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A SURVEY OF PERTURBED OSTROWSKI TYPE INEQUALITIES

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Abstract. In this paper we survey a number of recent perturbed versions of Ostrowski inequality that have been obtained by the author and provide their connections with numerous calssical results of interest

1. Introduction

1.1. **Ostrowski's Inequality.** As revealed by a simple search in MathSciNet database with the key words "Ostrowski" and "inequality" in the title, an exponential evolution of research papers devoted to this result has been registered in the last decade. There are now at least 360 papers that can be found by performing the above search. Numerous extensions, generalizations in both the integral and discrete case have been discovered. More general versions for n-time differentiable functions, the corresponding versions on time scales, for vector valued functions or multiple integrals have been established as well. Numerous applications in Numerical Analysis, Approximation Theory, Probability Theory & Statistics, Information Theory and other fields have been also given.

In this paper, after presenting some inequalities of Ostrowski type for absolutely continuous functions, we survey a number of recent perturbed versions of this inequality that have been obtained lately by the author and provide their connections with numerous classical results of interest. Complete proofs are provided and the necessary references where all these results have been obtained first are also given.

In 1938, A. Ostrowski [21] proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x\in[a,b]$ in the case of differentiable functions on an open interval:

Theorem 1 (Ostrowski, 1938 [21])**.** *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)|<\infty$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty (b-a), \tag{1.1}$$

for all $x\in[a,b]$ and the constant $\frac{1}{4}$ is the best possible.

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In [17], S.S. Dragomir and S. Wang, by the use of the *Montgomery integral identity* [19, p. 565],

$$f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{1}{b-a}\int_a^b p(x,t)\,f'(t)\,dt\,,\quad x\in[a,b]\,,$$

where $p:[a,b]^2\to\mathbb{R}$ is given by

$$p(x,t):=\begin{cases} t-a & \text{if} \quad t\in[a,x]\,,\\ \\ t-b & \text{if} \quad t\in(x,b],\end{cases}$$

gave a simple proof of Ostrowski's inequality and applied it for special means (identric mean, logarithmic mean, etc.) and to the problem of estimating the error bound in approximating the Riemann integral $\int_a^b f(t)\,dt$ by an arbitrary Riemann sum (see [17], Section 3).

1.2. **A Refinement for L_∞-norm.** The following result, which is an improvement on Ostrowski's inequality, holds.

Theorem 2 (Dragomir, 2002 [4])**.** *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$ whose derivative $f'\in L_\infty[a,b]$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.2}$$
$$\leq\frac{1}{2(b-a)}\left[\|f'\|_{[a,x],\infty}(x-a)^2+\|f'\|_{[x,b],\infty}(b-x)^2\right]$$
$$\leq\begin{cases}\|f'\|_{[a,b],\infty}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\,;\\ \\ \frac{1}{2}\left[\|f'\|^\alpha_{[a,x],\infty}+\|f'\|^\alpha_{[x,b],\infty}\right]^{\frac{1}{\alpha}}\left[\left(\frac{x-a}{b-a}\right)^{2\beta}+\left(\frac{b-x}{b-a}\right)^{2\beta}\right]^{\frac{1}{\beta}}(b-a)\,,\\ \\ \text{where}\quad \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\ \\ \frac{1}{2}\left[\|f'\|_{[a,x],\infty}+\|f'\|_{[x,b],\infty}\right]\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^2(b-a)\end{cases}$$

for all $x\in[a,b]$, where $\|\cdot\|_{[m,n],\infty}$ denotes the usual norm on $L_\infty[m,n]$, i.e., we recall that

$$\|g\|_{[m,n],\infty}=ess\sup_{t\in[m,n]}|g(t)|<\infty.$$

Proof. Using the integration by parts formula for absolutely continuous functions on $[a,b]$, we have

$$\int_a^x(t-a)\,f'(t)\,dt=(x-a)\,f(x)-\int_a^x f(t)\,dt \tag{1.3}$$

and

$$\int_x^b(t-b)\,f'(t)\,dt=(b-x)\,f(x)-\int_x^b f(t)\,dt, \tag{1.4}$$

for all $x\in[a,b]$.

Adding these two equalities, we obtain the Montgomery identity (see for example [19, p. 565]):

$$(b-a)f(x)-\int_a^b f(t)\,dt=\int_a^x (t-a)f'(t)\,dt+\int_x^b (t-b)f'(t)\,dt \tag{1.5}$$

for all $x\in[a,b]$.

Taking the modulus, we deduce

$$\begin{aligned}\left|(b-a)f(x)-\int_a^b f(t)\,dt\right| &\le \left|\int_a^x (t-a)f'(t)\,dt\right|+\left|\int_x^b (t-b)f'(t)\,dt\right| \\ &\le \int_a^x (t-a)\left|f'(t)\right|dt+\int_x^b (b-t)\left|f'(t)\right|dt \\ &\le \|f'\|_{[a,x],\infty}\int_a^x (t-a)\,dt+\|f'\|_{[x,b],\infty}\int_x^b (b-t)\,dt \\ &= \frac{1}{2}\left[\|f'\|_{[a,x],\infty}(x-a)^2+\|f'\|_{[x,b],\infty}(b-x)^2\right]\end{aligned} \tag{1.6}$$

and the first inequality in (1.2) is proved.

Now, let us observe that

$$\begin{aligned}&\|f'\|_{[a,x],\infty}(x-a)^2+\|f'\|_{[x,b],\infty}(b-x)^2 \\ &\le \max\left\{\|f'\|_{[a,x],\infty},\|f'\|_{[x,b],\infty}\right\}\left[(x-a)^2+(b-x)^2\right] \\ &= \max\left\{\|f'\|_{[a,x],\infty},\|f'\|_{[x,b],\infty}\right\}\left[\frac{1}{2}(b-a)^2+2\left(x-\frac{a+b}{2}\right)^2\right] \\ &= (b-a)^2\max\left\{\|f'\|_{[a,x],\infty},\|f'\|_{[x,b],\infty}\right\}\left[\frac{1}{2}+2\cdot\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right] \\ &= (b-a)^2\|f'\|_{[a,b],\infty}\left[\frac{1}{2}+2\cdot\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right],\end{aligned}$$

and the first part of the second inequality in (1.2) is proved.

For the second inequality, we employ the elementary inequality for real numbers which can be derived from *Hölder's discrete inequality*

$$0\le ms+nt\le\left(m^\alpha+n^\alpha\right)^{\frac{1}{\alpha}}\times\left(s^\beta+t^\beta\right)^{\frac{1}{\beta}}, \tag{1.7}$$

provided that $m,s,n,t\ge 0$, $\alpha>1$ and $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

Using (1.7), we obtain

$$\begin{aligned}&\|f'\|_{[a,x],\infty}(x-a)^2+\|f'\|_{[x,b],\infty}(b-x)^2 \\ &\le\left(\|f'\|_{[a,x],\infty}^\alpha+\|f'\|_{[x,b],\infty}^\alpha\right)^{\frac{1}{\alpha}}\left[(x-a)^{2\beta}+(b-x)^{2\beta}\right]^{\frac{1}{\beta}}\end{aligned}$$

and the second part of the second inequality in (1.2) is also obtained.

Finally, we observe that

$$\begin{aligned}
&\|f'\|_{[a,x],\infty}(x-a)^2+\|f'\|_{[x,b],\infty}(b-x)^2\\
&\le \max\left\{(x-a)^2,(b-x)^2\right\}\left[\|f'\|_{[a,x],\infty}+\|f'\|_{[x,b],\infty}\right]\\
&=\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^2\left[\|f'\|_{[a,x],\infty}+\|f'\|_{[x,b],\infty}\right]
\end{aligned}$$

and the last part of the second inequality in (1.2) is proved. □

The following corollary is also natural.

Corollary 1. *Under the above assumptions, we have the midpoint inequality*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \qquad (1.8)\\
&\le \frac{1}{8}(b-a)\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty}+\|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\right]\\
&\le\begin{cases}\frac{1}{4}(b-a)\|f'\|_{[a,b],\infty};\\[2mm]
\frac{1}{2^{\frac{3\beta-1}{2}}}(b-a)\left[\|f'\|^{\alpha}_{\left[a,\frac{a+b}{2}\right],\infty}+\|f'\|^{\alpha}_{\left[\frac{a+b}{2},b\right],\infty}\right]^{\frac{1}{\alpha}},\\[2mm]
\text{where }\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1.\end{cases}
\end{aligned}$$

2. Ostrowski for L_1-norm

2.1. L_1-norm Inequality.

In 1997, Dragomir and Wang proved the following Ostrowski type inequality [15].

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\|f'\|_{[a,b],1}, \qquad (2.1)$$

for all $x\in[a,b]$, where $\|\cdot\|_1$ is the Lebesgue norm on $L_1[a,b]$, i.e., we recall it $\|g\|_{[a,b],1}:=\int_a^b|g(t)|\,dt$. The constant $\frac{1}{2}$ is best possible.

Note that the fact that $\frac{1}{2}$ is the best constant for differentiable functions was proved in [20] and (2.1) can also be obtained from a more general result obtained by A. M. Fink in [18] choosing $n=1$ and doing some appropriate computation. However the inequality (2.1) was not stated explicitly in [18].

2.2. A Refinement for L_1-norm.

The following result, which is an improvement on the inequality (2.1), holds.

Theorem 4 (Dragomir, 2002 [3]). *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.2}$$
$$\le \frac{x-a}{b-a}\|f'\|_{[a,x],1}+\frac{b-x}{b-a}\|f'\|_{[x,b],1}$$
$$\le\begin{cases}\frac{1}{2}\left[\|f'\|_{[a,b],1}+\left|\|f'\|_{[a,x],1}-\|f'\|_{[x,b],1}\right|\right]\\[2mm] \left[\left(\frac{x-a}{b-a}\right)^{\beta}+\left(\frac{b-x}{b-a}\right)^{\beta}\right]^{\frac{1}{\beta}}\left(\left|\|f'\|^{\alpha}_{[a,x],1}\right|+\|f'\|^{\alpha}_{[x,b],1}\right)^{\frac{1}{\alpha}}\\ \text{where } \alpha>1 \text{ and } \frac{1}{\alpha}+\frac{1}{\beta}=1,\\[2mm] \left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\|f'\|_{[a,b],1}\end{cases}$$

for all $x\in[a,b]$.

Proof. Start with the Montgomery identity for absolutely continuous functions proved in Theorem 2

$$(b-a)f(x)-\int_a^b f(t)\,dt=\int_a^x (t-a)f'(t)\,dt+\int_x^b (t-b)f'(t)\,dt \tag{2.3}$$

for all $x\in[a,b]$.

Taking the modulus, we deduce

$$\left|(b-a)f(x)-\int_a^b f(t)\,dt\right|\le\left|\int_a^x (t-a)f'(t)\,dt\right|+\left|\int_x^b (t-b)f'(t)\,dt\right| \tag{2.4}$$
$$\le\int_a^x (t-a)|f'(t)|\,dt+\int_x^b (b-t)|f'(t)|\,dt$$
$$\le(x-a)\int_a^x |f'(t)|\,dt+(b-x)\int_x^b |f'(t)|\,dt$$
$$=(x-a)\|f'\|_{[a,x],1}+(b-x)\|f'\|_{[x,b],1}$$

and the first inequality in (2.2) is proved.

Now, let us observe that

$$(x-a)\|f'\|_{[a,x],1}+(b-x)\|f'\|_{[x,b],1}$$
$$\le\max\left\{\|f'\|_{[a,x],1},\|f'\|_{[x,b],1}\right\}(b-a)$$
$$=\frac{1}{2}\left[\|f'\|_{[a,b],1}+\left|\|f'\|_{[a,x],1}-\|f'\|_{[x,b],1}\right|\right](b-a)$$

and the first part of the second inequality is proved.

For the second inequality, we employ the elementary inequality for real numbers which can be derived from Hölder's discrete inequality

$$0\le ms+nt\le(m^{\alpha}+n^{\alpha})^{\frac{1}{\alpha}}\times\left(s^{\beta}+t^{\beta}\right)^{\frac{1}{\beta}}, \tag{2.5}$$

provided that $m,s,n,t\ge0$, $\alpha>1$ and $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

Using (2.5), we obtain

$$\begin{aligned}&(x-a)\|f'\|_{[a,x],1}+(b-x)\|f'\|_{[x,b],1}\\&\le\left(\|f'\|_{[a,x],1}^{\alpha}+\|f'\|_{[x,b],1}^{\alpha}\right)^{\frac{1}{\alpha}}\left[(x-a)^{\beta}+(b-x)^{\beta}\right]^{\frac{1}{\beta}}\end{aligned}$$

and the second part of the second inequality in (2.2) is also obtained.

Finally, we observe that

$$\begin{aligned}&(x-a)\|f'\|_{[a,x],1}+(b-x)\|f'\|_{[x,b],1}\\&\le\max\{x-a,b-x\}\left[\|f'\|_{[a,x],1}+\|f'\|_{[x,b],1}\right]\\&=\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\|f'\|_{[a,b],1}\end{aligned}$$

and the last part of the second inequality in (2.2) is proved. □

The following corollary is also natural.

Corollary 2. *Under the above assumptions, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{2}\|f'\|_{[a,b],1}. \tag{2.6}$$

Another interesting result is the following one.

Corollary 3. *Under the above assumptions, and if there is an $x_0\in[a,b]$ with*

$$\int_a^{x_0}|f'(t)|\,dt=\int_{x_0}^b|f'(t)|\,dt \tag{2.7}$$

then we have the inequality

$$\left|f(x_0)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{2}\|f'\|_{[a,b],1}. \tag{2.8}$$

3. Ostrowski for L_p-norm

3.1. **L_p-norm Inequality.** In 1998, Dragomir and Wang proved the following Ostrowski type inequality [16].

