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GENERALIZATION AND REVERSES OF THE LEFT FEJÉR INEQUALITY FOR CONVEX FUNCTIONS

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ABSTRACT. In this paper we establish a generalization of the left Fejér inequality for general Lebesgue integral on measurable spaces as well as various upper bounds for the difference

$$\frac{1}{\int_a^b g(x) dx} \int_a^b h(x) g(x) dx - h\left(\frac{a+b}{2}\right)$$

where $h : [a, b] \rightarrow \mathbb{R}$ is a convex function and $g : [a, b] \rightarrow [0, \infty)$ is an integrable weight. Applications for discrete means and Hermite-Hadamard type inequalities are also provided.

1. INTRODUCTION

The *Hermite-Hadamard* integral inequality for convex functions $f : [a, b] \rightarrow \mathbb{R}$

$$(HH) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

is well known in the literature and has many applications for Special Means, in Information Theory and in Probability Theory and Statistics.

For related results, see for instance the research papers [1], [9], [10], [11], [13], [12], [14], [15], [16], the monograph online [7] and the references therein.

In 1906, Fejér [8], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Theorem 1 (Fejér's Inequality). *Consider the integral $\int_a^b h(x) g(x) dx$, where h is a convex function in the interval (a, b) and g is a positive function in the same interval such that*

$$g(a+t) = g(b-t), \quad 0 \leq t \leq \frac{1}{2}(b-a),$$

i.e., $y = g(x)$ is a symmetric curve with respect to the straight line which contains the point $(\frac{1}{2}(a+b), 0)$ and is normal to the x -axis. Under those conditions the following inequalities are valid:

$$(1.1) \quad h\left(\frac{a+b}{2}\right) \int_a^b g(x) dx \leq \int_a^b h(x) g(x) dx \leq \frac{h(a) + h(b)}{2} \int_a^b g(x) dx.$$

If h is concave on (a, b) , then the inequalities reverse in (1.1).

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Clearly, for $g(x) \equiv 1$ on $[a, b]$ we get (HH).

Motivated by the above result, we establish in this paper a generalization of the left Fejér inequality (1.1) for general Lebesgue integral on measurable spaces as well as various *upper bounds* for the difference

$$\frac{1}{\int_a^b g(x) dx} \int_a^b h(x) g(x) dx - h\left(\frac{a+b}{2}\right),$$

where g and h is as above.

Applications for discrete means and Hermite-Hadamard type inequalities are also provided.

2. GENERAL RESULTS

Suppose that I is an interval of real numbers with interior $\overset{\circ}{I}$ and $\Phi : I \rightarrow \mathbb{R}$ is a convex function on I . Then Φ is continuous on $\overset{\circ}{I}$ and has finite left and right derivatives at each point of $\overset{\circ}{I}$. Moreover, if $x, y \in \overset{\circ}{I}$ and $x < y$, then $\Phi'_-(x) \leq \Phi'_+(x) \leq \Phi'_-(y) \leq \Phi'_+(y)$ which shows that both Φ'_- and Φ'_+ are nondecreasing function on $\overset{\circ}{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $\Phi : I \rightarrow \mathbb{R}$, the subdifferential of Φ denoted by $\partial\Phi$ is the set of all functions $\varphi : I \rightarrow [-\infty, \infty]$ such that $\varphi(\overset{\circ}{I}) \subset \mathbb{R}$ and

$$(2.1) \quad \Phi(x) \geq \Phi(a) + (x-a)\varphi(a) \text{ for any } x, a \in I.$$

It is also well known that if Φ is convex on I , then $\partial\Phi$ is nonempty, $\Phi'_-, \Phi'_+ \in \partial\Phi$ and if $\varphi \in \partial\Phi$, then

$$\Phi'_-(x) \leq \varphi(x) \leq \Phi'_+(x) \text{ for any } x \in \overset{\circ}{I}.$$

In particular, φ is a nondecreasing function.

If Φ is differentiable and convex on $\overset{\circ}{I}$, then $\partial\Phi = \{\Phi'\}$.

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω , a σ -algebra \mathcal{A} of parts of Ω and a countably additive and positive measure μ on \mathcal{A} with values in $\mathbb{R} \cup \{\infty\}$. For a μ -measurable function $w : \Omega \rightarrow \mathbb{R}$, with $w(x) \geq 0$ for μ -a.e.(almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega, \mu) := \{f : \Omega \rightarrow \mathbb{R}, f \text{ is } \mu\text{-measurable and } \int_{\Omega} |f(x)| w(x) d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_{\Omega} w d\mu$ instead of $\int_{\Omega} w(x) d\mu(x)$.

In what follows we assume that $w \geq 0$ μ -a.e. on Ω with $\int_{\Omega} w d\mu = 1$.

The following result holds:

Theorem 2. *Let $\Phi : [m, M] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on $[m, M]$, $\varphi \in \partial\Phi$ and $f : \Omega \rightarrow \mathbb{R}$ satisfying the condition*

$$(2.2) \quad -\infty < m \leq f \leq M < \infty$$

μ -a.e. on Ω and so that $\Phi \circ f, \varphi \circ f, (\varphi \circ f) f, f \in L_w(\Omega, \mu)$. Then we have the inequalities:

$$\begin{aligned}
(2.3) \quad 0 &\leq \int_{\Omega} (\Phi \circ f) w d\mu - \Phi\left(\frac{m+M}{2}\right) - \varphi\left(\frac{m+M}{2}\right) \int_{\Omega} \left(f - \frac{m+M}{2}\right) w d\mu \\
&\leq \int_{\Omega} \left(\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right) \left(f - \frac{m+M}{2}\right) w d\mu \\
&\leq \operatorname{essup}_{\Omega} \left|f - \frac{m+M}{2}\right| \int_{\Omega} \left|\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right| w d\mu \\
&\leq \begin{cases} \frac{1}{2}(M-m) \int_{\Omega} |\varphi \circ f - \varphi(\frac{m+M}{2})| w d\mu \\ \operatorname{essup}_{\Omega} |f - \frac{m+M}{2}| (\Phi'_-(M) - \Phi'_+(m)) \end{cases} \\
&\leq \frac{1}{2}(M-m) (\Phi'_-(M) - \Phi'_+(m)).
\end{aligned}$$

Proof. By the gradient inequality (2.1) we have

$$(2.4) \quad \Phi(t) - \Phi\left(\frac{m+M}{2}\right) \geq \left(t - \frac{m+M}{2}\right) \varphi\left(\frac{m+M}{2}\right)$$

for any $\varphi \in \partial\Phi$ and for any $t \in [m, M]$.

This inequality implies that

$$(2.5) \quad \Phi(f(x)) \geq \Phi\left(\frac{m+M}{2}\right) + \left(f(x) - \frac{m+M}{2}\right) \varphi\left(\frac{m+M}{2}\right)$$

for any $x \in \Omega$.

If we multiply (2.5) by $w \geq 0$ μ -a.e and integrate on Ω , we get the first inequality in (2.3).

