dt)^\alpha \right| \tag{3.8}$$
$$\begin{aligned} &\leq \frac{\left\| f - lf^{(\alpha)} \right\|_\infty}{\Gamma(\alpha+1)} \int_a^b |x-t|^\alpha (dt)^\alpha \\ &= \frac{\left\| f - lf^{(\alpha)} \right\|_\infty}{\Gamma(\alpha+1)} \left(\int_a^x (x-t)^\alpha (dt)^\alpha + \int_x^b (t-x)^\alpha (dt)^\alpha \right). \end{aligned}$$

Using the Lemma 4 and the inequality (3.8), we obtain

$$\begin{aligned}(3.9)\qquad & \left|\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(b^{2\alpha}-a^{2\alpha}\right)f(x)-x^{\alpha}\ {}_aI_b^{\alpha}f(t)\right| \\ \leq & \left\|f-lf^{(\alpha)}\right\|_{\infty}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[(x-a)^{2\alpha}+(b-x)^{2\alpha}\right] \\ = & \left\|f-lf^{(\alpha)}\right\|_{\infty}\frac{2^{\alpha}\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[\frac{(b-a)^{2\alpha}}{4^{\alpha}}+\left(x-\frac{a+b}{2}\right)^{2\alpha}\right].\end{aligned}$$

If we divide the inequality (3.9) with $x^{\alpha}(b-a)^{\alpha}$, we easily deduce required result. □

Corollary 1. *Under the same assumptions of Theorem 7 with $x=\frac{a+b}{2}$. Then, we have*

$$\begin{aligned}(3.10)\qquad & \left|\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}f\left(\frac{a+b}{2}\right)-\frac{1}{2^{\alpha}(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(t)\right| \\ \leq & \frac{\Gamma(1+\alpha)(b-a)^{\alpha}}{2^{\alpha}\Gamma(1+2\alpha)\left|a+b\right|^{\alpha}}\left\|f-lf^{(\alpha)}\right\|_{\infty}.\end{aligned}$$

We consider now the weighted integral case.

Theorem 8. *Let $f:[a,b]\subseteq\mathbb{R}\to\mathbb{R}^{\alpha}$ be a mapping such that $f\in C_{\alpha}[a,b]$ and $f\in D_{\alpha}(a,b)$,with $[a,b]$ not containing 0. If $w:[a,b]\to\mathbb{R}^{\alpha}$ is nonnegatiive on $[a,b]$ and $w(t)\in C_{\alpha}[a,b]$, then we have the inequality:*

$$\begin{aligned} & \left|{}_aI_b^{\alpha}f(t)w(t)-\frac{f(x)}{x^{\alpha}}\ {}_aI_b^{\alpha}t^{\alpha}w(t)\right| \\ \leq & \left\|f-lf^{(\alpha)}\right\|_{\infty}\left[sgn(x)\left({}_aI_x^{\alpha}w(t)-{}_xI_b^{\alpha}w(t)\right)\right. \\ & \left.+\frac{1}{\left|x\right|^{\alpha}}\left({}_xI_b^{\alpha}t^{\alpha}w(t)-{}_aI_x^{\alpha}t^{\alpha}w(t)\right)\right]\end{aligned}$$

for each $x\in[a,b]$ and where $l(t)=\frac{t^{\alpha}}{\Gamma(1+\alpha)}$ for all $t\in[a,b]$.

Proof. Using the inequality (3.7), we have

$$\begin{aligned}&\left|\frac{f(x)}{\Gamma(\alpha+1)}\int_a^b t^\alpha w(t)(dt)^\alpha-\frac{x^\alpha}{\Gamma(\alpha+1)}\int_a^b f(t)w(t)(dt)^\alpha\right|\\ \leq\;&\frac{\left\|f-lf^{(\alpha)}\right\|_\infty}{\Gamma(\alpha+1)}\int_a^b w(t)\left|x-t\right|^\alpha(dt)^\alpha\\ =\;&\frac{\left\|f-lf^{(\alpha)}\right\|_\infty}{\Gamma(\alpha+1)}x^\alpha\left(\int_a^x w(t)(dt)^\alpha-\int_x^b w(t)(dt)^\alpha\right)\\ &+\frac{\left\|f-lf^{(\alpha)}\right\|_\infty}{\Gamma(\alpha+1)}\left(\int_x^b t^\alpha w(t)(dt)^\alpha-\int_a^x t^\alpha w(t)(dt)^\alpha\right).\end{aligned}$$

Thus, the proof is completed. □

We prove an inequality of generalized Ostrowski type for $p-$norm.