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f'\in L_p[a,b]$, then we have the inequality*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\&\le\frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\|f'\|_{[a,b],p},\end{aligned} \tag{3.1}$$

for all $x\in[a,b]$, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\|\cdot\|_{[a,b],p}$ is the p-Lebesgue norm on $L_p[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],p}:=\left(\int_a^b|g(t)|^p\,dt\right)^{1/p}.$$

From (3.1) we get the following midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{2(q+1)^{1/q}}(b-a)^{1/q}\|f'\|_{[a,b],p}, \tag{3.2}$$

and $\frac{1}{2}$ is a best possible constant.

Indeed, if we take $f:[a,b]\to\mathbb{R}$ with $f(t)=\left|t-\frac{a+b}{2}\right|$, then f is absolutely continuous $\int_a^b f(t)\,dt=\frac{(b-a)^2}{4}$, $\|f'\|_{[a,b],p}=(b-a)^{1/p}$ and if we assume that (3.2) holds with a constant $C>0$ instead of $\frac{1}{2}$, then we get $\frac{1}{4}(b-a)\leq\frac{C}{(q+1)^{1/q}}(b-a)$ for any $q>1$. Letting $q\to 1+$, we obtain $C\geq\frac{1}{2}$, which proves the sharpness of the constant.

In the following, we provide some refinements of (3.1) and (3.2).

3.2. A Refinement for L_p-norm.

The following result, which is an improvement on the inequality (3.1), holds.

Theorem 6 (Dragomir, 2013 [10]). *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. If $f'\in L_p[a,b]$, then*

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}}\|f'\|_{[a,x],p}+\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}}\|f'\|_{[x,b],p}\right](b-a)^{1/q} \\
&\leq \frac{1}{(q+1)^{1/q}} \\
&\times\begin{cases}
\frac{1}{2}\left[\|f'\|_{[a,x],p}+\|f'\|_{[x,b],p}+\left|\|f'\|_{[a,x],p}-\|f'\|_{[x,b],p}\right|\right] \\
\times\left[\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}}+\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}}\right](b-a)^{1/q} \\
\\
\left(\|f'\|_{[a,x],p}^{\alpha}+\|f'\|_{[x,b],p}^{\alpha}\right)^{\frac{1}{\alpha}}\left[\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}\beta}+\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}\beta}\right]^{\frac{1}{\beta}}(b-a)^{1/q} \\
\text{where } \alpha>1 \text{ and } \frac{1}{\alpha}+\frac{1}{\beta}=1, \\
\\
\left[\|f'\|_{[a,x],p}+\|f'\|_{[x,b],p}\right]\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^{\frac{q+1}{q}}(b-a)^{1/q}
\end{cases}
\end{aligned} \tag{3.3}$$

for all $x\in[a,b]$, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Start with the Montgomery identity for absolutely continuous functions

$$(b-a)f(x)-\int_a^b f(t)\,dt=\int_a^x (t-a)f'(t)\,dt+\int_x^b (t-b)f'(t)\,dt \tag{3.4}$$

for all $x\in[a,b]$.

Taking the modulus, we deduce

$$\left|(b-a)f(x)-\int_a^b f(t)\,dt\right| \le \left|\int_a^x (t-a)f'(t)\,dt\right| + \left|\int_x^b (t-b)f'(t)\,dt\right| \le \int_a^x (t-a)\left|f'(t)\right|dt + \int_x^b (b-t)\left|f'(t)\right|dt. \tag{3.5}$$

Utilizing Hölder's integral inequality we have

$$\begin{aligned}
&\int_a^x (t-a)\left|f'(t)\right|dt + \int_x^b (b-t)\left|f'(t)\right|dt \\
&\le \left(\int_a^x (t-a)^q\,dt\right)^{1/q}\left(\int_a^x \left|f'(t)\right|^p dt\right)^{1/p} + \left(\int_x^b (b-t)^q\,dt\right)^{1/q}\left(\int_x^b \left|f'(t)\right|^p dt\right)^{1/p} \\
&= \frac{1}{(b-a)(q+1)^{1/q}}\left[(x-a)^{\frac{q+1}{q}}\left\|f'\right\|_{[a,x],p} + (b-x)^{\frac{q+1}{q}}\left\|f'\right\|_{[x,b],p}\right]
\end{aligned}$$

for all $x \in [a,b]$, and the first inequality in (3.3) is proved.

Now, let us observe that

$$\begin{aligned}
&(x-a)^{\frac{q+1}{q}}\left\|f'\right\|_{[a,x],p} + (b-x)^{\frac{q+1}{q}}\left\|f'\right\|_{[x,b],p} \\
&\le \max\left\{\left\|f'\right\|_{[a,x],p}, \left\|f'\right\|_{[x,b],p}\right\}\left[(x-a)^{\frac{q+1}{q}} + (b-x)^{\frac{q+1}{q}}\right] \\
&= \frac{1}{2}\left[\left\|f'\right\|_{[a,x],p} + \left\|f'\right\|_{[x,b],p} + \left|\left\|f'\right\|_{[a,x],p} - \left\|f'\right\|_{[x,b],p}\right|\right] \\
&\times \left[(x-a)^{\frac{q+1}{q}} + (b-x)^{\frac{q+1}{q}}\right]
\end{aligned}$$

and the first part of the second inequality is proved.

For the second inequality, we employ the elementary inequality for real numbers which can be derived from Hölder's discrete inequality

$$0 \le ms + nt \le \left(m^\alpha + n^\alpha\right)^{\frac{1}{\alpha}} \times \left(s^\beta + t^\beta\right)^{\frac{1}{\beta}}, \tag{3.6}$$

provided that $m, s, n, t \ge 0$, $\alpha > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Using (3.6), we obtain

$$\begin{aligned}
&(x-a)^{\frac{q+1}{q}}\left\|f'\right\|_{[a,x],p} + (b-x)^{\frac{q+1}{q}}\left\|f'\right\|_{[x,b],p} \\
&\le \left(\left\|f'\right\|_{[a,x],p}^\alpha + \left\|f'\right\|_{[x,b],p}^\alpha\right)^{\frac{1}{\alpha}}\left[(x-a)^{\frac{q+1}{q}\beta} + (b-x)^{\frac{q+1}{q}\beta}\right]^{\frac{1}{\beta}}
\end{aligned}$$

and the second part of the second inequality in (3.3) is also obtained.

Finally, we observe that

$$\begin{aligned}
&(x-a)^{\frac{q+1}{q}}\left\|f'\right\|_{[a,x],p} + (b-x)^{\frac{q+1}{q}}\left\|f'\right\|_{[x,b],p} \\
&\le \max\left\{(x-a)^{\frac{q+1}{q}}, (b-x)^{\frac{q+1}{q}}\right\}\left[\left\|f'\right\|_{[a,x],p} + \left\|f'\right\|_{[x,b],p}\right] \\
&= \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^{\frac{q+1}{q}}\left[\left\|f'\right\|_{[a,x],p} + \left\|f'\right\|_{[x,b],p}\right]
\end{aligned}$$

and the last part of the second inequality in (3.3) is proved. □

The following corollary is also natural.

Corollary 4. *Under the above assumptions, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{2^{(q+1)/q}(q+1)^{1/q}}\left[\|f'\|_{[a,\frac{a+b}{2}],p}+\|f'\|_{[\frac{a+b}{2},b],p}\right](b-a)^{1/q}. \tag{3.7}$$

Another interesting result is the following one.

Corollary 5. *Under the above assumptions, and if there is an $x_0\in[a,b]$ with*

$$\int_a^{x_0}|f'(t)|^p\,dt=\int_{x_0}^b|f'(t)|^p\,dt \tag{3.8}$$

then we have the inequality

$$\left|f(x_0)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x_0-a}{b-a}\right)^{\frac{q+1}{q}}+\left(\frac{b-x_0}{b-a}\right)^{\frac{q+1}{q}}\right]\|f'\|_{[a,x_0],p}(b-a)^{1/q}. \tag{3.9}$$

Remark 1. *If we take in (3.3) $\alpha=p$ and $\beta=q$, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then we get the following refinement of (3.1)*

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| &\leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}}\|f'\|_{[a,x],p}+\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}}\|f'\|_{[x,b],p}\right](b-a)^{1/q}\\ &\leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\|f'\|_{[a,b],p},\end{aligned} \tag{3.10}$$

for all $x\in[a,b]$.

This is true, since for $\alpha=p$ we have

$$\begin{aligned}\left(\|f'\|^{\alpha}_{[a,x],p}+\|f'\|^{\alpha}_{[x,b],p}\right)^{\frac{1}{\alpha}} &= \left(\|f'\|^{p}_{[a,x],p}+\|f'\|^{p}_{[x,b],p}\right)^{\frac{1}{p}}\\ &= \left(\int_a^x|f'(t)|^p\,dt+\int_x^b|f'(t)|^p\right)^{1/p}=\|f'\|_{[a,b],p}.\end{aligned}$$

4. Ostrowski for Bounded Derivatives

We start with the following result.

Theorem 7 (Dragomir, 2003 [8])**.** *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $x\in[a,b]$. Suppose that there exist the functions m_i, M_i : $[a,b]\to\mathbb{R}$ $\left(i=\overline{1,2}\right)$ with the properties:*

$$m_1(x)\leq f'(t)\leq M_1(x)\quad \textit{for a.e. } t\in[a,x] \tag{4.1}$$

and

$$m_2(x)\leq f'(t)\leq M_2(x)\quad \textit{for a.e. } t\in(x,b]. \tag{4.2}$$

Then we have the inequalities:

$$\begin{aligned}(4.3) \qquad & \frac{1}{2(b-a)}\left[m_1(x)(x-a)^2 - M_2(x)(b-x)^2\right] \\ & \leq f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ & \leq \frac{1}{2(b-a)}\left[M_1(x)(x-a)^2 - m_2(x)(b-x)^2\right].\end{aligned}$$

The constant $\frac{1}{2}$ is sharp on both sides.

Proof. Start with the Montgomery identity
(4.4)

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^x (t-a) f'(t)\,dt + \frac{1}{b-a}\int_x^b (t-b) f'(t)\,dt$$

for any $x \in [a,b]$.

Using the assumption (4.1) and (4.2), we have:

$$(4.5) \qquad m_1(x)(t-a) \leq (t-a) f'(t) \leq M_1(x)(t-a) \quad \text{for a.e. } t \in [a,x]$$

and

$$(4.6) \qquad M_2(x)(t-b) \leq f'(t)(t-b) \leq m_2(x)(t-b) \quad \text{for a.e. } t \in (x,b].$$

Integrating (4.5) on $[a,x]$ and (4.6) on $[x,b]$ and summing the obtained inequalities, we have

$$\begin{aligned}& \frac{1}{2}m_1(x)(x-a)^2 - \frac{1}{2}M_2(x)(b-x)^2 \\ & \leq \int_a^x (t-a) f'(t)\,dt + \int_x^b (t-b) f'(t)\,dt \\ & \leq \frac{1}{2}M_1(x)(x-a)^2 - \frac{1}{2}m_2(x)(b-x)^2.\end{aligned}$$

Using the representation (4.4), we deduce (4.3).

Assume that the first inequality in (4.3) holds with a constant $c > 0$; that is,

$$(4.7) \qquad \frac{c}{b-a}\left[m_1(x)(x-a)^2 - M_2(x)(b-x)^2\right] \leq f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Consider the function $f : [a,b] \to \mathbb{R}$, $f(t) = M|t-x|$, $M > 0$. Then f is absolutely continuous and

$$f'(t) = \begin{cases} -M & \text{if} \quad t \in [a,x] \\ M & \text{if} \quad t \in (x,b]. \end{cases}$$

Thus, if we choose $m_1 = -M$, $m_2 = M$ in (4.7), we get

$$\begin{aligned}-M\frac{c}{b-a}\left[(x-a)^2 + (b-x)^2\right] & \leq -\frac{M}{b-a}\int_a^b |t-x|\,dt \\ & = -\frac{M}{b-a}\left[\frac{(b-x)^2 + (x-a)^2}{2}\right]\end{aligned}$$

for all $x \in [a,b]$, implying that $c \geq \frac{1}{2}$, that is, $\frac{1}{2}$ is the best constant in the first member of (4.3).

Using a similar process, we may prove that $\frac{1}{2}$ is the best constant in the third member of (4.3) and the theorem is completely proved. □

Corollary 6. *If $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ and the derivative $f':[a,b]\to\mathbb{R}$ is bounded above and below, that is,*

$$-\infty<m\leq f'(t)\leq M<\infty \text{ for a.e. } t\in[a,b], \tag{4.8}$$

then we have the inequality

$$\frac{1}{2(b-a)}\left[m(x-a)^2-M(b-x)^2\right]\leq f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt \leq\frac{1}{2(b-a)}\left[M(x-a)^2-m(b-x)^2\right] \tag{4.9}$$

for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best in both inequalities.