By the gradient inequality (2.1) we also have

$$\varphi(t) \left(t - \frac{m+M}{2}\right) \geq \Phi(t) - \Phi\left(\frac{m+M}{2}\right)$$

for any $\varphi \in \partial\Phi$ and for any $t \in (m, M)$.

This inequality is equivalent to

$$\begin{aligned}
&\left(\varphi(t) - \varphi\left(\frac{m+M}{2}\right)\right) \left(t - \frac{m+M}{2}\right) \\
&\geq \Phi(t) - \Phi\left(\frac{m+M}{2}\right) - \varphi\left(\frac{m+M}{2}\right) \left(t - \frac{m+M}{2}\right)
\end{aligned}$$

for any $\varphi \in \partial\Phi$ and for any $t \in (m, M)$.

This inequality implies that

$$\begin{aligned}
(2.6) \quad &\left(\varphi(f(x)) - \varphi\left(\frac{m+M}{2}\right)\right) \left(f(x) - \frac{m+M}{2}\right) \\
&\geq \Phi(f(x)) - \Phi\left(\frac{m+M}{2}\right) - \varphi\left(\frac{m+M}{2}\right) \left(f(x) - \frac{m+M}{2}\right)
\end{aligned}$$

for μ -a.e $x \in \Omega$.

If we multiply (2.6) by $w \geq 0$ μ -a.e and integrate on Ω , we get

$$(2.7) \quad \int_{\Omega} \left(\varphi \circ f - \varphi \left(\frac{m+M}{2} \right) \right) \left(f - \frac{m+M}{2} \right) w d\mu \\ \geq \int_{\Omega} (\Phi \circ f) w d\mu - \Phi \left(\frac{m+M}{2} \right) - \varphi \left(\frac{m+M}{2} \right) \int_{\Omega} \left(f - \frac{m+M}{2} \right) w d\mu,$$

which proves the second inequality in (2.3).

Now, since φ is monotonic nondecreasing on $[m, M]$, then

$$\left(\varphi \circ f - \varphi \left(\frac{m+M}{2} \right) \right) \left(f - \frac{m+M}{2} \right) \geq 0$$

on Ω .

We then have

$$\int_{\Omega} \left(\varphi \circ f - \varphi \left(\frac{m+M}{2} \right) \right) \left(f - \frac{m+M}{2} \right) w d\mu \\ = \left| \int_{\Omega} \left(\varphi \circ f - \varphi \left(\frac{m+M}{2} \right) \right) \left(f - \frac{m+M}{2} \right) w d\mu \right| \\ \leq \int_{\Omega} \left| \left(\varphi \circ f - \varphi \left(\frac{m+M}{2} \right) \right) \left(f - \frac{m+M}{2} \right) \right| w d\mu \\ \leq \operatorname{esssup}_{\Omega} \left| f - \frac{m+M}{2} \right| \int_{\Omega} \left| \varphi \circ f - \varphi \left(\frac{m+M}{2} \right) \right| w d\mu$$

and the third inequality in (2.3) is proved.

Since for any $\varphi \in \partial\Phi$ we have $\Phi'_+(m) \leq \varphi(t) \leq \Phi'_-(M)$ for any $t \in (m, M)$, then $\Phi'_+(m) \leq \varphi \left(\frac{m+M}{2} \right) \leq \Phi'_-(M)$ and

$$(2.8) \quad \left| \varphi \circ f - \varphi \left(\frac{m+M}{2} \right) \right| \leq \Phi'_-(M) - \Phi'_+(m)$$

μ -a.e. on Ω .

If we multiply (2.8) by $w \geq 0$ μ -a.e and integrate on Ω , we get

$$\int_{\Omega} \left| \varphi \circ f - \varphi \left(\frac{m+M}{2} \right) \right| w d\mu \leq \Phi'_-(M) - \Phi'_+(m)$$

and the last part in (2.3) is proved. \square

We have the following result:

Corollary 1. *Let $\Phi : [m, M] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on $[m, M]$, $\varphi \in \partial\Phi$ and $f : \Omega \rightarrow \mathbb{R}$ satisfying the condition (2.2) μ -a.e. on Ω and so that $\Phi \circ f$, $\varphi \circ f$, $(\varphi \circ f) f$, $f \in L_w(\Omega, \mu)$. If*

$$(2.9) \quad \int_{\Omega} \left(f - \frac{m+M}{2} \right) w d\mu = 0,$$

then we have the inequalities:

$$\begin{aligned}
(2.10) \quad 0 &\leq \int_{\Omega} (\Phi \circ f) w d\mu - \Phi\left(\frac{m+M}{2}\right) \leq \int_{\Omega} (\varphi \circ f - \lambda) \left(f - \frac{m+M}{2}\right) w d\mu \\
&\leq \operatorname{esssup}_{\Omega} \left|f - \frac{m+M}{2}\right| \int_{\Omega} |\varphi \circ f - \lambda| w d\mu \\
&\leq \frac{1}{2} (M - m) \int_{\Omega} |\varphi \circ f - \lambda| w d\mu
\end{aligned}$$

for any $\lambda \in \mathbb{R}$.

It follows by Theorem 2 on observing that, if (2.9) holds true, then

$$\begin{aligned}
&\int_{\Omega} \left(\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right) \left(f - \frac{m+M}{2}\right) w d\mu \\
&= \int_{\Omega} (\varphi \circ f) \left(f - \frac{m+M}{2}\right) w d\mu = \int_{\Omega} (\varphi \circ f - \lambda) \left(f - \frac{m+M}{2}\right) w d\mu
\end{aligned}$$

for any $\lambda \in \mathbb{R}$.

Remark 1. From the inequality (2.10) we have

$$\begin{aligned}
(2.11) \quad 0 &\leq \int_{\Omega} (\Phi \circ f) w d\mu - \Phi\left(\frac{m+M}{2}\right) \leq \int_{\Omega} (\varphi \circ f) \left(f - \frac{m+M}{2}\right) w d\mu \\
&\leq \operatorname{esssup}_{\Omega} \left|f - \frac{m+M}{2}\right| \int_{\Omega} |\varphi \circ f| w d\mu \leq \frac{1}{2} (M - m) \int_{\Omega} |\varphi \circ f| w d\mu.
\end{aligned}$$

Since for any $\varphi \in \partial\Phi$ we have $\Phi'_+(m) \leq \varphi(t) \leq \Phi'_-(M)$ for any $t \in (m, M)$, then from (2.10) we have

$$\begin{aligned}
(2.12) \quad 0 &\leq \int_{\Omega} (\Phi \circ f) w d\mu - \Phi\left(\frac{m+M}{2}\right) \\
&\leq \int_{\Omega} \left(\varphi \circ f - \frac{\Phi'_-(M) + \Phi'_+(m)}{2}\right) \left(f - \frac{m+M}{2}\right) w d\mu \\
&\leq \operatorname{esssup}_{\Omega} \left|f - \frac{m+M}{2}\right| \int_{\Omega} \left|\varphi \circ f - \frac{\Phi'_-(M) + \Phi'_+(m)}{2}\right| w d\mu \\
&\leq \frac{1}{2} \left\{ \begin{array}{l} (M - m) \int_{\Omega} \left|\varphi \circ f - \frac{\Phi'_-(M) + \Phi'_+(m)}{2}\right| w d\mu \\ \operatorname{esssup}_{\Omega} \left|f - \frac{m+M}{2}\right| (\Phi'_-(M) - \Phi'_+(m)) \end{array} \right. \\
&\leq \frac{1}{4} (M - m) (\Phi'_-(M) - \Phi'_+(m)).
\end{aligned}$$