Theorem 9. *Let $f:[a,b]\subseteq\mathbb{R}\to\mathbb{R}^\alpha$ be a mapping such that $f\in C_\alpha[a,b]$ and $f\in D_\alpha(a,b)$, with $0<a<b$. Then for $\frac{1}{p}+\frac{1}{q}=1$, with $p\geq1$ and all $x\in[a,b]$, the following inequality holds:*

$$\begin{aligned}&\left|\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{f(x)}{x^\alpha}(a+b)^\alpha-\frac{1}{(b-a)^\alpha}\;{}_aI_b^\alpha f(t)\right|\\ \leq\;&\frac{(b-a)^{\left(\frac{1}{p}-1\right)\alpha}}{\Gamma(1+\alpha)^{\frac{1}{q}}}M_\alpha(x,q)\left\|f-lf^{(\alpha)}\right\|_p\end{aligned}\tag{3.11}$$

where $l(t)=\frac{t^\alpha}{\Gamma(1+\alpha)}$, $t\in[a,b]$,

$$\left\|f-lf^{(\alpha)}\right\|_p=\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b\left|f(s)-l(s)f^{(\alpha)}(s)\right|^p(ds)^\alpha\right)^{\frac{1}{p}}$$

and

$$\begin{aligned}M_\alpha(x,q)\;=\;&\left[\frac{\Gamma(1+(1-q)\alpha)}{\Gamma(1+(2-q)\alpha)}\frac{\left(a^{(2-q)}-x^{(2-q)}\right)^\alpha}{(1-2q)^\alpha}\right.\\ &\left.+\frac{\Gamma(1+q\alpha)}{\Gamma(1+(q+1)\alpha)}\frac{\left(x^{(2-q)}-x^{(1-2q)}a^{(q+1)}\right)^\alpha}{(1-2q)^\alpha}\right]^{\frac{1}{q}}\\ &+\left[\frac{\Gamma(1+(1-q)\alpha)}{\Gamma(1+(2-q)\alpha)}\frac{\left(b^{(2-q)}-x^{(2-q)}\right)^\alpha}{(1-2q)^\alpha}+\right.\\ &\left.\frac{\Gamma(1+q\alpha)}{\Gamma(1+(q+1)\alpha)}\frac{\left(x^{(2-q)}-x^{(1-2q)}b^{(q+1)}\right)^\alpha}{(1-2q)^\alpha}\right]^{\frac{1}{q}}.\end{aligned}$$

Proof. Because of $f \in C_\alpha[a,b]$ and $f \in D_\alpha(a,b)$, $H \in C_\alpha[a,b]$ and $H \in D_\alpha(a,b)$ defined as $H(s) = \frac{f(s)}{s^\alpha}$. Then, for any $t, x \in [a,b]$ with $x \neq t$, we have

$$\frac{1}{\Gamma(\alpha+1)} \int_t^x H^{(\alpha)}(s)(ds)^\alpha = \frac{f(x)}{x^\alpha} - \frac{f(t)}{t^\alpha}. \tag{3.12}$$

On the other hand, using the second and fifth items of Theorem 2, we obtain

$$H^{(\alpha)}(s) = \frac{s^\alpha f^{(\alpha)}(s) - \Gamma(1+\alpha) f(s)}{s^{2\alpha}}. \tag{3.13}$$