Applying Taylor's formula

$$g(x)=g\left(\frac{a+b}{2}\right)+\left(x-\frac{a+b}{2}\right)g'\left(\frac{a+b}{2}\right)+\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2g''\left(\frac{a+b}{2}\right)$$

for $g(x)=M(x-a)^2-m(b-x)^2$, we obtain

$$g(x)=\frac{1}{4}(M-m)(b-a)^2+2\left(x-\frac{a+b}{2}\right)\left(\frac{M+m}{2}\right)(b-a) +(M-m)\left(x-\frac{a+b}{2}\right)^2.$$

The same formula applied for $h(x)=m(x-a)^2-M(b-x)^2$, will reveal that

$$h(x)=\frac{1}{4}(M-m)(b-a)^2+2\left(x-\frac{a+b}{2}\right)\left(\frac{M+m}{2}\right)(b-a) -(M-m)\left(x-\frac{a+b}{2}\right)^2.$$

Consequently, we may rewrite Corollary 6 in the following equivalent manner:

Corollary 7. *With the assumptions on Corollary 6, we have:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\left(x-\frac{a+b}{2}\right)\left(\frac{M+m}{2}\right)\right| \leq\frac{1}{2}(M-m)(b-a)\left[\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2+\frac{1}{4}\right] \tag{4.10}$$

for all $x\in[a,b]$.

Remark 2. *If we assume that $\|f'\|_\infty:=ess\sup_{t\in[a,b]}|f'(t)|$, then obviously we may choose in (4.9) $m=\|f'\|_\infty$ and $M=\|f'\|_\infty$, obtaining Ostrowski's inequality for* *absolutely continuous functions whose derivatives are essentially bounded:*

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| &\le \frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2+(b-x)^2\right] \\ &= \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty,\end{aligned}$$

for all $x\in[a,b]$. *The constant* $\frac{1}{4}$ *here is best.*

Remark 3. *Ostrowski's inequality for absolutely continuous mappings in terms of* $\|f'\|_\infty$ *basically states that*

$$\begin{aligned}-\frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2+(b-x)^2\right] &\le f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le \frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2+(b-x)^2\right]\end{aligned} \tag{4.11}$$

for all $x\in[a,b]$.

Now, if we assume that (4.1) and (4.2) hold, then $-\|f'\|_\infty \le m_1(x),\ m_2(x)$ *and* $M_1(x),\ M_2(x)\le\|f'\|_\infty$, *which implies:*

$$\begin{aligned} & -\frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2+(b-x)^2\right] \\ & \le \frac{1}{2(b-a)}\left[m_1(x)(x-a)^2-M_2(x)(b-x)^2\right] \\ & \le f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt \\ & \le \frac{1}{2(b-a)}\left[M_1(x)(x-a)^2-m_2(x)(b-x)^2\right] \\ & \le \frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2+(b-x)^2\right].\end{aligned} \tag{4.12}$$

Thus, the inequality (4.3) may also be regarded as a refinement of the classical Ostrowski result.

An important particular case is $x=\frac{a+b}{2}$ providing the following corollary.

Corollary 8. *Assume that the derivative* $f':[a,b]\to\mathbb{R}$ *satisfy the conditions:*

$$-\infty<m_1\le f'(t)\le M_1<\infty \text{ for a.e. } t\in\left[a,\frac{a+b}{2}\right] \tag{4.13}$$

and

$$-\infty<m_2\le f'(t)\le M_2<\infty \text{ for a.e. } t\in\left(\frac{a+b}{2},b\right]. \tag{4.14}$$

Then we have the inequalities

$$\begin{aligned}\frac{1}{8}(m_1-M_2)(b-a) &\le f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le \frac{1}{8}(M_1-m_2)(b-a).\end{aligned} \tag{4.15}$$

The constant $\frac{1}{8}$ *is the best in both inequalities.*

Finally, if we know some global bounds for the derivative f' on $[a,b]$, then we may state the following corollary.

Corollary 9. *Under the assumptions of Corollary 6, we have the midpoint inequality:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{8}(M-m)(b-a). \tag{4.16}$$

The constant $\frac{1}{8}$ is best.

Proof. The inequality is obvious by Corollary 6 putting $x=\frac{a+b}{2}$. We observe that the function $f:[a,b]\to\mathbb{R}$, $f(x)=k\left|x-\frac{a+b}{2}\right|$, $k>0$ is absolutely continuous and $-k\le f'(t)\le k$ for all $t\in[a,b]$. Thus, we may choose $M=k$, $m=-k$ and as

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| = \frac{1}{8}(M-m)(b-a)=\frac{k(b-a)}{4},$$

we conclude that the constant $\frac{1}{8}$ is best in (4.16). □

5. Functional Ostrowski Inequality for Convex Mappings

5.1. A Generalization of Ostrowski's Inequality.

The following result holds:

Theorem 8 (Dragomir, 2013 [10])**.** *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. If $\Phi:\mathbb{R}\to\mathbb{R}$ is convex (concave) on $\mathbb{R}$ then we have the inequalities*

$$\begin{aligned}\Phi\left(f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right) \\ \le (\ge)\frac{1}{b-a}\left[\int_a^x \Phi\left[(t-a)f'(t)\right]dt+\int_x^b \Phi\left[(t-b)f'(t)\right]dt\right]\end{aligned} \tag{5.1}$$

for any $x\in[a,b]$.

Proof. Utilising the Montgomery identity

$$\begin{aligned} f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt &= \frac{1}{b-a}\left[\int_a^x (t-a)f'(t)\,dt+\int_x^b (t-b)f'(t)\,dt\right] \\ &= \frac{x-a}{b-a}\left(\frac{1}{x-a}\int_a^x (t-a)f'(t)\,dt\right) \\ &\quad+\frac{b-x}{b-a}\left(\frac{1}{b-x}\int_x^b (t-b)f'(t)\,dt\right),\end{aligned} \tag{5.2}$$

which holds for any $x\in(a,b)$ and the convexity of $\Phi:\mathbb{R}\to\mathbb{R}$, we have

$$\begin{aligned}&\Phi\left(f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right) \\ &\le \frac{x-a}{b-a}\Phi\left(\frac{1}{x-a}\int_a^x (t-a)f'(t)\,dt\right) \\ &+\frac{b-x}{b-a}\Phi\left(\frac{1}{b-x}\int_x^b (t-b)f'(t)\,dt\right)\end{aligned} \tag{5.3}$$

for any $x \in (a,b)$, which is an inequality of interest in itself as well.

If we use Jensen's integral inequality

$$\Phi\left(\frac{1}{d-c}\int_c^d g(t)\,dt\right) \leq \frac{1}{d-c}\int_c^d \Phi[g(t)]\,dt$$

we have

$$\Phi\left(\frac{1}{x-a}\int_a^x (t-a)\,f'(t)\,dt\right) \leq \frac{1}{x-a}\int_a^x \Phi[(t-a)\,f'(t)]\,dt \tag{5.4}$$

and

$$\Phi\left(\frac{1}{b-x}\int_x^b (t-b)\,f'(t)\,dt\right) \leq \frac{1}{b-x}\int_x^b \Phi[(t-b)\,f'(t)]\,dt \tag{5.5}$$

for any $x \in (a,b)$.

Making use of (5.3)-(5.5) we get the desired result (5.1) for the convex functions.

If $x = b$, then

$$f(b) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^b (t-a)\,f'(t)\,dt$$

and by Jensen's inequality we get

$$\Phi\left(f(b) - \frac{1}{b-a}\int_a^b f(t)\,dt\right) \leq \frac{1}{b-a}\int_a^b \Phi[(t-a)\,f'(t)]\,dt,$$

which proves the inequality (5.1) for $x = b$.

The same argument can be applied for $x = a$.

The case of concave functions goes likewise and the theorem is proved. □

Corollary 10 (Dragomir, 2013 [10])**.** *With the assumptions of Theorem 8 we have*

$$\begin{aligned} \Phi(0) &\leq (\geq)\, \frac{1}{b-a}\int_a^b \Phi\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right) dx \\ &\leq (\geq)\, \frac{1}{(b-a)^2}\left[\int_a^b (b-x)\,\Phi[(x-a)\,f'(x)]\,dx \right. \\ &\quad \left. + \int_a^b (x-a)\,\Phi[(x-b)\,f'(x)]\,dx\right]. \end{aligned} \tag{5.6}$$

Proof. By Jensen's integral inequality we have

$$\begin{aligned} &\frac{1}{b-a}\int_a^b \Phi\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right) dx \\ &\geq (\leq)\, \Phi\left[\frac{1}{b-a}\int_a^b \left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right) dx\right] = \Phi(0), \end{aligned}$$

which proves the first inequality in (5.6).

Integrating the inequality (5.1) over x we have

$$\frac{1}{b-a}\int_a^b \Phi\left(f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right)dx \\ \leq (\geq)\,\frac{1}{(b-a)^2}\int_a^b\left[\int_a^x \Phi\left[(t-a)f'(t)\right]dt+\int_x^b \Phi\left[(t-b)f'(t)\right]dt\right]dx. \tag{5.7}$$

Integrating by parts we have

$$\begin{aligned}&\int_a^b\left(\int_a^x \Phi\left[(t-a)f'(t)\right]dt\right)dx\\ &= x\int_a^x \Phi\left[(t-a)f'(t)\right]dt\Big|_a^b-\int_a^b x\,d\left(\int_a^x \Phi\left[(t-a)f'(t)\right]dt\right)\\ &= b\int_a^b \Phi\left[(t-a)f'(t)\right]dt-\int_a^b x\Phi\left[(x-a)f'(x)\right]dx\\ &= \int_a^b (b-x)\,\Phi\left[(x-a)f'(x)\right]dx\end{aligned}$$

and

$$\begin{aligned}&\int_a^b\left(\int_x^b \Phi\left[(t-b)f'(t)\right]dt\right)dx\\ &= x\left(\int_x^b \Phi\left[(t-b)f'(t)\right]dt\right)\Big|_a^b-\int_a^b x\,d\left(\int_x^b \Phi\left[(t-b)f'(t)\right]dt\right)\\ &= -a\left(\int_a^b \Phi\left[(t-b)f'(t)\right]dt\right)+\int_a^b x\Phi\left[(x-b)f'(x)\right]dx\\ &= \int_a^b (x-a)\,\Phi\left[(x-b)f'(x)\right]dx.\end{aligned}$$

Utilising the inequality (5.7) we deduce the desired inequality (5.6). □

Remark 4. *If we write the inequality (5.1) for the convex function $\Phi(x)=|x|^p$, $p\geq 1$ then we get the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|^p \\ \leq \frac{1}{b-a}\left[\int_a^x (t-a)^p\left|f'(t)\right|^p dt+\int_x^b (b-t)^p\left|f'(t)\right|^p dt\right] \tag{5.8}$$

for $x\in[a,b]$.

Utilising Hölder's inequality we have

$$B(x) := \int_a^x (t-a)^p |f'(t)|^p dt + \int_x^b (b-t)^p |f'(t)|^p dt \tag{5.9}$$
$$\leq \begin{cases} \frac{(x-a)^{p+1}}{p+1} \|f'\|_{[a,x],\infty}^p & \text{if } f' \in L_\infty[a,x]; \\ \frac{(x-a)^{p+1/\alpha}}{(p\alpha+1)^{1/\alpha}} \|f'\|_{[a,x],p\beta}^p & \begin{array}{c}\text{if } f' \in L_{p\beta}[a,x], \\ \alpha>1, 1/\alpha+1/\beta=1;\end{array} \\ (x-a)^p \|f'\|_{[a,x],p}^p \end{cases}$$
$$+ \begin{cases} \frac{(b-x)^{p+1}}{p+1} \|f'\|_{[x,b],\infty}^p & \text{if } f' \in L_\infty[x,b]; \\ \frac{(b-x)^{p+1/\alpha}}{(p\alpha+1)^{1/\alpha}} \|f'\|_{[x,b],p\beta}^p & \begin{array}{c}\text{if } f' \in L_{p\beta}[x,b], \\ \alpha>1, 1/\alpha+1/\beta=1;\end{array} \\ (b-x)^p \|f'\|_{[x,b],p}^p \end{cases}$$

for $x \in [a,b]$.

Utilising the inequalities (5.8) and (5.9) we have for $x \in [a,b]$ *that*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|^p \tag{5.10}$$
$$\leq \frac{1}{(b-a)(p+1)} \left[(x-a)^{p+1} \|f'\|_{[a,x],\infty}^p + (b-x)^{p+1} \|f'\|_{[x,b],\infty}^p \right]$$
$$\leq \frac{1}{(p+1)} \left[\left(\frac{x-a}{b-a}\right)^{p+1} + \left(\frac{b-x}{b-a}\right)^{p+1} \right] (b-a)^p \|f'\|_{[a,b],\infty}^p$$

provided $f' \in L_\infty[a,b]$,

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|^p \tag{5.11}$$
$$\leq \frac{1}{(b-a)(p\alpha+1)^{1/\alpha}} \left[(x-a)^{p+1/\alpha} \|f'\|_{[a,x],p\beta}^p + (b-x)^{p+1/\alpha} \|f'\|_{[x,b],p\beta}^p \right]$$
$$\leq \frac{1}{(p\alpha+1)^{1/\alpha}} \left[\left(\frac{x-a}{b-a}\right)^{p+1/\alpha} + \left(\frac{b-x}{b-a}\right)^{p+1/\alpha} \right] (b-a)^{p-1/\beta} \|f'\|_{[a,b],p\beta}^p$$

provided $f' \in L_{p\beta}[a,b], \alpha > 1, 1/\alpha + 1/\beta = 1$ *and*

$$\begin{aligned}
(5.12)\qquad & \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|^p \\
&\leq \frac{1}{b-a}\left[(x-a)^p \|f'\|^p_{[a,x],p} + (b-x)^p \|f'\|^p_{[x,b],p} \right] \\
&\leq \max\left\{ \left(\frac{x-a}{b-a}\right)^p, \left(\frac{b-x}{b-a}\right)^p \right\} (b-a)^{p-1} \|f'\|^p_{[a,b],p} \\
&= \left\{ \frac{1}{2} + \left| \frac{x-\frac{a+b}{2}}{b-a} \right| \right\}^p (b-a)^{p-1} \|f'\|^p_{[a,b],p}
\end{aligned}$$

provided $f' \in L_p[a,b]$.