We have:

Theorem 3. Let $\Phi : [m, M] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on $[m, M]$, $\varphi \in \partial\Phi$ and $f : \Omega \rightarrow \mathbb{R}$ satisfying the condition (2.2) μ -a.e. on Ω and so that $\Phi \circ f, (\varphi \circ f)^q$,

$f^p \in L_w(\Omega, \mu)$, for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then we have the inequalities:

$$\begin{aligned}
(2.13) \quad 0 &\leq \int_{\Omega} (\Phi \circ f) w d\mu - \Phi\left(\frac{m+M}{2}\right) - \varphi\left(\frac{m+M}{2}\right) \int_{\Omega} \left(f - \frac{m+M}{2}\right) w d\mu \\
&\leq \int_{\Omega} \left(\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right) \left(f - \frac{m+M}{2}\right) w d\mu \\
&\leq \left(\int_{\Omega} \left|f - \frac{m+M}{2}\right|^p w d\mu\right)^{1/p} \left(\int_{\Omega} \left|\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right|^q w d\mu\right)^{1/q} \\
&\leq \begin{cases} \frac{1}{2}(M-m) \left(\int_{\Omega} \left|\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right|^q w d\mu\right)^{1/q} \\ (\Phi'_-(M) - \Phi'_+(m)) \left(\int_{\Omega} \left|f - \frac{m+M}{2}\right|^p w d\mu\right)^{1/p} \end{cases} \\
&\leq \frac{1}{2}(M-m) (\Phi'_-(M) - \Phi'_+(m)).
\end{aligned}$$

Proof. By Hölder's integral inequality we have for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\begin{aligned}
&\int_{\Omega} \left(\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right) \left(f - \frac{m+M}{2}\right) w d\mu \\
&\leq \int_{\Omega} \left|\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right| \left|f - \frac{m+M}{2}\right| w d\mu \\
&\leq \left(\int_{\Omega} \left|f - \frac{m+M}{2}\right|^p w d\mu\right)^{1/p} \left(\int_{\Omega} \left|\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right|^q w d\mu\right)^{1/q},
\end{aligned}$$

which proves the third inequality in (2.13).

Also, we have

$$\begin{aligned}
\left(\int_{\Omega} \left|f - \frac{m+M}{2}\right|^p w d\mu\right)^{1/p} &\leq \left(\left|\frac{M-m}{2}\right|^p \int_{\Omega} w d\mu\right)^{1/p} \\
&= \frac{1}{2}(M-m)
\end{aligned}$$

and

$$\left(\int_{\Omega} \left|\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right|^q w d\mu\right)^{1/q} \leq \left((\Phi'_-(M) - \Phi'_+(m))^q \int_{\Omega} w d\mu\right)^{1/q}$$

that proves the last part of (2.13). \square

Remark 2. *In a similar way, the inequality (2.13) can be extended for $p = 1$ and $q = \infty$ to obtain the result:*

$$\begin{aligned}
(2.14) \quad 0 &\leq \int_{\Omega} (\Phi \circ f) w d\mu - \Phi\left(\frac{m+M}{2}\right) - \varphi\left(\frac{m+M}{2}\right) \int_{\Omega} \left(f - \frac{m+M}{2}\right) w d\mu \\
&\leq \int_{\Omega} \left(\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right) \left(f - \frac{m+M}{2}\right) w d\mu \\
&\leq \operatorname{esssup}_{\Omega} \left| \varphi \circ f - \varphi\left(\frac{m+M}{2}\right) \right| \int_{\Omega} \left| f - \frac{m+M}{2} \right| w d\mu \\
&\leq \begin{cases} \frac{1}{2} (M - m) \operatorname{esssup}_{\Omega} \left| \varphi \circ f - \varphi\left(\frac{m+M}{2}\right) \right| \\ (\Phi'_-(M) - \Phi'_+(m)) \int_{\Omega} \left| f - \frac{m+M}{2} \right| w d\mu \end{cases} \\
&\leq \frac{1}{2} (M - m) (\Phi'_-(M) - \Phi'_+(m)),
\end{aligned}$$

provided that $\varphi \circ f$ is essentially bounded on Ω .

Corollary 2. *Under the assumptions of Theorem 3 and if the condition (2.9) holds, then we have the simpler inequalities*

$$\begin{aligned}
(2.15) \quad 0 &\leq \int_{\Omega} (\Phi \circ f) w d\mu - \Phi\left(\frac{m+M}{2}\right) \leq \int_{\Omega} (\varphi \circ f - \lambda) \left(f - \frac{m+M}{2}\right) w d\mu \\
&\leq \left(\int_{\Omega} \left| f - \frac{m+M}{2} \right|^p w d\mu \right)^{1/p} \left(\int_{\Omega} |\varphi \circ f - \lambda|^q w d\mu \right)^{1/q} \\
&\leq \frac{1}{2} (M - m) \left(\int_{\Omega} |\varphi \circ f - \lambda|^q w d\mu \right)^{1/q}
\end{aligned}$$

for any $\lambda \in \mathbb{R}$.

In particular we have

$$\begin{aligned}
(2.16) \quad 0 &\leq \int_{\Omega} (\Phi \circ f) w d\mu - \Phi\left(\frac{m+M}{2}\right) \leq \int_{\Omega} (\varphi \circ f) \left(f - \frac{m+M}{2}\right) w d\mu \\
&\leq \left(\int_{\Omega} \left| f - \frac{m+M}{2} \right|^p w d\mu \right)^{1/p} \left(\int_{\Omega} |\varphi \circ f|^q w d\mu \right)^{1/q} \\
&\leq \frac{1}{2} (M - m) \left(\int_{\Omega} |\varphi \circ f|^q w d\mu \right)^{1/q}
\end{aligned}$$

and

$$\begin{aligned}
(2.17) \quad 0 &\leq \int_{\Omega} (\Phi \circ f) w d\mu - \Phi \left(\frac{m+M}{2} \right) \\
&\leq \int_{\Omega} \left(\varphi \circ f - \frac{\Phi'_-(M) + \Phi'_+(m)}{2} \right) \left(f - \frac{m+M}{2} \right) w d\mu \\
&\leq \left(\int_{\Omega} \left| f - \frac{m+M}{2} \right|^p w d\mu \right)^{1/p} \\
&\quad \times \left(\int_{\Omega} \left| \varphi \circ f - \frac{\Phi'_-(M) + \Phi'_+(m)}{2} \right|^q w d\mu \right)^{1/q} \\
&\leq \frac{1}{2} \left\{ \begin{array}{l} (M-m) \left(\int_{\Omega} \left| \varphi \circ f - \frac{\Phi'_-(M) + \Phi'_+(m)}{2} \right|^q w d\mu \right)^{1/q} \\ \left(\int_{\Omega} \left| f - \frac{m+M}{2} \right|^p w d\mu \right)^{1/p} (\Phi'_-(M) - \Phi'_+(m)) \end{array} \right. \\
&\leq \frac{1}{4} (M-m) (\Phi'_-(M) - \Phi'_+(m)).
\end{aligned}$$