From (3.12) and (3.13), we get

$$t^\alpha f(x) - x^\alpha f(t) = x^\alpha t^\alpha \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)} \int_x^t \frac{f(s) - \frac{s^\alpha}{\Gamma(\alpha+1)} f^{(\alpha)}(s)}{s^{2\alpha}} (ds)^\alpha. \tag{3.14}$$

Integrating both sides of (3.14) with respect to t from a to b for local fractional integrals, we obtain

$$\begin{aligned} & \frac{f(x)}{x^\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left(b^2 - a^2\right)^\alpha - {}_aI_b^\alpha f(t) \\ = \; & \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)^2} \int_a^b t^\alpha \int_x^t \frac{f(s) - l(s) f^{(\alpha)}(s)}{s^{2\alpha}} (ds)^\alpha (dt)^\alpha \end{aligned}$$

and therefore

$$\begin{aligned} & \left| \frac{f(x)}{x^\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left(b^2 - a^2\right)^\alpha - {}_aI_b^\alpha f(t) \right| \\ \leq \; & \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)} \int_a^b \left| \frac{t^\alpha}{\Gamma(1+\alpha)} \int_x^t \left| \frac{f(s) - l(s) f^{(\alpha)}(s)}{s^{2\alpha}} \right| (ds)^\alpha \right| (dt)^\alpha \\ = \; & \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)} \int_a^x \left| \frac{t^\alpha}{\Gamma(1+\alpha)} \int_t^x \left| \frac{f(s) - l(s) f^{(\alpha)}(s)}{s^{2\alpha}} \right| (ds)^\alpha \right| (dt)^\alpha \\ & + \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)} \int_x^b \left| \frac{t^\alpha}{\Gamma(1+\alpha)} \int_x^t \left| \frac{f(s) - l(s) f^{(\alpha)}(s)}{s^{2\alpha}} \right| (ds)^\alpha \right| (dt)^\alpha. \end{aligned} \tag{3.15}$$

Applying generalized Hölder's inequality, the sum in the last line of (3.15), we find that

$$\begin{aligned}
&\left|\frac{f(x)}{x^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(b^{2}-a^{2}\right)^{\alpha}-{}_{a}I_{b}^{\alpha}f(t)\right| \qquad (3.16)\\
&\leq \Gamma(1+\alpha)\left(\frac{1}{\Gamma(1+\alpha)^{2}}\int_{a}^{x}\int_{t}^{x}\left|f(s)-l(s)f^{(\alpha)}(s)\right|^{p}(ds)^{\alpha}(dt)^{\alpha}\right)^{\frac{1}{p}}\\
&\quad\times\left(\frac{1}{\Gamma(1+\alpha)^{2}}\int_{a}^{x}\int_{t}^{x}\frac{t^{q\alpha}}{s^{2q\alpha}}(ds)^{\alpha}(dt)^{\alpha}\right)^{\frac{1}{q}}\\
&\quad+\Gamma(1+\alpha)\left(\frac{1}{\Gamma(1+\alpha)^{2}}\int_{x}^{b}\int_{x}^{t}\left|f(s)-l(s)f^{(\alpha)}(s)\right|^{p}(ds)^{\alpha}(dt)^{\alpha}\right)^{\frac{1}{p}}\\
&\quad\times\left(\frac{1}{\Gamma(1+\alpha)^{2}}\int_{x}^{b}\int_{x}^{t}\frac{t^{q\alpha}}{s^{2q\alpha}}(ds)^{\alpha}(dt)^{\alpha}\right)^{\frac{1}{q}}\\
&\leq \Gamma(1+\alpha)\left(\frac{1}{\Gamma(1+\alpha)^{2}}\int_{a}^{b}\int_{a}^{b}\left|f(s)-l(s)f^{(\alpha)}(s)\right|^{p}(ds)^{\alpha}(dt)^{\alpha}\right)^{\frac{1}{p}}\\
&\quad\times\left[\left(\frac{1}{\Gamma(1+\alpha)^{2}}\int_{a}^{x}\int_{t}^{x}\frac{t^{q\alpha}}{s^{2q\alpha}}(ds)^{\alpha}(dt)^{\alpha}\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\frac{1}{\Gamma(1+\alpha)^{2}}\int_{x}^{b}\int_{x}^{t}\frac{t^{q\alpha}}{s^{2q\alpha}}(ds)^{\alpha}(dt)^{\alpha}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