Remark 5. *If we take* $p = 1$ *in the above inequalities (5.11)-(5.12), then we obtain*

$$\begin{aligned}
(5.13)\qquad & \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
&\leq \frac{1}{2(b-a)}\left[(x-a)^2 \|f'\|_{[a,x],\infty} + (b-x)^2 \|f'\|_{[x,b],\infty} \right] \\
&\leq \frac{1}{2}\left[\left(\frac{x-a}{b-a}\right)^2 + \left(\frac{b-x}{b-a}\right)^2 \right] (b-a) \|f'\|_{[a,b],\infty} \\
&= \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f'\|_{[a,b],\infty}
\end{aligned}$$

for $x \in [a,b]$, *provided* $f' \in L_\infty[a,b]$,

$$\begin{aligned}
(5.14)\qquad & \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
&\leq \frac{1}{(b-a)(\alpha+1)^{1/\alpha}}\left[(x-a)^{1+1/\alpha} \|f'\|_{[a,x],\beta} + (b-x)^{1+1/\alpha} \|f'\|_{[x,b],\beta} \right] \\
&\leq \frac{1}{(\alpha+1)^{1/\alpha}}\left[\left(\frac{x-a}{b-a}\right)^{1+1/\alpha} + \left(\frac{b-x}{b-a}\right)^{1+1/\alpha} \right] (b-a)^{1/\alpha} \|f'\|_{[a,b],\beta}
\end{aligned}$$

for $x \in [a,b]$, *provided* $f' \in L_\beta[a,b], \alpha > 1, 1/\alpha + 1/\beta = 1$ *and*

$$\begin{aligned}
(5.15)\qquad & \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
&\leq \frac{1}{b-a}\left[(x-a) \|f'\|_{[a,x],1} + (b-x) \|f'\|_{[x,b],1} \right] \\
&= \left\{ \frac{1}{2} + \left| \frac{x-\frac{a+b}{2}}{b-a} \right| \right\} \|f'\|_{[a,b],1}
\end{aligned}$$

for $x \in [a,b]$.

5.2. **Applications for p-Norms.** We have the following inequalities for Lebesgue norms of the deviation of a function from its integral mean:

Theorem 9 (Dragomir, 2013 [10]). *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$.*

(i) *If $f'\in L_{\infty}[a,b]$, then*

$$\left\| f-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],p}\le\left[\frac{2}{(p+1)(p+2)}\right]^{1/p}(b-a)^{1+\frac{1}{p}}\|f'\|_{[a,b],\infty}. \tag{5.16}$$

(ii) *If $f'\in L_{p\beta}[a,b]$, with $\alpha>1, 1/\alpha+1/\beta=1$ then*

$$\begin{aligned}&\left\| f-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],p}\\ &\le\left[\frac{2}{(p\alpha+1)^{1/\alpha}(p+1/\alpha+1)}\right]^{1/p}\|f'\|_{[a,b],p\beta}(b-a)^{1+\frac{1}{\alpha p}}.\end{aligned} \tag{5.17}$$

(iii) *We have*

$$\left\| f-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],p}\le\frac{1}{2}\left(\frac{2^{p+1}-1}{p+1}\right)^{1/p}(b-a)\|f'\|_{[a,b],p}. \tag{5.18}$$

Proof. Integrating on $[a,b]$ the inequality (5.10) we have

$$\begin{aligned}&\int_a^b\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|^p dx\\ &\le\frac{1}{(b-a)(p+1)}\|f'\|_{[a,b],\infty}^p\int_a^b\left[(x-a)^{p+1}+(b-x)^{p+1}\right]dx\\ &=\frac{1}{(b-a)(p+1)}\|f'\|_{[a,b],\infty}^p\left[\frac{2(b-a)^{p+2}}{p+2}\right]\\ &=\frac{2}{(p+1)(p+2)}\|f'\|_{[a,b],\infty}^p(b-a)^{p+1}\end{aligned} \tag{5.19}$$

which is equivalent with (5.16).

Integrating the inequality (5.11)

$$\begin{aligned}&\int_a^b\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|^p dx\\ &\le\frac{1}{(b-a)(p\alpha+1)^{1/\alpha}}\|f'\|_{[a,b],p\beta}^p\int_a^b\left[(x-a)^{p+1/\alpha}+(b-x)^{p+1/\alpha}\right]dx\\ &=\frac{1}{(b-a)(p\alpha+1)^{1/\alpha}}\|f'\|_{[a,b],p\beta}^p\left[\frac{2(b-a)^{p+1/\alpha+1}}{p+1/\alpha+1}\right]\\ &=\frac{2}{(p\alpha+1)^{1/\alpha}(p+1/\alpha+1)}\|f'\|_{[a,b],p\beta}^p(b-a)^{p+1/\alpha}\end{aligned} \tag{5.20}$$

which is equivalent with (5.17).

Integrating the inequality (5.12) we have

$$\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|^p dx \leq \frac{1}{b-a}\|f'\|_{[a,b],p}^p \int_a^b \max\{(x-a)^p, (b-x)^p\}\,dx. \tag{5.21}$$

Since

$$\begin{aligned}
&\int_a^b \max\{(x-a)^p, (b-x)^p\}\,dx \\
&= \int_a^{\frac{a+b}{2}} (b-x)^p\,dx + \int_{\frac{a+b}{2}}^b (x-a)^p\,dx \\
&= -\frac{\left(\frac{b-a}{2}\right)^{p+1}}{p+1} + \frac{(b-a)^{p+1}}{p+1} + \frac{(b-a)^{p+1}}{p+1} - \frac{\left(\frac{b-a}{2}\right)^{p+1}}{p+1} \\
&= \frac{1}{p+1}\left(\frac{2^{p+1}-1}{2^p}\right)(b-a)^{p+1}
\end{aligned}$$

then from (5.21) we get

$$\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|^p dx \leq \frac{1}{p+1}\left(\frac{2^{p+1}-1}{2^p}\right)(b-a)^p \|f'\|_{[a,b],p}^p,$$

which is equivalent with (5.18). □

5.3. **Applications for the Exponential.** If we write the inequality (5.1) for the convex function $\Phi(x) = \exp(x)$ then we get the inequality

$$\exp\left[f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right] \leq \frac{1}{b-a}\left[\int_a^x \exp[(t-a)f'(t)]\,dt + \int_x^b \exp[(t-b)f'(t)]\,dt\right] \tag{5.22}$$

for $x \in [a,b]$.

If we write the inequality (5.1) for the convex function $\Phi(x) = \cosh(x) := \frac{e^x + e^{-x}}{2}$ then we get the inequality

$$\cosh\left[f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right] \leq \frac{1}{b-a}\left[\int_a^x \cosh[(t-a)f'(t)]\,dt + \int_x^b \cosh[(t-b)f'(t)]\,dt\right] \tag{5.23}$$

for $x \in [a,b]$.

Utilising the inequality (5.22) we have the following multiplicative version of Ostrowski's inequality:

Theorem 10 (Dragomir, 2013 [10])**.** *Let $f:[a,b]\to(0,\infty)$ be absolutely continuous on $[a,b]$. Then we have the inequalities*

$$\frac{f(x)}{\exp\left[\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right]} \leq \frac{1}{b-a}\left[\int_a^x \exp\left[(t-a)\frac{f'(t)}{f(t)}\right]dt+\int_x^b \exp\left[(t-b)\frac{f'(t)}{f(t)}\right]dt\right] \tag{5.24}$$

for any $x\in[a,b]$ and

$$\frac{\int_a^b f(x)\,dx}{\exp\left[\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right]} \leq \frac{1}{b-a}\left[\int_a^b (b-x)\exp\left[(x-a)\frac{f'(x)}{f(x)}\right]dx + \int_a^b (x-a)\exp\left[(x-b)\frac{f'(x)}{f(x)}\right]dx\right]. \tag{5.25}$$

Proof. If we replace f by $\ln f$ in (5.22) we get

$$\exp\left[\ln f(x)-\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right] \leq \frac{1}{b-a}\left[\int_a^x \exp\left[(t-a)\frac{f'(t)}{f(t)}\right]dt+\int_x^b \exp\left[(t-b)\frac{f'(t)}{f(t)}\right]dt\right] \tag{5.26}$$

for any $x\in[a,b]$.

Since

$$\begin{aligned}
&\exp\left[\ln f(x)-\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right] \\
&=\exp\left[\ln f(x)-\ln\left\{\exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right)\right\}\right] \\
&=\exp\left[\ln\left(\frac{f(x)}{\exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right)}\right)\right] \\
&=\frac{f(x)}{\exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right)}
\end{aligned}$$

for any $x\in[a,b]$, then we get from (5.26) the desired inequality (5.24).

If we integrate the inequality (5.24) we get

$$\frac{\int_a^b f(x)\,dx}{\exp\left[\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right]} \leq \frac{1}{b-a}\int_a^b\left[\int_a^x \exp\left[(t-a)\frac{f'(t)}{f(t)}\right]dt+\int_x^b \exp\left[(t-b)\frac{f'(t)}{f(t)}\right]dt\right]dx. \tag{5.27}$$

Integrating by parts we have

$$\begin{aligned}
&\int_a^b \left(\int_a^x \exp\left[(t-a)\frac{f'(t)}{f(t)}\right]dt\right)dx \\
&= x\int_a^x \exp\left[(t-a)\frac{f'(t)}{f(t)}\right]dt\Big|_a^b - \int_a^b x d\left(\int_a^x \exp\left[(t-a)\frac{f'(t)}{f(t)}\right]dt\right) \\
&= b\int_a^b \exp\left[(t-a)\frac{f'(t)}{f(t)}\right]dt - \int_a^b x\exp\left[(x-a)\frac{f'(x)}{f(x)}\right]dx \\
&= \int_a^b (b-x)\exp\left[(x-a)\frac{f'(x)}{f(x)}\right]dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b \left(\int_x^b \exp\left[(t-b)\frac{f'(t)}{f(t)}\right]dt\right)dx \\
&= x\int_x^b \exp\left[(t-b)\frac{f'(t)}{f(t)}\right]dt\Big|_a^b - \int_a^b x d\left(\int_x^b \exp\left[(t-b)\frac{f'(t)}{f(t)}\right]dt\right) \\
&= -a\int_a^b \exp\left[(t-b)\frac{f'(t)}{f(t)}\right]dt + \int_a^b x\exp\left[(x-b)\frac{f'(x)}{f(x)}\right]dx \\
&= \int_a^b (x-a)\exp\left[(x-b)\frac{f'(x)}{f(x)}\right]dx,
\end{aligned}$$

then by (5.27) we deduce the desired inequality (5.25). □

5.4. **Applications for Midpoint-Inequalities.** We have from the inequality (5.1) written for $-f$ the following result:

Proposition 1 (Dragomir, 2013 [10])**.** *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. If $\Phi:\mathbb{R}\to\mathbb{R}$ is convex (concave) on $\mathbb{R}$ then from (5.1) we have the inequalities*

$$\begin{aligned}
&\Phi\left(\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right) \\
&\leq (\geq)\,\frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^b \Phi\left[(b-t)f'(t)\right]dt + \int_a^{\frac{a+b}{2}} \Phi\left[(a-t)f'(t)\right]dt\right].
\end{aligned} \tag{5.28}$$

If $f:[a,b]\to\mathbb{R}$ is convex on $[a,b]$, then by Hermite-Hadamard inequality we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt \geq f\left(\frac{a+b}{2}\right).$$

We can state the following result in which the function Φ is assumed be convex only on $[0,\infty)$ or $(0,\infty)$.

Proposition 2. *If $f:[a,b]\to\mathbb{R}$ is convex on $[a,b]$, monotonic nondecreasing on $\left[a,\frac{a+b}{2}\right]$ and monotonic nonincreasing $\left[a,\frac{a+b}{2}\right]$ .If $\Phi:[0,\infty),(0,\infty)\to\mathbb{R}$ is convex (concave) on $[0,\infty)$ or $(0,\infty)$, then (5.28) holds true.*

If $f:[a,b]\to\mathbb{R}$ is strictly convex on $[a,b]$, monotonic nondecreasing on $\left[a,\frac{a+b}{2}\right]$ and monotonic nonincreasing $\left[a,\frac{a+b}{2}\right]$, then by taking $\Phi(x)=\ln x$, which is strictly concave on $(0,\infty)$, we get the logarithmic inequality

$$\ln\left(\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\right) \geq \frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^b \ln\left[(b-t)f'(t)\right]dt+\int_a^{\frac{a+b}{2}}\ln\left[(a-t)f'(t)\right]dt\right]. \tag{5.29}$$

If $f:[a,b]\to\mathbb{R}$ is convex on $[a,b]$, monotonic nondecreasing on $\left[a,\frac{a+b}{2}\right]$ and monotonic nonincreasing $\left[a,\frac{a+b}{2}\right]$, then by taking $\Phi(x)=x^q$, with $q\in(0,1)$ we also have

$$\left(\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\right)^q \geq \frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^b \left[(b-t)f'(t)\right]^q dt+\int_a^{\frac{a+b}{2}}\left[(a-t)f'(t)\right]^q dt\right]. \tag{5.30}$$

If $\Phi:[0,\infty),(0,\infty)\to\mathbb{R}$ is convex (concave) on $[0,\infty)$ or $(0,\infty)$, and if we take $f(t):=\left|t-\frac{a+b}{2}\right|^p$, $p\geq 1$, then we get from (5.28)

$$\Phi\left(\frac{(b-a)^p}{2^p(p+1)}\right)\leq(\geq)\frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^b\Phi\left[p(b-t)\left(t-\frac{a+b}{2}\right)^{p-1}\right]dt +\int_a^{\frac{a+b}{2}}\Phi\left[(t-a)\left(\frac{a+b}{2}-t\right)^{p-1}\right]dt\right]. \tag{5.31}$$

Assume that $\Phi:\mathbb{R}\to\mathbb{R}$ is convex (concave) on $\mathbb{R}$.