We observe that if, under the assumptions of Remark 2, we choose f and w such that the condition (2.9) holds, then we have

$$\begin{aligned}
(2.18) \quad 0 &\leq \int_{\Omega} (\Phi \circ f) w d\mu - \Phi \left(\frac{m+M}{2} \right) \leq \int_{\Omega} (\varphi \circ f - \lambda) \left(f - \frac{m+M}{2} \right) w d\mu \\
&\leq \operatorname{esssup}_{\Omega} |\varphi \circ f - \lambda| \int_{\Omega} \left| f - \frac{m+M}{2} \right| w d\mu \\
&\leq \frac{1}{2} (M-m) \operatorname{esssup}_{\Omega} |\varphi \circ f - \lambda|
\end{aligned}$$

for any $\lambda \in \mathbb{R}$.

In particular, we have

$$\begin{aligned}
(2.19) \quad 0 &\leq \int_{\Omega} (\Phi \circ f) w d\mu - \Phi \left(\frac{m+M}{2} \right) \leq \int_{\Omega} (\varphi \circ f) \left(f - \frac{m+M}{2} \right) w d\mu \\
&\leq \operatorname{esssup}_{\Omega} |\varphi \circ f| \int_{\Omega} \left| f - \frac{m+M}{2} \right| w d\mu \leq \frac{1}{2} (M-m) \operatorname{esssup}_{\Omega} |\varphi \circ f|
\end{aligned}$$

and

$$\begin{aligned}
(2.20) \quad 0 &\leq \int_{\Omega} (\Phi \circ f) w d\mu - \Phi \left(\frac{m+M}{2} \right) \\
&\leq \int_{\Omega} \left(\varphi \circ f - \frac{\Phi'_-(M) + \Phi'_+(m)}{2} \right) \left(f - \frac{m+M}{2} \right) w d\mu \\
&\leq \operatorname{esssup}_{\Omega} \left| \varphi \circ f - \frac{\Phi'_-(M) + \Phi'_+(m)}{2} \right| \int_{\Omega} \left| f - \frac{m+M}{2} \right| w d\mu \\
&\leq \frac{1}{2} \left\{ \begin{array}{l} (\Phi'_-(M) - \Phi'_+(m)) \int_{\Omega} \left| f - \frac{m+M}{2} \right| w d\mu \\ (M-m) \operatorname{esssup}_{\Omega} \left| \varphi \circ f - \frac{\Phi'_-(M) + \Phi'_+(m)}{2} \right| \end{array} \right. \\
&\leq \frac{1}{4} (M-m) (\Phi'_-(M) - \Phi'_+(m)).
\end{aligned}$$

We observe that, if Φ is differentiable on (m, M) then we can replace in all inequalities above φ by Φ' . We omit the details.

3. APPLICATIONS FOR DISCRETE INEQUALITIES

Let $\mathbf{x} = (x_1, \dots, x_n)$ be an n -tuple with $x_i \in \mathbb{R}$, $i \in \{1, \dots, n\}$ and $\mathbf{p} = (p_1, \dots, p_n)$ a probability distribution, i.e. $p_i \geq 0$, $i \in \{1, \dots, n\}$ with $\sum_{i=1}^n p_i = 1$.

If $\Phi : [m, M] \subset \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on $[m, M]$ and $\varphi \in \partial\Phi$, then for any $\mathbf{x} = (x_1, \dots, x_n)$ with $x_i \in (m, M) \subset \mathbb{R}$, $i \in \{1, \dots, n\}$ and any probability distribution $\mathbf{p} = (p_1, \dots, p_n)$ we have from (2.3), (2.13) and (2.14) that

$$\begin{aligned}
(3.1) \quad 0 &\leq \sum_{i=1}^n p_i \Phi(x_i) - \Phi\left(\frac{m+M}{2}\right) - \varphi\left(\frac{m+M}{2}\right) \sum_{i=1}^n p_i \left(x_i - \frac{m+M}{2}\right) \\
&\leq \sum_{i=1}^n p_i \left(\varphi(x_i) - \varphi\left(\frac{m+M}{2}\right)\right) \left(x_i - \frac{m+M}{2}\right) \\
&\leq \max_{i \in \{1, \dots, n\}} \left| x_i - \frac{m+M}{2} \right| \sum_{i=1}^n p_i \left| \varphi(x_i) - \varphi\left(\frac{m+M}{2}\right) \right| \\
&\leq \begin{cases} \frac{1}{2} (M-m) \sum_{i=1}^n p_i \left| \varphi(x_i) - \varphi\left(\frac{m+M}{2}\right) \right| \\ \max_{i \in \{1, \dots, n\}} \left| x_i - \frac{m+M}{2} \right| (\Phi'_-(M) - \Phi'_+(m)) \end{cases} \\
&\leq \frac{1}{2} (M-m) (\Phi'_-(M) - \Phi'_+(m)),
\end{aligned}$$

$$\begin{aligned}
(3.2) \quad 0 &\leq \sum_{i=1}^n p_i \Phi(x_i) - \Phi\left(\frac{m+M}{2}\right) - \varphi\left(\frac{m+M}{2}\right) \sum_{i=1}^n p_i \left(x_i - \frac{m+M}{2}\right) \\
&\leq \sum_{i=1}^n p_i \left(\varphi(x_i) - \varphi\left(\frac{m+M}{2}\right)\right) \left(x_i - \frac{m+M}{2}\right) \\
&\leq \left(\sum_{i=1}^n p_i \left| x_i - \frac{m+M}{2} \right|^p \right)^{1/p} \left(\sum_{i=1}^n p_i \left| \varphi(x_i) - \varphi\left(\frac{m+M}{2}\right) \right|^q \right)^{1/q} \\
&\leq \frac{1}{2} \times \begin{cases} (M-m) \left(\sum_{i=1}^n p_i \left| \varphi(x_i) - \varphi\left(\frac{m+M}{2}\right) \right|^q \right)^{1/q} \\ (\Phi'_-(M) - \Phi'_+(m)) \left(\sum_{i=1}^n p_i \left| x_i - \frac{m+M}{2} \right|^p \right)^{1/p} \end{cases} \\
&\leq \frac{1}{2} (M-m) (\Phi'_-(M) - \Phi'_+(m)).
\end{aligned}$$

and

$$\begin{aligned}
(3.3) \quad 0 &\leq \sum_{i=1}^n p_i \Phi(x_i) - \Phi\left(\frac{m+M}{2}\right) - \varphi\left(\frac{m+M}{2}\right) \sum_{i=1}^n p_i \left(x_i - \frac{m+M}{2}\right) \\
&\leq \sum_{i=1}^n p_i \left(\varphi(x_i) - \varphi\left(\frac{m+M}{2}\right)\right) \left(x_i - \frac{m+M}{2}\right) \\
&\leq \max_{i \in \{1, \dots, n\}} \left| \varphi(x_i) - \varphi\left(\frac{m+M}{2}\right) \right| \sum_{i=1}^n p_i \left| x_i - \frac{m+M}{2} \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2} \times \begin{cases} (M-m) \max_{i \in \{1, \dots, n\}} |\varphi(x_i) - \varphi(\frac{m+M}{2})| \\ (\Phi'_-(M) - \Phi'_+(m)) \sum_{i=1}^n p_i |x_i - \frac{m+M}{2}| \end{cases} \\
&\leq \frac{1}{2} (M-m) (\Phi'_-(M) - \Phi'_+(m)).
\end{aligned}$$