The first factor in (3.16) equals

$$\begin{aligned}
&\left(\frac{1}{\Gamma(1+\alpha)^{2}}\int_{a}^{b}\int_{a}^{b}\left|f(s)-l(s)f^{(\alpha)}(s)\right|^{p}(ds)^{\alpha}(dt)^{\alpha}\right)^{\frac{1}{p}} \qquad (3.17)\\
&= \frac{(b-a)^{\frac{\alpha}{p}}}{\Gamma(1+\alpha)}\left\|f-lf^{(\alpha)}\right\|_{p}
\end{aligned}$$

and for the second factor, applying the change of the variable

$$\frac{1}{s^{2q-1}}=u \text{ and from } \frac{1}{s^{2q\alpha}}(ds)^{\alpha}=\frac{(du)^{\alpha}}{(1-2q)^{\alpha}},$$

we write

$$\begin{aligned}
\frac{1}{\Gamma(1+\alpha)}\int_{t}^{x}\frac{1}{s^{2q\alpha}}(ds)^{\alpha} &= \frac{1}{(1-2q)^{\alpha}}\frac{1}{\Gamma(1+\alpha)}\int_{t^{1-2q}}^{x^{1-2q}}(du)^{\alpha} \qquad (3.18)\\
&= \frac{1}{\Gamma(1+\alpha)(1-2q)^{\alpha}}\left(x^{1-2q}-t^{1-2q}\right)^{\alpha}
\end{aligned}$$

and similarly,

$$\frac{1}{\Gamma(1+\alpha)}\int_x^t \frac{1}{s^{2q\alpha}}(ds)^\alpha = \frac{1}{\Gamma(1+\alpha)\,(1-2q)^\alpha}\left(t^{1-2q}-x^{1-2q}\right)^\alpha. \tag{3.19}$$

Using the equalities (3.18) and (3.19), we obtain

$$\begin{aligned}
&\left(\frac{1}{\Gamma(1+\alpha)^2}\int_a^x\int_t^x \frac{t^{q\alpha}}{s^{2q\alpha}}(ds)^\alpha(dt)^\alpha\right)^{\frac{1}{q}} + \left(\frac{1}{\Gamma(1+\alpha)^2}\int_x^b\int_x^t \frac{t^{q\alpha}}{s^{2q\alpha}}(ds)^\alpha(dt)^\alpha\right)^{\frac{1}{q}} \qquad (3.20)\\
&= \left(\frac{1}{(1-2q)^\alpha\,\Gamma(1+\alpha)^2}\left[x^{(1-2q)\alpha}\int_a^x t^{q\alpha}(dt)^\alpha - \int_a^x t^{(1-q)\alpha}(dt)^\alpha\right]\right)^{\frac{1}{q}}\\
&\quad + \left(\frac{1}{(1-2q)^\alpha\,\Gamma(1+\alpha)^2}\left[\int_x^b t^{(1-q)\alpha}(dt)^\alpha - x^{(1-2q)\alpha}\int_x^b t^{q\alpha}(dt)^\alpha\right]\right)^{\frac{1}{q}}\\
&= \frac{M_\alpha(x,q)}{\Gamma(1+\alpha)^{\frac{1}{q}}}.
\end{aligned}$$

Substituting (3.17) and (3.20) in (3.16), we deduce the required inequality (3.11) which completes the proof. □

Finally, we establish weighted case of the Theorem 9.