Now, if we take $f(t)=\frac{1}{t}$ in (5.28), where $t\in[a,b]\subset(0,\infty)$, then we have

$$\Phi\left(\frac{A(a,b)-L(a,b)}{A(a,b)L(a,b)}\right) \leq(\geq)\frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^b\Phi\left(\frac{t-b}{t^2}\right)dt+\int_a^{\frac{a+b}{2}}\Phi\left(\frac{t-a}{t^2}\right)dt\right]. \tag{5.32}$$

If we take $f(t)=-\ln t$ in (5.28), where $t\in[a,b]\subset(0,\infty)$, then we have

$$\Phi\left(\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right) \leq(\geq)\frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^b\Phi\left(\frac{t-b}{t}\right)dt+\int_a^{\frac{a+b}{2}}\Phi\left(\frac{t-a}{t}\right)dt\right]. \tag{5.33}$$

If we take $f(t)=t^p$, $p\in\mathbb{R}\setminus\{0,-1\}$ in (5.28), where $t\in[a,b]\subset(0,\infty)$, then we have

$$\Phi\left(L_p^p(a,b)-A^p(a,b)\right) \leq(\geq)\frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^b\Phi\left[p(b-t)t^{p-1}\right]dt+\int_a^{\frac{a+b}{2}}\Phi\left[p(a-t)t^{p-1}\right]dt\right]. \tag{5.34}$$

6. Perturbed Ostrowski Type Inequalities for Functions of Bounded Variation

6.1. **Some Identities.**

We start with the following identity that will play an important role in the following:

Lemma 1 (Dragomir, 2013 [11])**.** *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and $x\in[a,b]$. Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers, we have*

$$f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{6.1}$$
$$=\frac{1}{b-a}\int_a^x (t-a)\,d[f(t)-\lambda_1(x)t]+\frac{1}{b-a}\int_x^b (t-b)\,d[f(t)-\lambda_2(x)t],$$

where the integrals in the right hand side are taken in the Riemann-Stieltjes sense.

Proof. Utilising the integration by parts formula in the Riemann-Stieltjes integral, we have

$$\int_a^x (t-a)\,d[f(t)-\lambda_1(x)t] \tag{6.2}$$
$$=(t-a)[f(t)-\lambda_1(x)t]\Big|_a^x-\int_a^x[f(t)-\lambda_1(x)t]\,dt$$
$$=(x-a)[f(x)-\lambda_1(x)x]-\int_a^x f(t)\,dt+\frac{1}{2}\lambda_1(x)\left(x^2-a^2\right)$$
$$=(x-a)f(x)-\lambda_1(x)x(x-a)-\int_a^x f(t)\,dt+\frac{1}{2}\lambda_1(x)\left(x^2-a^2\right)$$
$$=(x-a)f(x)-\int_a^x f(t)\,dt-\frac{1}{2}(x-a)^2\lambda_1(x)$$

and

$$\int_x^b (t-b)\,d[f(t)-\lambda_2(x)t] \tag{6.3}$$
$$=(t-b)[f(t)-\lambda_2(x)t]\Big|_x^b-\int_x^b[f(t)-\lambda_2(x)t]\,dt$$
$$=(b-x)[f(x)-\lambda_2(x)x]-\int_x^b f(t)\,dt+\frac{1}{2}\lambda_2(x)\left(b^2-x^2\right)$$
$$=(b-x)f(x)-\int_x^b f(t)\,dt-(b-x)\lambda_2(x)x+\frac{1}{2}\lambda_2(x)\left(b^2-x^2\right)$$
$$=(b-x)f(x)-\int_x^b f(t)\,dt+\frac{1}{2}(b-x)^2\lambda_2(x).$$

By adding the equalities (6.2) and (6.3) and dividing by $b-a$ we get the desired representation (6.1). □

Corollary 11. *With the assumption in Lemma 1, we have for any $\lambda(x) \in \mathbb{C}$ that*

$$f(x) + \left(\frac{a+b}{2} - x\right)\lambda(x) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^x (t-a)\,d[f(t) - \lambda(x)t] + \frac{1}{b-a}\int_x^b (t-b)\,d[f(t) - \lambda(x)t]. \tag{6.4}$$

We have the following midpoint representation:

Corollary 12. *With the assumption in Lemma 1, we have for any $\lambda_1, \lambda_2 \in \mathbb{C}$ that*

$$\begin{aligned} &f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)(\lambda_2 - \lambda_1) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\,d[f(t) - \lambda_1 t] \\ &+ \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\,d[f(t) - \lambda_2 t]. \end{aligned} \tag{6.5}$$

In particular, if $\lambda_1 = \lambda_2 = \lambda$, then we have the equality

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\,d[f(t) - \lambda t] + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\,d[f(t) - \lambda t]. \tag{6.6}$$

Remark 6. *If we take $\lambda(x) = 0$ in (6.4) we recapture the Montgomery type identity established in* [2].

6.2. **Inequalities for Functions of Bounded Variation.** The following lemma will be used in the sequel and is of interest in itself as well [1, p. 177]. For a simple proof see also [9].

Lemma 2. *Let $f, u : [a,b] \to \mathbb{C}$. If f is continuous on $[a,b]$ and u is of bounded variation on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and*

$$\left|\int_a^b f(t)\,du(t)\right| \le \int_a^b |f(t)|\,d\left(\bigvee_a^t (u)\right) \le \max_{t\in[a,b]} |f(t)| \bigvee_a^b (u). \tag{6.7}$$

We denote by $\ell : [a,b] \to [a,b]$ the identity function, namely $\ell(t) = t$ for any $t \in [a,b]$.

We have the following result:

Theorem 11 (Dragomir, 2013 [11])**.** *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$ and $x \in [a,b]$. Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers,* *we have the inequality*

$$\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{6.8}$$
$$\leq \frac{1}{b-a}\left[\int_a^x\left(\bigvee_t^x(f-\lambda_1(x)\ell)\right)dt+\int_x^b\left(\bigvee_x^t(f-\lambda_2(x)\ell)\right)dt\right]$$
$$\leq \frac{1}{b-a}\left[(x-a)\bigvee_a^x(f-\lambda_1(x)\ell)+(b-x)\bigvee_x^b(f-\lambda_2(x)\ell)\right]$$
$$\leq\begin{cases}\max\left\{\bigvee_a^x(f-\lambda_1(x)\ell),\bigvee_x^b(f-\lambda_2(x)\ell)\right\}\\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left(\bigvee_a^x(f-\lambda_1(x)\ell)+\bigvee_x^b(f-\lambda_2(x)\ell)\right),\end{cases}$$

where $\bigvee_c^d(g)$ *denotes the total variation of* g *on the interval* $[c,d]$.

Proof. Taking the modulus in (6.1) and using the property (6.7) we have

$$\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{6.9}$$
$$\leq\frac{1}{b-a}\left|\int_a^x(t-a)\,d\left[f(t)-\lambda_1(x)t\right]\right|$$
$$+\frac{1}{b-a}\left|\int_x^b(t-b)\,d\left[f(t)-\lambda_2(x)t\right]\right|$$
$$\leq\frac{1}{b-a}\int_a^x(t-a)\,d\left(\bigvee_a^t(f-\lambda_1(x)\ell)\right)$$
$$+\frac{1}{b-a}\int_x^b(b-t)\,d\left(\bigvee_a^t(f-\lambda_2(x)\ell)\right).$$

Integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\int_a^x (t-a)\,d\left(\bigvee_a^t (f-\lambda_1(x)\ell)\right) \\
&= (t-a)\bigvee_a^t (f-\lambda_1(x)\ell)\Big|_a^x - \int_a^x \left(\bigvee_a^t (f-\lambda_1(x)\ell)\right)dt \\
&= (x-a)\bigvee_a^x (f-\lambda_1(x)\ell) - \int_a^x \left(\bigvee_a^t (f-\lambda_1(x)\ell)\right)dt \\
&= \int_a^x \left(\bigvee_t^x (f-\lambda_1(x)\ell)\right)dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_x^b (b-t)\,d\left(\bigvee_a^t (f-\lambda_2(x)\ell)\right) \\
&= (b-t)\bigvee_a^t (f-\lambda_2(x)\ell)\Big|_x^b + \int_x^b \left(\bigvee_a^t (f-\lambda_2(x)\ell)\right)dt \\
&= \int_x^b \left(\bigvee_a^t (f-\lambda_2(x)\ell)\right)dt - (b-x)\bigvee_a^x (f-\lambda_2(x)\ell) \\
&= \int_x^b \left(\bigvee_x^t (f-\lambda_2(x)\ell)\right)dt.
\end{aligned}$$

Using (6.9) we deduce the first inequality in (6.8).

We also have

$$\int_a^x \left(\bigvee_t^x (f-\lambda_1(x)\ell)\right)dt \le (x-a)\bigvee_a^x (f-\lambda_1(x)\ell)$$

and

$$\int_x^b \left(\bigvee_x^t (f-\lambda_2(x)\ell)\right)dt \le (b-x)\bigvee_x^b (f-\lambda_2(x)\ell),$$

which prove the second inequality in (6.8).

The last part is obvious. □

The following result holds:

Corollary 13. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and $x\in[a,b]$. Then for any $\lambda(x)$ a complex number, we have the inequality*

$$\begin{aligned}(6.10)\quad &\left|f(x)+\left(\frac{a+b}{2}-x\right)\lambda(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ &\le\frac{1}{b-a}\left[\int_a^x\left(\bigvee_t^x(f-\lambda(x)\ell)\right)dt+\int_x^b\left(\bigvee_x^t(f-\lambda(x)\ell)\right)dt\right]\\ &\le\frac{1}{b-a}\left[(x-a)\bigvee_a^x(f-\lambda(x)\ell)+(b-x)\bigvee_x^b(f-\lambda(x)\ell)\right]\\ &\le\begin{cases}\frac{1}{2}\bigvee_a^b(f-\lambda(x)\ell)+\frac{1}{2}\left|\bigvee_x^b(f-\lambda(x)\ell)-\bigvee_a^x(f-\lambda(x)\ell)\right|\\[2ex] \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f-\lambda(x)\ell).\end{cases}\end{aligned}$$

Remark 7. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Then for any $\lambda\in\mathbb{C}$ we have the inequalities*

$$\begin{aligned}(6.11)\quad &\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ &\le\frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_t^{\frac{a+b}{2}}(f-\lambda\ell)\right)dt+\int_{\frac{a+b}{2}}^b\left(\bigvee_{\frac{a+b}{2}}^t(f-\lambda\ell)\right)dt\right]\\ &\le\frac{1}{2}\bigvee_a^b(f-\lambda\ell).\end{aligned}$$

This is equivalent to

$$\begin{aligned}(6.12)\quad &\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ &\le\frac{1}{b-a}\inf_{\lambda\in\mathbb{C}}\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_t^{\frac{a+b}{2}}(f-\lambda\ell)\right)dt+\int_{\frac{a+b}{2}}^b\left(\bigvee_{\frac{a+b}{2}}^t(f-\lambda\ell)\right)dt\right]\\ &\le\frac{1}{2}\inf_{\lambda\in\mathbb{C}}\left[\bigvee_a^b(f-\lambda\ell)\right].\end{aligned}$$

6.3. **Inequalities for Lipshitzian Functions.** We can state the following result:

Theorem 12 (Dragomir, 2013 [11])**.** *Let $f:[a,b]\to\mathbb{C}$ be a bounded function on $[a,b]$ and $x\in(a,b)$. If $\lambda_1(x)$ and $\lambda_2(x)$ are complex numbers and there exist the positive numbers $L_1(x)$ and $L_2(x)$ such that $f-\lambda_1(x)\ell$ is Lipschitzian with the constant $L_1(x)$ on the interval $[a,x]$ and $f-\lambda_2(x)\ell$ is Lipschitzian with the* *constant* $L_2(x)$ *on the interval* $[x,b]$, *then*

$$\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{6.13}$$

$$\leq \frac{1}{2}\left[\left(\frac{x-a}{b-a}\right)^2 L_1(x)+\left(\frac{b-x}{b-a}\right)^2 L_2(x)\right](b-a)$$

$$\leq \begin{cases} \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\max\{L_1(x),L_2(x)\}(b-a), \\ \\ \frac{1}{2}\left[\left(\frac{x-a}{b-a}\right)^{2q}+\left(\frac{b-x}{b-a}\right)^{2q}\right]^{1/q}\left(L_1^p(x)+L_2^p(x)\right)^{1/p}(b-a), \\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\frac{L_1(x)+L_2(x)}{2}(b-a). \end{cases}$$

Proof. It is known that, if $g:[c,d]\to\mathbb{C}$ is Riemann integrable and $u:[c,d]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, then the Riemann-Stieltjes integral $\int_c^d g(t)\,du(t)$ exists and

$$\left|\int_c^d g(t)\,du(t)\right|\leq L\int_c^d |g(t)|\,dt. \tag{6.14}$$

Taking the modulus in (6.1) and using the property (6.14) we have

$$\begin{aligned} &\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{b-a}\left|\int_a^x (t-a)\,d\left[f(t)-\lambda_1(x)t\right]\right| \\ &+\frac{1}{b-a}\left|\int_x^b (t-b)\,d\left[f(t)-\lambda_2(x)t\right]\right| \\ &\leq \frac{1}{b-a}\left[L_1(x)\int_a^x (t-a)\,dt+L_2(x)\int_x^b (b-t)\,dt\right] \\ &=\frac{L_1(x)(x-a)^2+L_2(x)(b-x)^2}{2(b-a)} \\ &=\frac{1}{2}\left[L_1(x)\left(\frac{x-a}{b-a}\right)^2+L_2(x)\left(\frac{b-x}{b-a}\right)^2\right](b-a), \end{aligned}$$

and the first inequality in (6.13) is proved.