If $\mathbf{x} = (x_1, \dots, x_n)$ with $x_i \in (m, M) \subset \mathbb{R}$, $i \in \{1, \dots, n\}$ and the probability distribution $\mathbf{p} = (p_1, \dots, p_n)$ are such that

$$(3.4) \quad \sum_{i=1}^n p_i x_i = \frac{m+M}{2},$$

then from (2.10), (2.15) and (2.18) we have

$$\begin{aligned}
(3.5) \quad 0 &\leq \sum_{i=1}^n p_i \Phi(x_i) - \Phi\left(\frac{m+M}{2}\right) \leq \sum_{i=1}^n p_i (\varphi(x_i) - \lambda) \left(x_i - \frac{m+M}{2}\right) \\
&\leq \max_{i \in \{1, \dots, n\}} \left| x_i - \frac{m+M}{2} \right| \sum_{i=1}^n p_i |\varphi(x_i) - \lambda| \\
&\leq \frac{1}{2} (M-m) \sum_{i=1}^n p_i |\varphi(x_i) - \lambda|
\end{aligned}$$

$$\begin{aligned}
(3.6) \quad 0 &\leq \sum_{i=1}^n p_i \Phi(x_i) - \Phi\left(\frac{m+M}{2}\right) \leq \sum_{i=1}^n p_i (\varphi(x_i) - \lambda) \left(x_i - \frac{m+M}{2}\right) \\
&\leq \left(\sum_{i=1}^n p_i \left| x_i - \frac{m+M}{2} \right|^p \right)^{1/p} \left(\sum_{i=1}^n p_i |\varphi(x_i) - \lambda|^q \right)^{1/q} \\
&\leq \frac{1}{2} (M-m) \left(\sum_{i=1}^n p_i |\varphi(x_i) - \lambda|^q \right)^{1/q}
\end{aligned}$$

and

$$\begin{aligned}
(3.7) \quad 0 &\leq \sum_{i=1}^n p_i \Phi(x_i) - \Phi\left(\frac{m+M}{2}\right) \leq \sum_{i=1}^n p_i (\varphi(x_i) - \lambda) \left(x_i - \frac{m+M}{2}\right) \\
&\leq \max_{i \in \{1, \dots, n\}} |\varphi(x_i) - \lambda| \sum_{i=1}^n p_i \left| x_i - \frac{m+M}{2} \right| \\
&\leq \frac{1}{2} (M-m) \max_{i \in \{1, \dots, n\}} |\varphi(x_i) - \lambda|.
\end{aligned}$$

for any $\lambda \in \mathbb{R}$.

If we take in (3.5)-(3.7) $\lambda = \frac{\Phi'_-(M) + \Phi'_+(m)}{2}$, then we get

$$\begin{aligned}
(3.8) \quad 0 &\leq \sum_{i=1}^n p_i \Phi(x_i) - \Phi\left(\frac{m+M}{2}\right) \\
&\leq \sum_{i=1}^n p_i \left(\varphi(x_i) - \frac{\Phi'_-(M) + \Phi'_+(m)}{2} \right) \left(x_i - \frac{m+M}{2} \right) \\
&\leq \max_{i \in \{1, \dots, n\}} \left| x_i - \frac{m+M}{2} \right| \left| \sum_{i=1}^n p_i \left(\varphi(x_i) - \frac{\Phi'_-(M) + \Phi'_+(m)}{2} \right) \right| \\
&\leq \frac{1}{2} \begin{cases} (M-m) \sum_{i=1}^n p_i \left| \varphi(x_i) - \frac{\Phi'_-(M) + \Phi'_+(m)}{2} \right| \\ (\Phi'_-(M) - \Phi'_+(m)) \max_{i \in \{1, \dots, n\}} \left| x_i - \frac{m+M}{2} \right| \end{cases} \\
&\leq \frac{1}{4} (M-m) (\Phi'_-(M) - \Phi'_+(m))
\end{aligned}$$

$$\begin{aligned}
(3.9) \quad 0 &\leq \sum_{i=1}^n p_i \Phi(x_i) - \Phi\left(\frac{m+M}{2}\right) \\
&\leq \sum_{i=1}^n p_i \left(\varphi(x_i) - \frac{\Phi'_-(M) + \Phi'_+(m)}{2} \right) \left(x_i - \frac{m+M}{2} \right) \\
&\leq \left(\sum_{i=1}^n p_i \left| x_i - \frac{m+M}{2} \right|^p \right)^{1/p} \left(\sum_{i=1}^n p_i \left| \varphi(x_i) - \frac{\Phi'_-(M) + \Phi'_+(m)}{2} \right|^q \right)^{1/q} \\
&\leq \frac{1}{2} \begin{cases} (M-m) \left(\sum_{i=1}^n p_i \left| \varphi(x_i) - \frac{\Phi'_-(M) + \Phi'_+(m)}{2} \right|^q \right)^{1/q} \\ \left(\sum_{i=1}^n p_i \left| x_i - \frac{m+M}{2} \right|^p \right)^{1/p} (\Phi'_-(M) - \Phi'_+(m)) \end{cases} \\
&\leq \frac{1}{4} (M-m) (\Phi'_-(M) - \Phi'_+(m))
\end{aligned}$$

and

$$\begin{aligned}
(3.10) \quad 0 &\leq \sum_{i=1}^n p_i \Phi(x_i) - \Phi\left(\frac{m+M}{2}\right) \\
&\leq \sum_{i=1}^n p_i \left(\varphi(x_i) - \frac{\Phi'_-(M) + \Phi'_+(m)}{2} \right) \left(x_i - \frac{m+M}{2} \right) \\
&\leq \max_{i \in \{1, \dots, n\}} \left| \varphi(x_i) - \frac{\Phi'_-(M) + \Phi'_+(m)}{2} \right| \left| \sum_{i=1}^n p_i \left| x_i - \frac{m+M}{2} \right| \right| \\
&\leq \frac{1}{2} \begin{cases} (M-m) \max_{i \in \{1, \dots, n\}} \left| \varphi(x_i) - \frac{\Phi'_-(M) + \Phi'_+(m)}{2} \right| \\ (\Phi'_-(M) - \Phi'_+(m)) \sum_{i=1}^n p_i \left| x_i - \frac{m+M}{2} \right| \end{cases} \\
&\leq \frac{1}{4} (M-m) (\Phi'_-(M) - \Phi'_+(m)).
\end{aligned}$$

for $\mathbf{x} = (x_1, \dots, x_n)$ with $x_i \in (m, M) \subset \mathbb{R}$, $i \in \{1, \dots, n\}$ and the probability distribution $\mathbf{p} = (p_1, \dots, p_n)$ with the property that (3.4) holds true.