Theorem 10. *Let $f:[a,b]\subseteq\mathbb{R}\to\mathbb{R}^\alpha$ be a mapping such that $f\in C_\alpha[a,b]$ and $f\in D_\alpha(a,b)$, with $0<a<b$, and let $w:[a,b]\to\mathbb{R}^\alpha$ be a nonnegatiive function on $[a,b]$ and $w(t)\in C_\alpha[a,b]$. Then for $\frac{1}{p}+\frac{1}{q}=1$, with $p\geq 1$ and all $x\in[a,b]$, the following inequality holds:*

$$\begin{aligned}
&\left|{}_aI_b^\alpha f(t)w(t) - \frac{f(x)}{x^\alpha}\;{}_aI_b^\alpha t^\alpha w(t)\right|\\
\leq\;& \frac{\left\|f-lf^{(\alpha)}\right\|_p (b-a)^{\frac{\alpha}{p}}}{\Gamma(1+\alpha)^{\frac{1}{q}}\,(1-2q)^{\frac{\alpha}{q}}}\\
&\times\left[\left(x^{(1-2q)\alpha}\;{}_aI_x^\alpha t^{q\alpha}\,(w(t))^q - {}_aI_x^\alpha t^{(1-q)\alpha}\,(w(t))^q\right)^{\frac{1}{q}}\right.\\
&\left.\left({}_xI_b^\alpha t^{(1-q)\alpha}\,(w(t))^q - x^{(1-2q)\alpha}\;{}_xI_b^\alpha t^{q\alpha}\,(w(t))^q\right)^{\frac{1}{q}}\right]
\end{aligned}$$

where $l(t)=\frac{t^\alpha}{\Gamma(1+\alpha)}$, $t\in[a,b]$, and $\left\|f-lf^{(\alpha)}\right\|_p$ is defined by

$$\left\|f-lf^{(\alpha)}\right\|_p = \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|f(s)-l(s)f^{(\alpha)}(s)\right|^p (ds)^\alpha\right)^{\frac{1}{p}}.$$

Proof. Multiplying (3.14) by $\frac{w(t)}{x^{\alpha}}$ and integrating on t for local fractional integrals, we obtain

$$\begin{aligned}
&\frac{f(x)}{x^{\alpha}}\ {}_aI_b^{\alpha}t^{\alpha}w(t)-\ {}_aI_b^{\alpha}f(t)\\
&= \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)^2}\int_a^b t^{\alpha}w(t)\int_x^t \frac{f(s)-l(s)f^{(\alpha)}(s)}{s^{2\alpha}}(ds)^{\alpha}(dt)^{\alpha}
\end{aligned}$$

and therefore

$$\begin{aligned}
&\left|{}_aI_b^{\alpha}f(t)w(t)-\frac{f(x)}{x^{\alpha}}\ {}_aI_b^{\alpha}t^{\alpha}w(t)\right| \qquad (3.21)\\
&\le \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)^2}\int_a^x \left|t^{\alpha}w(t)\int_t^x \left|\frac{f(s)-l(s)f^{(\alpha)}(s)}{s^{2\alpha}}\right|(ds)^{\alpha}\right|(dt)^{\alpha}\\
&\quad +\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)^2}\int_x^b \left|t^{\alpha}w(t)\int_x^t \left|\frac{f(s)-l(s)f^{(\alpha)}(s)}{s^{2\alpha}}\right|(ds)^{\alpha}\right|(dt)^{\alpha}.
\end{aligned}$$