By Hölder's inequality we have

$$L_1(x)\left(\frac{x-a}{b-a}\right)^2+L_2(x)\left(\frac{b-x}{b-a}\right)^2$$

$$\leq\begin{cases}\left[\left(\frac{x-a}{b-a}\right)^2+\left(\frac{b-x}{b-a}\right)^2\right]\max\{L_1(x),L_2(x)\}\\ \\ \left[\left(\frac{x-a}{b-a}\right)^{2q}+\left(\frac{b-x}{b-a}\right)^{2q}\right]^{1/q}\left(L_1^p(x)+L_2^p(x)\right)^{1/p},\\ p>1,\ \frac{1}{p}+\frac{1}{q}=1\\ \\ \max\left\{\left(\frac{x-a}{b-a}\right)^2,\left(\frac{b-x}{b-a}\right)^2\right\}[L_1(x)+L_2(x)],\end{cases}$$

which proves, upon simple calculations, the last part of the inequality (6.13). □

Corollary 14. *Let $f:[a,b]\to\mathbb{C}$ be a bounded function on $[a,b]$ and $x\in(a,b)$. If $\lambda(x)$ is a complex number and there exist the positive number $L(x)$ such that $f-\lambda(x)\ell$ is Lipschitzian with the constant $L(x)$ on the interval $[a,b]$, then*

$$\left|f(x)+\left(\frac{a+b}{2}-x\right)\lambda(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]L(x)(b-a). \tag{6.15}$$

Remark 8. *If λ is a complex number and there exist the positive number L such that $f-\lambda\ell$ is Lipschitzian with the constant L on the interval $[a,b]$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{4}L(b-a). \tag{6.16}$$

6.4. **Inequalities for Monotonic Functions.** Now, the case of monotonic integrators is as follows:

Theorem 13 (Dragomir, 2013 [11])**.** *Let $f:[a,b]\to\mathbb{R}$ be a bounded function on $[a,b]$ and $x\in(a,b)$. If $\lambda_1(x)$ and $\lambda_2(x)$ are real numbers such that $f-\lambda_1(x)\ell$ is monotonic nondecreasing on the interval $[a,x]$ and $f-\lambda_2(x)\ell$ is monotonic* *nondecreasing on the interval* $[x,b]$, *then*

$$\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{6.17}$$
$$\le \frac{1}{b-a}\left[(2x-a-b)f(x)+\int_a^b sgn(t-x)f(t)\,dt\right.$$
$$\left.-\frac{1}{2}\left[\lambda_1(x)(x-a)^2+\lambda_2(x)(b-x)^2\right]\right]$$
$$\le \frac{1}{b-a}\{(x-a)[f(x)-f(a)-\lambda_1(x)(x-a)]$$
$$+(b-x)[f(b)-f(x)-\lambda_2(x)(b-x)]\}$$
$$\le \begin{cases} \frac{1}{2}[f(b)-f(a)-\lambda_1(x)(x-a)-\lambda_2(x)(b-x)] \\ +\left|f(x)-\frac{f(a)+f(b)}{2}-\frac{1}{2}\lambda_1(x)(x-a)+\frac{1}{2}\lambda_2(x)(b-x)\right|, \\ \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right] \\ \times[f(b)-f(a)-\lambda_1(x)(x-a)-\lambda_2(x)(b-x)]. \end{cases}$$

Proof. It is known that, if $g:[c,d]\to\mathbb{C}$ is continuous and $u:[c,d]\to\mathbb{C}$ is monotonic nondecreasing, then the Riemann-Stieltjes integral $\int_c^d g(t)\,du(t)$ exists and

$$\left|\int_c^d g(t)\,du(t)\right|\le\int_c^d |g(t)|\,du(t). \tag{6.18}$$

Taking the modulus in (6.1) and using the property (6.18) we have

$$\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{6.19}$$
$$\le\frac{1}{b-a}\left|\int_a^x (t-a)\,d[f(t)-\lambda_1(x)t]\right|$$
$$+\frac{1}{b-a}\left|\int_x^b (t-b)\,d[f(t)-\lambda_2(x)t]\right|$$
$$\le\frac{1}{b-a}\int_a^x (t-a)\,d[f(t)-\lambda_1(x)t]$$
$$+\frac{1}{b-a}\int_x^b (b-t)\,d[f(t)-\lambda_2(x)t].$$

Integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned}
&\int_a^x (t-a)\, d\left[f(t)-\lambda_1(x)\,t\right] \\
&= (t-a)\left[f(t)-\lambda_1(x)\,t\right]\Big|_a^x - \int_a^x \left[f(t)-\lambda_1(x)\,t\right] dt \\
&= (x-a)\left[f(x)-\lambda_1(x)\,x\right] - \int_a^x \left[f(t)-\lambda_1(x)\,t\right] dt \\
&= (x-a)\,f(x) - \lambda_1(x)\,x\,(x-a) - \int_a^x f(t)\,dt + \lambda_1(x)\,\frac{x^2-a^2}{2} \\
&= (x-a)\,f(x) - \int_a^x f(t)\,dt - \frac{1}{2}\lambda_1(x)\,(x-a)^2
\end{aligned}$$

and

$$\begin{aligned}
&\int_x^b (b-t)\, d\left[f(t)-\lambda_2(x)\,t\right] \\
&= (b-t)\left[f(t)-\lambda_2(x)\,t\right]\Big|_x^b + \int_x^b \left[f(t)-\lambda_2(x)\,t\right] dt \\
&= \int_x^b f(t)\,dt - \lambda_2(x)\int_x^b t\,dt - (b-x)\left[f(x)-\lambda_2(x)\,x\right] \\
&= \int_x^b f(t)\,dt - \lambda_2(x)\,\frac{b^2-x^2}{2} - (b-x)\,f(x) + (b-x)\,\lambda_2(x)\,x \\
&= \int_x^b f(t)\,dt - (b-x)\,f(x) - \frac{1}{2}\lambda_2(x)\,(b-x)^2 .
\end{aligned}$$

If we add these equalities, we get

$$\begin{aligned}
&\int_a^x (t-a)\, d\left[f(t)-\lambda_1(x)\,t\right] + \int_x^b (b-t)\, d\left[f(t)-\lambda_2(x)\,t\right] \\
&= (x-a)\,f(x) - \int_a^x f(t)\,dt - \frac{1}{2}\lambda_1(x)\,(x-a)^2 \\
&+ \int_x^b f(t)\,dt - (b-x)\,f(x) - \frac{1}{2}\lambda_2(x)\,(b-x)^2 \\
&= (2x-a-b)\,f(x) + \int_a^b sgn(t-x)\,f(t)\,dt \\
&- \frac{1}{2}\left[\lambda_1(x)\,(x-a)^2 + \lambda_2(x)\,(b-x)^2\right]
\end{aligned}$$

and by (6.19) we get the first inequality in (6.17).

Now, since $f-\lambda_1(x)\,\ell$ is monotonic nondecreasing on the interval $[a,x]$, then

$$\begin{aligned}
&\int_a^x (t-a)\, d\left[f(t)-\lambda_1(x)\,t\right] \\
&\leq (x-a)\left[f(x)-\lambda_1(x)\,x-f(a)+\lambda_1(x)\,a\right] \\
&= (x-a)\left[f(x)-f(a)-\lambda_1(x)\,(x-a)\right]
\end{aligned}$$

and, since $f-\lambda_2(x)\ell$ is monotonic nondecreasing on the interval $[x,b]$, then also

$$\begin{aligned}&\int_x^b (b-t)\,d[f(t)-\lambda_2(x)t]\\ &\leq (b-x)[f(b)-\lambda_2(x)b-f(x)+\lambda_2(x)x]\\ &=(b-x)[f(b)-f(x)-\lambda_2(x)(b-x)].\end{aligned}$$

These prove the second inequality in (6.17).

The last part follows by the properties of maximum and the details are omitted. □

Corollary 15. *Let $f:[a,b]\to\mathbb{R}$ be a bounded function on $[a,b]$ and $x\in(a,b)$. If $\lambda(x)$ is a real number such that $f-\lambda(x)\ell$ is monotonic nondecreasing on the interval $[a,b]$, then*

$$\begin{aligned}&\left|f(x)+\left(\frac{a+b}{2}-x\right)\lambda(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \quad (6.20)\\ &\leq\frac{1}{b-a}\left[(2x-a-b)f(x)+\int_a^b sgn(t-x)f(t)\,dt\right.\\ &\left.-\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2\lambda(x)\right]\\ &\leq\frac{1}{b-a}\{(x-a)[f(x)-f(a)-\lambda(x)(x-a)]\\ &+(b-x)[f(b)-f(x)-\lambda(x)(b-x)]\}\\ &\leq\begin{cases}\frac{f(b)-f(a)}{2}-\frac{1}{2}\lambda(x)(b-a)\\ +\left|f(x)-\frac{f(a)+f(b)}{2}-\frac{1}{2}\lambda(x)(2x-a-b)\right|,\\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\\ \times[f(b)-f(a)-\lambda(x)(b-a)].\end{cases}\end{aligned}$$

Remark 9. *If λ is a real number such that $f-\lambda\ell$ is monotonic nondecreasing on the interval $[a,b]$, then*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \quad (6.21)\\ &\leq\frac{1}{b-a}\left[\int_a^b sgn\left(t-\frac{a+b}{2}\right)f(t)\,dt-\frac{1}{4}\lambda(b-a)^2\right]\\ &\leq\frac{1}{2}[f(b)-f(a)-\lambda(b-a)].\end{aligned}$$

7. Some Perturbed Ostrowski Type Inequalities for Absolutely Continuous Functions

7.1. **Some Identities.** We start with the following identity that will play an important role in the following:

Lemma 3 (Dragomir, 2013 [12])**.** *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous on $[a,b]$ and $x\in[a,b]$. Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers, we have*

$$f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \\ =\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\lambda_1(x)\right]dt+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-\lambda_2(x)\right]dt, \tag{7.1}$$

where the integrals in the right hand side are taken in the Lebesgue sense.

Proof. Utilising the integration by parts formula in the Lebesgue integral, we have

$$\begin{aligned} &\int_a^x (t-a)\left[f'(t)-\lambda_1(x)\right]dt \\ &= (t-a)\left[f(t)-\lambda_1(x)t\right]\Big|_a^x-\int_a^x\left[f(t)-\lambda_1(x)t\right]dt \\ &=(x-a)\left[f(x)-\lambda_1(x)x\right]-\int_a^x f(t)\,dt+\frac{1}{2}\lambda_1(x)\left(x^2-a^2\right) \\ &=(x-a)f(x)-\lambda_1(x)x(x-a)-\int_a^x f(t)\,dt+\frac{1}{2}\lambda_1(x)\left(x^2-a^2\right) \\ &=(x-a)f(x)-\int_a^x f(t)\,dt-\frac{1}{2}(x-a)^2\lambda_1(x) \end{aligned} \tag{7.2}$$

and

$$\begin{aligned} &\int_x^b (t-b)\left[f'(t)-\lambda_2(x)\right]dt \\ &= (t-b)\left[f(t)-\lambda_2(x)t\right]\Big|_x^b-\int_x^b\left[f(t)-\lambda_2(x)t\right]dt \\ &=(b-x)\left[f(x)-\lambda_2(x)x\right]-\int_x^b f(t)\,dt+\frac{1}{2}\lambda_2(x)\left(b^2-x^2\right) \\ &=(b-x)f(x)-\int_x^b f(t)\,dt-(b-x)\lambda_2(x)x+\frac{1}{2}\lambda_2(x)\left(b^2-x^2\right) \\ &=(b-x)f(x)-\int_x^b f(t)\,dt+\frac{1}{2}(b-x)^2\lambda_2(x). \end{aligned} \tag{7.3}$$

If we add the identities (7.2) and (7.3) and divide by $b-a$ we deduce the desired identity (7.1). □

Corollary 16. *With the assumption in Lemma 3, we have for any $\lambda(x)\in\mathbb{C}$ that*

$$f(x)+\left(\frac{a+b}{2}-x\right)\lambda(x)-\frac{1}{b-a}\int_a^b f(t)\,dt \\ =\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\lambda(x)\right]dt+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-\lambda(x)\right]dt. \tag{7.4}$$

Remark 10. *If we take $\lambda(x) = 0$ in (7.4), then we get Montgomery's identity for absolutely continuous functions, i.e.*

$$f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^x (t-a) f'(t)\,dt + \frac{1}{b-a}\int_x^b (t-b) f'(t)\,dt, \tag{7.5}$$

for $x \in [a,b]$.