Consider the function $\Phi : [m, M] \subset (0, \infty) \rightarrow (0, \infty)$ defined by $\Phi(x) = x^p$ with $p \in (-\infty, 0) \cup [1, \infty)$. This function is convex and for n -tuple any $\mathbf{x} = (x_1, \dots, x_n)$ with $x_i \in (m, M)$ and any probability distribution $\mathbf{p} = (p_1, \dots, p_n)$ we have by (3.1) that

$$\begin{aligned}
(3.11) \quad 0 &\leq \sum_{i=1}^n p_i x_i^p - p \left(\frac{m+M}{2} \right)^{p-1} \sum_{i=1}^n p_i x_i + (p-1) \left(\frac{m+M}{2} \right)^p \\
&\leq p \sum_{i=1}^n p_i \left(x_i^{p-1} - \left(\frac{m+M}{2} \right)^{p-1} \right) \left(x_i - \frac{m+M}{2} \right) \\
&\leq |p| \max_{i \in \{1, \dots, n\}} \left| x_i - \frac{m+M}{2} \right| \left| \sum_{i=1}^n p_i \left(x_i^{p-1} - \left(\frac{m+M}{2} \right)^{p-1} \right) \right| \\
&\leq \begin{cases} \frac{1}{2} |p| (M-m) \sum_{i=1}^n p_i \left| x_i^{p-1} - \left(\frac{m+M}{2} \right)^{p-1} \right| \\ p \max_{i \in \{1, \dots, n\}} \left| x_i - \frac{m+M}{2} \right| (M^{p-1} - m^{p-1}) \end{cases} \\
&\leq \frac{1}{2} p (M-m) (M^{p-1} - m^{p-1}).
\end{aligned}$$

Similar inequalities may be stated by utilizing the other two inequalities (3.2) and (3.3).

Consider the function $\Phi : [m, M] \subset (0, \infty) \rightarrow (0, \infty)$ defined by $\Phi(x) = -\ln x$. This function is convex and for n -tuple any $\mathbf{x} = (x_1, \dots, x_n)$ with $x_i \in (m, M)$ and any probability distribution $\mathbf{p} = (p_1, \dots, p_n)$ we have by (3.1) that

$$\begin{aligned}
(3.12) \quad 0 &\leq \ln \left(\frac{m+M}{2} \right) + \frac{2}{m+M} \sum_{i=1}^n p_i \left(x_i - \frac{m+M}{2} \right) - \sum_{i=1}^n p_i \ln x_i \\
&\leq \frac{2}{m+M} \sum_{i=1}^n p_i \frac{(x_i - \frac{m+M}{2})^2}{x_i} \\
&\leq \frac{2}{m+M} \max_{i \in \{1, \dots, n\}} \left| x_i - \frac{m+M}{2} \right| \left| \sum_{i=1}^n p_i \left| x_i - \frac{m+M}{2} \right| \right| \\
&\leq \begin{cases} \frac{M-m}{m+M} \sum_{i=1}^n p_i \left| x_i - \frac{m+M}{2} \right| \\ \frac{M-m}{mM} \max_{i \in \{1, \dots, n\}} \left| x_i - \frac{m+M}{2} \right| \end{cases} \leq \frac{1}{2mM} (M-m)^2,
\end{aligned}$$

or, equivalently

$$\begin{aligned}
(3.13) \quad 1 &\leq \frac{\frac{m+M}{2} \exp \left[\frac{2}{m+M} \sum_{i=1}^n p_i \left(x_i - \frac{m+M}{2} \right) \right]}{\prod_{i=1}^n x_i^{p_i}} \\
&\leq \exp \left(\frac{2}{m+M} \sum_{i=1}^n \frac{p_i}{x_i} \left(x_i - \frac{m+M}{2} \right)^2 \right)
\end{aligned}$$

$$\begin{aligned} &\leq \exp \left[\frac{2}{m+M} \max_{i \in \{1, \dots, n\}} \left| x_i - \frac{m+M}{2} \right| \sum_{i=1}^n p_i \left| x_i - \frac{m+M}{2} \right| \right] \\ &\leq \begin{cases} \exp \left[\frac{M-m}{m+M} \sum_{i=1}^n p_i \left| x_i - \frac{m+M}{2} \right| \right] \\ \exp \left[\frac{M-m}{mM} \max_{i \in \{1, \dots, n\}} \left| x_i - \frac{m+M}{2} \right| \right] \end{cases} \leq \exp \left[\frac{1}{2mM} (M-m)^2 \right]. \end{aligned}$$

4. APPLICATIONS FOR FEJÉR'S INEQUALITY

Let $h : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function, $\varphi \in \partial h$ and $g : [a, b] \rightarrow [0, \infty)$ a Lebesgue integrable function on $[a, b]$. Then from Theorems 2, 3 and Remark 2 we have the inequalities

$$\begin{aligned} (4.1) \quad 0 &\leq \frac{1}{\int_a^b g(x) dx} \int_a^b h(x) g(x) dx - h\left(\frac{a+b}{2}\right) \\ &\quad - \varphi\left(\frac{a+b}{2}\right) \frac{1}{\int_a^b g(x) dx} \int_a^b \left(x - \frac{a+b}{2}\right) g(x) dx \\ &\leq \frac{1}{\int_a^b g(x) dx} \int_a^b \left(\varphi(x) - \varphi\left(\frac{a+b}{2}\right)\right) \left(x - \frac{a+b}{2}\right) g(x) dx \\ &\leq \frac{1}{2} (b-a) \frac{1}{\int_a^b g(x) dx} \int_a^b \left| \varphi(x) - \varphi\left(\frac{a+b}{2}\right) \right| g(x) dx \\ &\leq \frac{1}{2} (b-a) (h'_-(b) - h'_+(a)), \end{aligned}$$

$$\begin{aligned} (4.2) \quad 0 &\leq \frac{1}{\int_a^b g(x) dx} \int_a^b h(x) g(x) dx - h\left(\frac{a+b}{2}\right) \\ &\quad - \varphi\left(\frac{a+b}{2}\right) \frac{1}{\int_a^b g(x) dx} \int_a^b \left(x - \frac{a+b}{2}\right) g(x) dx \\ &\leq \left(\frac{1}{\int_a^b g(x) dx} \int_a^b \left| x - \frac{a+b}{2} \right|^p g(x) dx \right)^{1/p} \\ &\quad \times \left(\frac{1}{\int_a^b g(x) dx} \int_a^b \left| \varphi(x) - \frac{a+b}{2} \right|^q g(x) dx \right)^{1/q} \\ &\leq \begin{cases} \frac{1}{2} (b-a) \left(\frac{1}{\int_a^b g(x) dx} \int_a^b \left| \varphi(x) - \frac{a+b}{2} \right|^q g(x) dx \right)^{1/q} \\ (h'_-(b) - h'_+(a)) \left(\frac{1}{\int_a^b g(x) dx} \int_a^b \left| x - \frac{a+b}{2} \right|^p g(x) dx \right)^{1/p} \end{cases} \\ &\leq \frac{1}{2} (b-a) (h'_-(b) - h'_+(a)) \end{aligned}$$