Applying generalized Hölder's inequality the inequality (3.21), we get

$$\begin{aligned}
&\left|{}_aI_b^{\alpha}f(t)w(t)-\frac{f(x)}{x^{\alpha}}\ {}_aI_b^{\alpha}t^{\alpha}w(t)\right|\\
&\le \Gamma(1+\alpha)\left(\frac{1}{\Gamma(1+\alpha)^2}\int_a^x\int_t^x \left|f(s)-l(s)f^{(\alpha)}(s)\right|^p(ds)^{\alpha}(dt)^{\alpha}\right)^{\frac{1}{p}}\\
&\quad\times\left(\frac{1}{\Gamma(1+\alpha)^2}\int_a^x\int_t^x \frac{t^{q\alpha}\left(w(t)\right)^q}{s^{2q\alpha}}(ds)^{\alpha}(dt)^{\alpha}\right)^{\frac{1}{q}}\\
&\quad+\Gamma(1+\alpha)\left(\frac{1}{\Gamma(1+\alpha)^2}\int_x^b\int_x^t \left|f(s)-l(s)f^{(\alpha)}(s)\right|^p(ds)^{\alpha}(dt)^{\alpha}\right)^{\frac{1}{p}}\\
&\quad\times\left(\frac{1}{\Gamma(1+\alpha)^2}\int_x^b\int_x^t \frac{t^{q\alpha}\left(w(t)\right)^q}{s^{2q\alpha}}(ds)^{\alpha}(dt)^{\alpha}\right)^{\frac{1}{q}}\\
&\le \Gamma(1+\alpha)\left(\frac{1}{\Gamma(1+\alpha)^2}\int_a^b\int_a^b \left|f(s)-l(s)f^{(\alpha)}(s)\right|^p(ds)^{\alpha}(dt)^{\alpha}\right)^{\frac{1}{p}}\\
&\quad\left[\left(\frac{1}{\Gamma(1+\alpha)^2}\int_a^x\int_t^x \frac{t^{q\alpha}\left(w(t)\right)^q}{s^{2q\alpha}}(ds)^{\alpha}(dt)^{\alpha}\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\frac{1}{\Gamma(1+\alpha)^2}\int_x^b\int_x^t \frac{t^{q\alpha}\left(w(t)\right)^q}{s^{2q\alpha}}(ds)^{\alpha}(dt)^{\alpha}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

If we use the equalities (3.18) and (3.19) we obtain required inequality. The proof is thus completed. ☐

4. Applications to Numerical Integration

We now consider applications of the integral inequalities involving local fractional integral developed in the previous section, to obtain estimates of composite quadrature rules which, it turns out have a markedly smaller error than that which may be obtained by the classical results.

Consider the division of the inteval $[a,b]$, $0<a<b$, given by

$$I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$$

and $\xi_i \in [x_i, x_{i+1}]$, $i = 0, ..., n-1$ a sequance of intermediate points. Define the quadrature

$$S(f, I_n, \xi) := \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \sum_{i=0}^{n-1} \frac{f(\xi_i)}{\xi_i^{\alpha}} \left(x_{i+1}^2 - x_i^2\right)^{\alpha} = \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \sum_{i=0}^{n-1} \frac{f(\xi_i)}{\xi_i^{\alpha}} (x_{i+1}+x_i)^{\alpha} h_i^{\alpha} \tag{4.1}$$

where $h_i = (x_{i+1} - x_i)$, $i = 0, ..., n-1$.

Theorem 11. *Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}^{\alpha}$ be a mapping such that $f \in C_{\alpha}[a,b]$ and $f \in D_{\alpha}(a,b)$. Then we have the representation*

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^{\alpha} = S(f, I_n, \xi) + R(f, I_n, \xi)$$

where $S(f, I_n, \xi)$ is as defined in (4.1) and the remainder satisfies the astimation:

$$|R(f, I_n, \xi)| \leq \frac{2^{\alpha}\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left\| f - lf^{(\alpha)} \right\|_{\infty} \sum_{i=0}^{n-1} \frac{h_i^{2\alpha}}{|\xi_i|^{\alpha}} \left[\frac{1}{4^{\alpha}} + \frac{\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^{2\alpha}}{h_i^{2\alpha}} \right]. \tag{4.2}$$

Proof. Applying Theorem 7 on the interval $[x_i, x_{i+1}]$ for the internadiate points ξ_i, we obtain

$$\left| \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \frac{f(\xi_i)}{\xi_i^{\alpha}} (x_{i+1}+x_i)^{\alpha} h_i^{\alpha} - {}_{x_i}I_{x_{i+1}}^{\alpha} f(t) \right| \leq \frac{2^{\alpha}\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \frac{h_i^{2\alpha}}{|\xi_i|^{\alpha}} \left[\frac{1}{4^{\alpha}} + \frac{\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^{2\alpha}}{h_i^{2\alpha}} \right] \left\| f - lf^{(\alpha)} \right\|_{\infty}$$

for all $i = 0, ..., n-1$. Summing over i from 0 to $n-1$ and using the triangle inequality we obtain the estimation (4.2). ☐

Now, define the mid-point rule as the following:

$$M(f, I_n) := \frac{2^{\alpha}\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i^{\alpha}$$

where $h_i = (x_{i+1} - x_i)$, $i = 0, ..., n-1$.