We have the following midpoint representation:

Corollary 17. *With the assumption in Lemma 3, we have for any $\lambda_1, \lambda_2 \in \mathbb{C}$ that*

$$f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)(\lambda_2 - \lambda_1) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)[f'(t) - \lambda_1]\,dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)[f'(t) - \lambda_2]\,dt. \tag{7.6}$$

In particular, if $\lambda_1 = \lambda_2 = \lambda$, then we have the equality

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)[f'(t) - \lambda]\,dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)[f'(t) - \lambda]\,dt. \tag{7.7}$$

Remark 11. *The identity (7.1) has many particular cases of interest.*

If we assume that the derivatives $f'_+(a)$, $f'_-(b)$ and $f'(x)$ exist and are finite, then by taking

$$\lambda_1(x) = \frac{f'_+(a) + f'(x)}{2} \text{ and } \lambda_2(x) = \frac{f'(x) + f'_-(b)}{2}$$

in (7.1) we get

$$\begin{aligned} & f(x) + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ & + \frac{1}{4(b-a)}\left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a)\right] \\ & = \frac{1}{b-a}\int_a^x (t-a)\left[f'(t) - \frac{f'_+(a) + f'(x)}{2}\right] dt \\ & + \frac{1}{b-a}\int_x^b (t-b)\left[f'(t) - \frac{f'(x) + f'_-(b)}{2}\right] dt. \end{aligned} \tag{7.8}$$

In particular, we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) + \frac{1}{16}(b-a)\left[f'_-(b) - f'_+(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \\ & = \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t) - \frac{f'_+(a) + f'\left(\frac{a+b}{2}\right)}{2}\right] dt \\ & + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t) - \frac{f'\left(\frac{a+b}{2}\right) + f'_-(b)}{2}\right] dt. \end{aligned} \tag{7.9}$$

7.2. **Inequalities for Bounded Derivatives.** Now, for $\gamma, \Gamma \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\begin{aligned}&\bar{U}_{[a,b]}(\gamma,\Gamma)\\&:=\left\{f:[a,b]\to\mathbb{C}|\operatorname{Re}\left[(\Gamma-f(t))\left(\overline{f(t)}-\overline{\gamma}\right)\right]\geq 0 \text{ for almost every } t\in[a,b]\right\}\end{aligned}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma):=\left\{f:[a,b]\to\mathbb{C}|\ \left|f(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|\ \text{ for a.e. } t\in[a,b]\right\}.$$

The following representation result may be stated.

Proposition 3 (Dragomir, 2013 [12]). *For any $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, we have that $\bar{U}_{[a,b]}(\gamma,\Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma)=\bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{7.10}$$

Proof. We observe that for any $z\in\mathbb{C}$ we have the equivalence

$$\left|z-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma-z)(\bar{z}-\bar{\gamma})\right]\geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma-\gamma|^2-\left|z-\frac{\gamma+\Gamma}{2}\right|^2=\operatorname{Re}\left[(\Gamma-z)(\bar{z}-\bar{\gamma})\right]$$

that holds for any $z\in\mathbb{C}$.

The equality (7.10) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 18. *For any $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, we have that*

$$\begin{aligned}\bar{U}_{[a,b]}(\gamma,\Gamma)=\{f:[a,b]\to\mathbb{C}\ |\ &(\operatorname{Re}\Gamma-\operatorname{Re}f(t))(\operatorname{Re}f(t)-\operatorname{Re}\gamma)\\&+(\operatorname{Im}\Gamma-\operatorname{Im}f(t))(\operatorname{Im}f(t)-\operatorname{Im}\gamma)\geq 0 \textit{ for a.e. } t\in[a,b]\}.\end{aligned} \tag{7.11}$$

Now, if we assume that $\operatorname{Re}(\Gamma)\geq\operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma)\geq\operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_{[a,b]}(\gamma,\Gamma):=\{f:[a,b]\to\mathbb{C}\ |\ &\operatorname{Re}(\Gamma)\geq\operatorname{Re}f(t)\geq\operatorname{Re}(\gamma)\\&\text{and } \operatorname{Im}(\Gamma)\geq\operatorname{Im}f(t)\geq\operatorname{Im}(\gamma)\ \text{ for a.e. } t\in[a,b]\}.\end{aligned} \tag{7.12}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset\neq\bar{S}_{[a,b]}(\gamma,\Gamma)\subseteq\bar{U}_{[a,b]}(\gamma,\Gamma). \tag{7.13}$$

Theorem 14 (Dragomir, 2013 [12]). *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous on $[a,b]$ and $x\in(a,b)$. Suppose that $\gamma_i,\Gamma_i\in\mathbb{C}$ with $\gamma_i\neq\Gamma_i$, $i=1,2$ and $f'\in$*

$\bar{U}_{[a,x]}(\gamma_1,\Gamma_1)\cap\bar{U}_{[x,b]}(\gamma_2,\Gamma_2)$, *then we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right. \tag{7.14}$$
$$\left.+\frac{1}{2(b-a)}\left[(b-x)^2\frac{\Gamma_2+\gamma_2}{2}-(x-a)^2\frac{\Gamma_1+\gamma_1}{2}\right]\right|$$
$$\leq\frac{1}{4}\left[|\Gamma_1-\gamma_1|\left(\frac{x-a}{b-a}\right)^2+|\Gamma_2-\gamma_2|\left(\frac{b-x}{b-a}\right)^2\right](b-a)$$
$$\leq\frac{1}{4}(b-a)$$
$$\times\begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\max\left\{|\Gamma_1-\gamma_1|,|\Gamma_2-\gamma_2|\right\}.\\[2ex]\left[\left(\frac{x-a}{b-a}\right)^{2p}+\left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p}\left[|\Gamma_1-\gamma_1|^q+|\Gamma_2-\gamma_2|^q\right]^{1/q}\\ p>1,\frac{1}{p}+\frac{1}{q}=1,\\[2ex]\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[|\Gamma_1-\gamma_1|+|\Gamma_2-\gamma_2|\right].\end{cases}$$

Proof. Since $f'\in\bar{U}_{[a,x]}(\gamma_1,\Gamma_1)\cap\bar{U}_{[x,b]}(\gamma_2,\Gamma_2)$, then by taking the modulus in (7.1) for $\lambda_1(x)=\frac{\Gamma_1+\gamma_1}{2}$ and $\lambda_2(x)=\frac{\Gamma_2+\gamma_2}{2}$ we get

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right.$$
$$\left.+\frac{1}{2(b-a)}\left[(b-x)^2\frac{\Gamma_2+\gamma_2}{2}-(x-a)^2\frac{\Gamma_1+\gamma_1}{2}\right]\right|$$
$$\leq\frac{1}{b-a}\left|\int_a^x(t-a)\left[f'(t)-\frac{\Gamma_1+\gamma_1}{2}\right]dt\right|$$
$$+\frac{1}{b-a}\left|\int_x^b(t-b)\left[f'(t)-\frac{\Gamma_2+\gamma_2}{2}\right]dt\right|$$
$$\leq\frac{1}{b-a}\int_a^x(t-a)\left|f'(t)-\frac{\Gamma_1+\gamma_1}{2}\right|dt$$
$$+\frac{1}{b-a}\int_x^b(t-b)\left|f'(t)-\frac{\Gamma_2+\gamma_2}{2}\right|dt$$
$$\leq\frac{1}{b-a}\frac{|\Gamma_1-\gamma_1|}{2}\int_a^x(t-a)\,dt+\frac{1}{b-a}\frac{|\Gamma_2-\gamma_2|}{2}\int_x^b(b-t)\,dt$$
$$=\frac{1}{4}\left[|\Gamma_1-\gamma_1|\left(\frac{x-a}{b-a}\right)^2+|\Gamma_2-\gamma_2|\left(\frac{b-x}{b-a}\right)^2\right](b-a)$$

and the first inequality in (7.14) is proved.

The last part follows by Hölder's inequality

$$mn+pq\leq(m^\alpha+p^\alpha)^{1/\alpha}\left(n^\beta+q^\beta\right)^{1/\beta},$$

where $m,n,p,q\geq 0$ and $\alpha>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$. □

Corollary 19. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous on $[a,b]$ and $x\in(a,b)$. Suppose that $\gamma,\Gamma\in\mathbb{C}$ with $\gamma\neq\Gamma$, and $f'\in\bar{U}_{[a,b]}(\gamma,\Gamma)$, then we have*

$$\left|f(x)+\left(\frac{a+b}{2}-x\right)\frac{\Gamma+\gamma}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{2}\left|\Gamma-\gamma\right|\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a). \tag{7.15}$$

In particular, we have

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{8}\left|\Gamma-\gamma\right|(b-a). \tag{7.16}$$

Remark 12. *If the derivative $f':[a,b]\to\mathbb{R}$ is bounded above and below, that is, there exists the constants $M>m$ such that*

$$-\infty<m\leq f'(t)\leq M<\infty \text{ for a.e. } t\in[a,b],$$

then we recapture from (7.15) the inequality [7]

$$\left|f(x)+\left(\frac{a+b}{2}-x\right)\frac{M+m}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq\frac{1}{2}(M-m)\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a).$$

Remark 13. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous on $[a,b]$. Suppose that $\gamma_i,\Gamma_i\in\mathbb{C}$ with $\gamma_i\neq\Gamma_i$, $i=1,2$ and $f'\in\bar{U}_{\left[a,\frac{a+b}{2}\right]}(\gamma_1,\Gamma_1)\cap\bar{U}_{\left[\frac{a+b}{2},b\right]}(\gamma_2,\Gamma_2)$, then we have from (7.14) that*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{8}(b-a)\left(\frac{\Gamma_2+\gamma_2}{2}-\frac{\Gamma_1+\gamma_1}{2}\right)\right| \leq\frac{1}{16}\left[\left|\Gamma_1-\gamma_1\right|+\left|\Gamma_2-\gamma_2\right|\right](b-a). \tag{7.17}$$

7.3. **Inequalities for Derivatives of Bounded Variation.** Assume that the function $f:I\to\mathbb{C}$ is differentiable on the interior of I, denoted $\mathring{I}$, and $[a,b]\subset\mathring{I}$. Then, as in (7.8), we have the equality

$$\begin{aligned} &f(x)+\frac{1}{2}\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &+\frac{1}{4(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right] \\ &=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\frac{f'(a)+f'(x)}{2}\right]dt \\ &+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-\frac{f'(x)+f'(b)}{2}\right]dt, \end{aligned} \tag{7.18}$$

for any $x\in[a,b]$.

Theorem 15 (Dragomir, 2013 [12])**.** *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) \right. \\ \left. + \frac{1}{4(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] \right| \tag{7.19}$$

$$\le \frac{1}{4}\left[\left(\frac{x-a}{b-a}\right)^2 \bigvee_a^x (f') + \left(\frac{b-x}{b-a}\right)^2 \bigvee_x^b (f')\right](b-a)$$
$$\le \frac{1}{4}(b-a) \times \begin{cases} \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{2}\bigvee_a^b (f') + \frac{1}{2}\left|\bigvee_a^x (f') - \bigvee_x^b (f')\right|\right], \\ \\ \left[\left(\frac{x-a}{b-a}\right)^{2p} + \left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p}\left[\left[\bigvee_a^x (f')\right]^q + \left[\bigvee_x^b (f')\right]^q\right]^{1/q} \\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b (f'), \end{cases}$$

for any $x \in [a,b]$.

Proof. Taking the modulus in (7.18) we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) \right. \\ \left. + \frac{1}{4(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] \right| \\ \le \frac{1}{b-a}\left|\int_a^x (t-a)\left[f'(t) - \frac{f'(a)+f'(x)}{2}\right]dt\right| \\ + \frac{1}{b-a}\left|\int_x^b (t-b)\left[f'(t) - \frac{f'(x)+f'(b)}{2}\right]dt\right| \\ \le \frac{1}{b-a}\int_a^x (t-a)\left|f'(t) - \frac{f'(a)+f'(x)}{2}\right|dt \\ + \frac{1}{b-a}\int_x^b (b-t)\left|f'(t) - \frac{f'(x)+f'(b)}{2}\right|dt. \tag{7.20}$$

Since $f':\mathring{I}\to\mathbb{C}$ is of bounded variation on $[a,x]$ and $[x,b]$, then

$$\begin{aligned}\left|f'(t)-\frac{f'(a)+f'(x)}{2}\right| &= \frac{|f'(t)-f'(a)+f'(t)-f'(x)|}{2}\\ &\le \frac{1}{2}\left[|f'(t)-f'(a)|+|f'(x)-f'(t)|\right]\\ &\le \frac{1}{2}\bigvee_a^x (f')\end{aligned}$$

for any $t\in[a,x]$ and, similarly,

$$\left|f'(t)-\frac{f'(x)+f'(b)}{2}\right|\le\frac{1}{2}\bigvee_x^b (f')$$

for any $t\in[x,b]$.

Then

$$\begin{aligned}\int_a^x (t-a)\left|f'(t)-\frac{f'(a)+f'(x)}{2}\right|dt &\le \frac{1}{2}\bigvee_a^x (f')\int_a^x (t-a)\,dt\\ &= \frac{1}{4}(x-a)^2\bigvee_a^x (f')\end{aligned}$$

and

$$\begin{aligned}\int_x^b (b-t)\left|f'(t)-\frac{f'(x)+f'(b)}{2}\right|dt &\le \frac{1}{2}\bigvee_x^b (f')\int_x^b (b-t)\,dt\\ &= \frac{1}{4}(b-x)^2\bigvee_x^b (f')\end{aligned}$$

and by (7.20) we get the desired inequality (7.19).