and

$$\begin{aligned}
(4.3) \quad 0 &\leq \frac{1}{\int_a^b g(x) dx} \int_a^b h(x) g(x) dx - h\left(\frac{a+b}{2}\right) \\
&\quad - \varphi\left(\frac{a+b}{2}\right) \frac{1}{\int_a^b g(x) dx} \int_a^b \left(x - \frac{a+b}{2}\right) g(x) dx \\
&\leq \frac{1}{\int_a^b g(x) dx} \int_a^b \left(\varphi(x) - \varphi\left(\frac{a+b}{2}\right)\right) \left(x - \frac{a+b}{2}\right) g(x) dx \\
&\leq \operatorname{esssup}_{x \in [a,b]} \left| \varphi(x) - \varphi\left(\frac{a+b}{2}\right) \right| \frac{1}{\int_a^b g(x) dx} \int_a^b \left|x - \frac{a+b}{2}\right| g(x) dx \\
&\leq \begin{cases} \frac{1}{2}(b-a) \operatorname{esssup}_{x \in [a,b]} \left| \varphi(x) - \varphi\left(\frac{a+b}{2}\right) \right| \\ (h'_-(b) - h'_+(a)) \frac{1}{\int_a^b g(x) dx} \int_a^b \left|x - \frac{a+b}{2}\right| g(x) dx \end{cases} \\
&\leq \frac{1}{2}(b-a)(h'_-(b) - h'_+(a)).
\end{aligned}$$

Now, if we assume that $g : [a, b] \rightarrow \mathbb{R}$ is integrable and symmetric on the interval $[a, b]$, then

$$(4.4) \quad \int_a^b \left(x - \frac{a+b}{2}\right) g(x) dx = 0.$$

The converse is obviously not true.

Now, if $g : [a, b] \rightarrow [0, \infty)$ is integrable and satisfies the condition (4.4), then from (2.12), (2.17) and 2.20 we have for any convex function $h : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ and $\varphi \in \partial h$ that

$$\begin{aligned}
(4.5) \quad 0 &\leq \frac{1}{\int_a^b g(x) dx} \int_a^b h(x) g(x) dx - h\left(\frac{a+b}{2}\right) \\
&\leq \frac{1}{\int_a^b g(x) dx} \int_a^b \left(\varphi(x) - \frac{h'_-(b) + h'_+(a)}{2}\right) \left(x - \frac{a+b}{2}\right) g(x) dx \\
&\leq \frac{1}{2}(b-a) \frac{1}{\int_a^b g(x) dx} \int_a^b \left| \varphi(x) - \frac{h'_-(b) + h'_+(a)}{2} \right| g(x) dx \\
&\leq \frac{1}{4}(b-a)(h'_-(b) - h'_+(a)),
\end{aligned}$$

$$\begin{aligned}
(4.6) \quad 0 &\leq \frac{1}{\int_a^b g(x) dx} \int_a^b h(x) g(x) dx - h\left(\frac{a+b}{2}\right) \\
&\leq \frac{1}{\int_a^b g(x) dx} \int_a^b \left(\varphi(x) - \frac{h'_-(b) + h'_+(a)}{2}\right) \left(x - \frac{a+b}{2}\right) g(x) dx \\
&\leq \left(\frac{1}{\int_a^b g(x) dx} \int_a^b \left|x - \frac{a+b}{2}\right|^p g(x) dx\right)^{1/p} \\
&\quad \times \left(\frac{1}{\int_a^b g(x) dx} \int_a^b \left|\varphi(x) - \frac{h'_-(b) + h'_+(a)}{2}\right|^q g(x) dx\right)^{1/q} \\
&\leq \frac{1}{2} \begin{cases} (b-a) \left(\frac{1}{\int_a^b g(x) dx} \int_a^b \left|\varphi(x) - \frac{h'_-(b) + h'_+(a)}{2}\right|^q g(x) dx\right)^{1/q} \\ (\Phi'_-(b) - \Phi'_+(a)) \left(\frac{1}{\int_a^b g(x) dx} \int_a^b \left|x - \frac{a+b}{2}\right|^p g(x) dx\right)^{1/p} \end{cases} \\
&\leq \frac{1}{4} (b-a) (\Phi'_-(b) - \Phi'_+(a))
\end{aligned}$$

and

$$\begin{aligned}
(4.7) \quad 0 &\leq \frac{1}{\int_a^b g(x) dx} \int_a^b h(x) g(x) dx - h\left(\frac{a+b}{2}\right) \\
&\leq \frac{1}{\int_a^b g(x) dx} \int_a^b \left(\varphi(x) - \frac{h'_-(b) + h'_+(a)}{2}\right) \left(x - \frac{a+b}{2}\right) g(x) dx \\
&\leq \operatorname{ess\,sup}_{x \in [a,b]} \left|\varphi(x) - \frac{h'_-(b) + h'_+(a)}{2}\right| \frac{1}{\int_a^b g(x) dx} \int_a^b \left|x - \frac{a+b}{2}\right| g(x) dx \\
&\leq \frac{1}{2} \begin{cases} (h'_-(b) - h'_+(a)) \frac{1}{\int_a^b g(x) dx} \int_a^b \left|x - \frac{a+b}{2}\right| g(x) dx \\ (b-a) \operatorname{ess\,sup}_{x \in [a,b]} \left|\varphi(x) - \frac{h'_-(b) + h'_+(a)}{2}\right| \end{cases} \\
&\leq \frac{1}{4} (b-a) (h'_-(b) - h'_+(a)).
\end{aligned}$$

The above inequalities provide both a generalization and several reverses for the left Fejér inequality (1.1) as announced in the introduction.