Corollary 2. *Under the same assumptions of Theorem 11 with* $\xi_i = \frac{x_i + x_{i+1}}{2}$. *Then, we have*

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha = M(f, I_n) + R(f, I_n),$$

where the remainder satisfies the astimation:

$$|R(f, I_n)| \leq \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left\| f - lf^{(\alpha)} \right\|_\infty \sum_{i=0}^{n-1} \frac{h_i^{2\alpha}}{(x_i + x_{i+1})^\alpha}.$$

5. Applications to Some Special Means

Let us recall some generalized means:

$$A(a,b) = \frac{a^\alpha + b^\alpha}{2^\alpha};$$

$$L_n(a,b) = \left[\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)} \left[\frac{b^{(n+1)\alpha} - a^{(n+1)\alpha}}{(b-a)^\alpha} \right] \right]^{\frac{1}{n}}, \ n \in Z \backslash \{-1, 0\}, \ a, b \in R, \ a \neq b.$$

Now, let us reconsider the inequality (3.10):

$$\left| \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} f(\frac{a+b}{2}) - \frac{1}{2^\alpha (b-a)^\alpha} \ {}_aI_b^\alpha f(t) \right|$$

$$\leq \frac{\Gamma(1+\alpha)(b-a)^\alpha}{2^\alpha \Gamma(1+2\alpha) |a+b|^\alpha} \left\| f - lf^{(\alpha)} \right\|_\infty.$$

Consider the mapping $f : (0, \infty) \to R^\alpha$, $f(t) = t^{n\alpha}$, $n \in Z \backslash \{-1, 0\}$. Then, $0 < a < b$, we have

$$f\left(\frac{a+b}{2}\right) = [A(a,b)]^n$$

and

$$\frac{1}{(b-a)^\alpha} \ {}_aI_b^\alpha f(t) = [L_n(a,b)]^n.$$

Using Lemma 4, we obtain

$$\left\| f - lf^{(\alpha)} \right\|_\infty = \begin{cases} \left| 1 - \frac{\Gamma(1+n\alpha)}{\Gamma(1+\alpha)\Gamma(1+(n-1)\alpha)} \right| b^{n\alpha}, & n > 1 \\ \left| 1 - \frac{\Gamma(1+n\alpha)}{\Gamma(1+\alpha)\Gamma(1+(n-1)\alpha)} \right| a^{n\alpha}, & n \in (-\infty, 1] \backslash \{-1, 0\} \end{cases}$$

and then we deduce that

$$\left| \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} [A(a,b)]^n - \frac{1}{2^\alpha} [L_n(a,b)]^n \right|$$

$$\leq \frac{\Gamma(1+\alpha)(b-a)^\alpha}{4^\alpha \Gamma(1+2\alpha) A(a,b)} \delta_n(a,b)$$

where

$$\delta_n(a,b) = \begin{cases} \left| 1 - \frac{\Gamma(1+n\alpha)}{\Gamma(1+\alpha)\Gamma(1+(n-1)\alpha)} \right| b^{n\alpha}, & n > 1 \\ \left| 1 - \frac{\Gamma(1+n\alpha)}{\Gamma(1+\alpha)\Gamma(1+(n-1)\alpha)} \right| a^{n\alpha}, & n \in (-\infty, 1] \backslash \{-1, 0\}. \end{cases}$$

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Department of Mathematics, Faculty of Science, Bartin University, BARTIN-TURKEY
E-mail address: erdensmt@gmail.com

Department of Mathematics, Faculty of Science and Arts, Düzce University, Konuralp Campus, Düzce-TURKEY
E-mail address: sarikayamz@gmail.com