5. APPLICATIONS FOR HERMITE-HADAMARD'S INEQUALITY

If we take $g(x) = 1$, $x \in [a, b]$ in (4.5)-(4.7), then we get for any convex function $h : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ and $\varphi \in \partial h$ that

$$\begin{aligned}
 (5.1) \quad 0 &\leq \frac{1}{b-a} \int_a^b h(x) dx - h\left(\frac{a+b}{2}\right) \\
 &\leq \frac{1}{b-a} \int_a^b \left(\varphi(x) - \frac{h'_-(b) + h'_+(a)}{2} \right) \left(x - \frac{a+b}{2} \right) dx \\
 &\leq \frac{1}{2} \int_a^b \left| \varphi(x) - \frac{h'_-(b) + h'_+(a)}{2} \right| dx \\
 &\leq \frac{1}{4} (b-a) (h'_-(b) - h'_+(a)),
 \end{aligned}$$

$$\begin{aligned}
 (5.2) \quad 0 &\leq \frac{1}{b-a} \int_a^b h(x) dx - h\left(\frac{a+b}{2}\right) \\
 &\leq \frac{1}{b-a} \int_a^b \left(\varphi(x) - \frac{h'_-(b) + h'_+(a)}{2} \right) \left(x - \frac{a+b}{2} \right) dx \\
 &\leq \frac{1}{2(p+1)^{1/p}} (b-a)^{1/p} \left(\int_a^b \left| \varphi(x) - \frac{h'_-(b) + h'_+(a)}{2} \right|^q dx \right)^{1/q} \\
 &\leq \frac{1}{4} (b-a) (h'_-(b) - h'_+(a))
 \end{aligned}$$

and

$$\begin{aligned}
 (5.3) \quad 0 &\leq \frac{1}{b-a} \int_a^b h(x) dx - h\left(\frac{a+b}{2}\right) \\
 &\leq \frac{1}{b-a} \int_a^b \left(\varphi(x) - \frac{h'_-(b) + h'_+(a)}{2} \right) \left(x - \frac{a+b}{2} \right) dx \\
 &\leq \frac{1}{4} (b-a) \operatorname{esssup}_{x \in [a,b]} \left| \varphi(x) - \frac{h'_-(b) + h'_+(a)}{2} \right| \\
 &\leq \frac{1}{8} (b-a) (h'_-(b) - h'_+(a)).
 \end{aligned}$$

For instance, if we take in the inequality (5.3) $h : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$, $h(x) = -\ln x$ then we get

$$(5.4) \quad 1 \leq \frac{A(a, b)}{I(a, b)} \leq \exp\left(\frac{A(a, b)}{L(a, b)} - 1\right) \leq \exp\left(\frac{1}{8} \cdot \frac{(b-a)^2}{G^2(a, b)}\right),$$

where $A(a, b) := \frac{a+b}{2}$ is the *arithmetic mean*, $G(a, b) := \sqrt{ab}$ is the *geometric mean*,

$$L(a, b) := \begin{cases} \frac{b-a}{\ln b - \ln a} & \text{if } b \neq a \\ b & \text{if } b = a \end{cases}$$

is the *logarithmic mean* and

$$I(a, b) := \begin{cases} \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } b \neq a \\ b & \text{if } b = a \end{cases}$$

is the *identric mean*.

Also, if we take in the inequality (5.3) $h : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$, $h(x) = \frac{1}{x}$ then we get

$$(5.5) \quad 0 \leq \frac{A(a, b) - L(a, b)}{A(a, b)L(a, b)} \leq \frac{L(a, b)A(a, b) - G^2(a, b)}{G^2(a, b)L(a, b)} \\ \leq \frac{1}{4} \frac{A(a, b)}{G^4(a, b)} (b - a)^2.$$

We define the *p-logarithmic mean* by

$$L_p(a, b) := \begin{cases} \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{1/p} & \text{if } b \neq a \\ b & \text{if } b = a \end{cases}$$

where $p \in \mathbb{R} \setminus \{-1, 0\}$.

We extend this mean for $\{-1, 0\}$ by putting $L_{-1}(a, b) := L(a, b)$ and $L_0(a, b) := I(a, b)$. With this convention we have that the function $\mathbb{R} \ni p \mapsto L_p(a, b)$ is monotonic increasing.

If we take in the inequality (5.3) $h : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$, $h(x) = x^p$, $p \in (-\infty, 0) \cup (1, \infty)$ then we get:

$$(5.6) \quad 0 \leq L_p^p(a, b) - A^p(a, b) \leq p \left(L_p^{p-1}(a, b) - L_{p-1}^{p-1}(a, b) A(a, b) \right) \\ \leq \frac{1}{8} p(p-1)(b-a)^2 L_{p-2}^{p-2}(a, b).$$

REFERENCES

- [1] A. G. AZPEITIA, Convex functions and the Hadamard inequality. *Rev. Colombiana Mat.* **28** (1994), no. 1, 7–12.
- [2] S. S. DRAGOMIR, A mapping in connection to Hadamard's inequalities, *An. Öster. Akad. Wiss. Math.-Natur.*, (Wien), **128**(1991), 17-20. MR 934:26032. ZBL No. 747:26015.
- [3] S. S. DRAGOMIR, Two mappings in connection to Hadamard's inequalities, *J. Math. Anal. Appl.*, **167**(1992), 49-56. MR:934:26038, ZBL No. 758:26014.
- [4] S. S. DRAGOMIR, On Hadamard's inequalities for convex functions, *Mat. Balkanica*, **6**(1992), 215-222. MR: 934: 26033.
- [5] S. S. DRAGOMIR, Bounds for the normalized Jensen functional, *Bull. Austral. Math. Soc.* **74**(3)(2006), 471-476.
- [6] S. S. DRAGOMIR, D. S. MILOŠEVIĆ and J. SÁNDOR, On some refinements of Hadamard's inequalities and applications, *Univ. Belgrad, Publ. Elek. Fak. Sci. Math.*, **4**(1993), 21-24.
- [7] S. S. DRAGOMIR and C. E. M. PEARCE, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000. [Online http://rgmia.org/monographs/hermite_hadamard.html].
- [8] L. FÉJÉR, Über die Fourierreihen, II, *Math. Naturwiss, Anz. Ungar. Akad. Wiss.*, **24** (1906), 369-390. (In Hungarian).
- [9] A. GUESSAB and G. SCHMEISSER, Sharp integral inequalities of the Hermite-Hadamard type. *J. Approx. Theory* **115** (2002), no. 2, 260–288.

- [10] E. KILIANITY and S. S. DRAGOMIR, Hermite-Hadamard's inequality and the p-HH-norm on the Cartesian product of two copies of a normed space. *Math. Inequal. Appl.* **13** (2010), no. 1, 1–32.
- [11] M. MERKLE, Remarks on Ostrowski's and Hadamard's inequality. *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.* **10** (1999), 113–117.
- [12] C. E. M. PEARCE and A. M. RUBINOV, P -functions, quasi-convex functions, and Hadamard type inequalities. *J. Math. Anal. Appl.* **240** (1999), no. 1, 92–104.
- [13] J. PEČARIĆ and A. VUKELIĆ, Hadamard and Dragomir-Agarwal inequalities, the Euler formulae and convex functions. *Functional equations, inequalities and applications*, 105–137, Kluwer Acad. Publ., Dordrecht, 2003.
- [14] G. TOADER, Superadditivity and Hermite-Hadamard's inequalities. *Studia Univ. Babeş-Bolyai Math.* **39** (1994), no. 2, 27–32.
- [15] G.-S. YANG and M.-C. HONG, A note on Hadamard's inequality. *Tamkang J. Math.* **28** (1997), no. 1, 33–37.
- [16] G.-S. YANG and K.-L. TSENG, On certain integral inequalities related to Hermite-Hadamard inequalities. *J. Math. Anal. Appl.* **239** (1999), no. 1, 180–187.

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