The last part follows by Hölder's inequality

$$mn+pq\le(m^\alpha+p^\alpha)^{1/\alpha}\left(n^\beta+q^\beta\right)^{1/\beta},$$

where $m,n,p,q\ge 0$ and $\alpha>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$. □

Corollary 20. *Let $f:I\to\mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b]\subset\mathring{I}$. If the derivative $f':\mathring{I}\to\mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{16}(b-a)\left[f'(b)-f'(a)\right]\right| \\ &\le \frac{1}{16}(b-a)\bigvee_a^b (f').\end{aligned}\tag{7.21}$$

Remark 14. *If* $p \in (a,b)$ *is a median point in bounded variation for the derivative, i.e.* $\bigvee_a^p (f') = \bigvee_p^b (f')$, *then under the assumptions of Theorem 15, we have*

$$\left| f(p) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - p\right) f'(p) + \frac{1}{4(b-a)}\left[(b-p)^2 f'(b) - (p-a)^2 f'(a)\right] \right| \leq \frac{1}{8}(b-a)\left[\frac{1}{4} + \left(\frac{p-\frac{a+b}{2}}{b-a}\right)^2\right] \bigvee_a^b (f'). \tag{7.22}$$

7.4. **Inequalities for Lipschitzian Derivatives.** We say that $v : [a,b] \to \mathbb{C}$ is *Lipschitzian* with the constant $L > 0$, if it satisfies the condition

$$|v(t) - v(s)| \leq L|t-s| \text{ for any } t,s \in [a,b].$$

Theorem 16 (Dragomir, 2013 [12]). *Let* $f : I \to \mathbb{C}$ *be a differentiable function on* $\mathring{I}$ *and* $[a,b] \subset \mathring{I}$. *Let* $x \in (a,b)$. *If the derivative* $f' : \mathring{I} \to \mathbb{C}$ *is Lipschitzian with the constant* $K_1(x)$ *on* $[a,x]$ *and constant* $K_2(x)$ *on* $[x,b]$, *then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) + \frac{1}{4(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] \right| \tag{7.23}$$

$$\leq \frac{1}{8}\left[\left(\frac{x-a}{b-a}\right)^3 K_1(x) + \left(\frac{b-x}{b-a}\right)^3 K_2(x)\right](b-a)^2$$

$$\leq \frac{1}{8}(b-a)^2 \times \begin{cases} \left[\left(\frac{x-a}{b-a}\right)^3 + \left(\frac{b-x}{b-a}\right)^3\right] \max\{K_1(x), K_2(x)\}, \\ \\ \left[\left(\frac{x-a}{b-a}\right)^{2p} + \left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p} \left[K_1^q(x) + K_2^q(x)\right]^{1/q} \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \left[\frac{1}{2} + \left|\frac{x - \frac{a+b}{2}}{b-a}\right|\right]^3 \left[K_1(x) + K_2(x)\right]. \end{cases}$$

Proof. Since $f' : \mathring{I} \to \mathbb{C}$ is Lipschitzian with the constant $K_1(x)$ on $[a,x]$ and constant $K_2(x)$ on $[x,b]$, then

$$\begin{aligned}\left|f'(t) - \frac{f'(a)+f'(x)}{2}\right| &= \frac{|f'(t)-f'(a)+f'(t)-f'(x)|}{2} \\ &\le \frac{1}{2}\left[|f'(t)-f'(a)| + |f'(x)-f'(t)|\right] \\ &\le \frac{1}{2}K_1(x)\left[|t-a|+|x-t|\right] \\ &= \frac{1}{2}K_1(x)(x-a)\end{aligned}$$

for any $t \in [a,x]$ and, similarly,

$$\begin{aligned}\left|f'(t) - \frac{f'(x)+f'(b)}{2}\right| &\le \frac{1}{2}K_2(x)\left[|t-x|+|b-t|\right] \\ &= \frac{1}{2}K_2(x)(b-x)\end{aligned}$$

for any $t \in [x,b]$.

Then

$$\begin{aligned}\int_a^x (t-a)\left|f'(t) - \frac{f'(a)+f'(x)}{2}\right| dt &\le \frac{1}{2}K_1(x)(x-a)\int_a^x (t-a)\,dt \\ &= \frac{1}{8}(x-a)^3 K_1(x)\end{aligned}$$

and

$$\begin{aligned}\int_x^b (b-t)\left|f'(t) - \frac{f'(x)+f'(b)}{2}\right| dt &\le \frac{1}{2}K_2(x)(b-x)\int_x^b (b-t)\,dt \\ &= \frac{1}{8}(b-x)^3 K_2(x).\end{aligned}$$

Making use of the inequality (7.20) we deduce the first bound in (7.23).

The second part is obvious. □

Corollary 21. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is Lipschitzian with the constant K on $[a,b]$ then*

$$\begin{aligned}&\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2}-x\right)f'(x)\right. \\ &\left.+\frac{1}{4(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right]\right| \\ &\le \frac{1}{8}\left[\left(\frac{x-a}{b-a}\right)^3 + \left(\frac{b-x}{b-a}\right)^3\right]K(b-a)^2\end{aligned} \tag{7.24}$$

for any $x \in [a,b]$.

In particular, we have

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right) + \frac{1}{16}(b-a)\left[f'(b)-f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\le \frac{1}{32}K(b-a)^2.\end{aligned} \tag{7.25}$$

8. Other Perturbed Ostrowski Type Inequalities for Absolutely Continuous Function

8.1. **Inequalities for Derivatives of Bounded Variation.** Assume that the function $f : I \to \mathbb{C}$ is differentiable on the interior of I, denoted $\mathring{I}$, and $[a,b] \subset \mathring{I}$. Then, we have the equality [13]

$$
\begin{aligned}
(8.1)\quad & f(x) + \left(\frac{a+b}{2} - x\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\
&= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t) - f'(x)\right] dt + \frac{1}{b-a}\int_x^b (t-b)\left[f'(t) - f'(x)\right] dt
\end{aligned}
$$

for any $x \in [a,b]$.

We have the following result:

Theorem 17 (Dragomir, 2013 [13]). *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$
\begin{aligned}
(8.2)\quad & \left| f(x) + \left(\frac{a+b}{2} - x\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
&\le \frac{1}{b-a}\left[\int_a^x (t-a)\bigvee_t^x (f')\,dt + \int_x^b (b-t)\bigvee_x^t (f')\,dt\right] \\
&\le \frac{1}{2}(b-a)\left[\left(\frac{x-a}{b-a}\right)^2 \bigvee_a^x (f')\,dt + \left(\frac{b-x}{b-a}\right)^2 \bigvee_x^b (f')\right]
\end{aligned}
$$

$$
\le \frac{1}{2}(b-a) \times \begin{cases}
\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{2}\bigvee_a^b (f') + \frac{1}{2}\left|\bigvee_a^x (f') - \bigvee_x^b (f')\right|\right], \\[2ex]
\left[\left(\frac{x-a}{b-a}\right)^{2p} + \left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p}\left[\left[\bigvee_a^x (f')\right]^q + \left[\bigvee_x^b (f')\right]^q\right]^{1/q} \\
p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\[2ex]
\left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b (f'),
\end{cases}
$$

for any $x \in [a,b]$.

Proof. Taking the modulus in (8.1) we have

$$\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{8.3}$$
$$\leq \frac{1}{b-a}\left|\int_a^x (t-a)\left[f'(t)-f'(x)\right]dt\right| + \frac{1}{b-a}\left|\int_x^b (t-b)\left[f'(t)-f'(x)\right]dt\right|$$
$$\leq \frac{1}{b-a}\int_a^x (t-a)\left|f'(t)-f'(x)\right|dt + \frac{1}{b-a}\int_x^b (b-t)\left|f'(t)-f'(x)\right|dt.$$

Since the derivative $f':\mathring{I}\to\mathbb{C}$ is of bounded variation on $[a,x]$ and $[x,b]$, then

$$\left|f'(t)-f'(x)\right| \leq \bigvee_t^x (f') \text{ for } t\in[a,x]$$

and

$$\left|f'(t)-f'(x)\right| \leq \bigvee_x^t (f') \text{ for } t\in[x,b].$$

Therefore

$$\int_a^x (t-a)\left|f'(t)-f'(x)\right|dt \leq \int_a^x (t-a)\bigvee_t^x (f')\,dt \leq \frac{1}{2}(x-a)^2\bigvee_a^x (f')\,dt$$

and

$$\int_x^b (b-t)\left|f'(t)-f'(x)\right|dt \leq \int_x^b (b-t)\bigvee_x^t (f')\,dt \leq \frac{1}{2}(b-x)^2\bigvee_x^b (f'),$$

which, by (8.3) produce the first two inequalities in (8.2).

The last part follows by Hölder's inequality

$$mn+pq \leq (m^\alpha+p^\alpha)^{1/\alpha}\left(n^\beta+q^\beta\right)^{1/\beta},$$

where $m,n,p,q\geq 0$ and $\alpha>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$. □

Corollary 22. *With the assumptions of Theorem 17, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{8.4}$$
$$\leq \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}(t-a)\bigvee_t^{\frac{a+b}{2}}(f')\,dt+\int_{\frac{a+b}{2}}^b(b-t)\bigvee_{\frac{a+b}{2}}^t(f')\,dt\right]$$
$$\leq \frac{1}{8}(b-a)\bigvee_a^b(f')\,dt.$$

Remark 15. *If $p\in(a,b)$ is a median point in bounded variation for the derivative, i.e. $\bigvee_a^p(f')=\bigvee_p^b(f')$, then under the assumptions of Theorem 17 we have*

$$\left|f(p)+\left(\frac{a+b}{2}-p\right)f'(p)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{8.5}$$
$$\leq \frac{1}{b-a}\left[\int_a^p(t-a)\bigvee_t^p(f')\,dt+\int_p^b(b-t)\bigvee_p^t(f')\,dt\right]$$
$$\leq \frac{1}{4}(b-a)\left[\frac{1}{4}+\left(\frac{p-\frac{a+b}{2}}{b-a}\right)^2\right]\bigvee_a^b(f').$$

8.2. **Inequalities for Lipschitzian Derivatives.** We start with the following result.

Theorem 18 (Dragomir, 2013 [13])**.** *Let $f:I\to\mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b]\subset\mathring{I}$. Let $x\in(a,b)$. If $\alpha_i>-1$ and $L_{\alpha_i}>0$ with $i=1,2$ are such that*

$$|f'(t)-f'(x)|\leq L_{\alpha_1}(x-t)^{\alpha_1}\ \text{ for any } t\in[a,x) \tag{8.6}$$

and

$$|f'(t)-f'(x)|\leq L_{\alpha_2}(t-x)^{\alpha_2}\ \text{ for any } t\in(x,b], \tag{8.7}$$

then we have

$$\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{8.8}$$
$$\leq \frac{1}{b-a}\left[\frac{L_{\alpha_1}}{(\alpha_1+1)(\alpha_1+2)}(x-a)^{\alpha_1+2}+\frac{L_{\alpha_2}}{(\alpha_2+1)(\alpha_2+2)}(b-x)^{\alpha_2+2}\right].$$

Proof. Taking the modulus in (8.1) we have

$$\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{8.9}$$
$$\leq \frac{1}{b-a}\int_a^x(t-a)|f'(t)-f'(x)|\,dt$$
$$+\frac{1}{b-a}\int_x^b(b-t)|f'(t)-f'(x)|\,dt.$$

Using the properties (8.6) and (8.7) we have

$$\begin{aligned}\int_a^x (t-a)\left|f'(t)-f'(x)\right| dt &\le L_{\alpha_1}\int_a^x (t-a)(x-t)^{\alpha_1} dt\\ &= L_{\alpha_1}(x-a)^{\alpha_1+2}\int_0^1 u(1-u)^{\alpha_1} du\\ &= L_{\alpha_1}(x-a)^{\alpha_1+2}\int_0^1 u^{\alpha_1}(1-u) du\\ &= \frac{1}{(\alpha_1+1)(\alpha_1+2)}L_{\alpha_1}(x-a)^{\alpha_1+2}\end{aligned}$$

and

$$\begin{aligned}\int_x^b (b-t)\left|f'(t)-f'(x)\right| dt &\le L_{\alpha_2}\int_x^b (b-t)(t-x)^{\alpha_2} dt\\ &= \frac{1}{(\alpha_2+1)(\alpha_2+2)}L_{\alpha_2}(b-x)^{\alpha_2+2}.\end{aligned}$$

Utilising (8.9) we get the desired result (8.8). □

Corollary 23. *Let $f: I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative is f' of r-H-Hölder type on $[a,b]$, i.e. we have the condition*

$$\left|f'(t)-f'(s)\right| \le H\left|t-s\right|^r$$

for any $t,s \in [a,b]$, where $r \in (0,1]$ and $H>0$ are given, then

$$\begin{aligned}&\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ &\le \frac{H}{(r+1)(r+2)}\left[\left(\frac{x-a}{b-a}\right)^{r+2}+\left(\frac{b-x}{b-a}\right)^{r+2}\right](b-a)^{r+1},\end{aligned}\tag{8.10}$$

for any $x \in [a,b]$.

In particular, if f' is Lipschitzian with the constant $L>0$, then

$$\begin{aligned}&\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ &\le \frac{1}{6}L\left[\left(\frac{x-a}{b-a}\right)^{3}+\left(\frac{b-x}{b-a}\right)^{3}\right](b-a)^{2},\end{aligned}\tag{8.11}$$

for any $x \in [a,b]$.

8.3. **Inequalities for Differentiable Convex Functions.** The case of convex functions is as follows:

Theorem 19 (Dragomir, 2013 [13])**.** *Let $f: I \to \mathbb{C}$ be a differentiable convex function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. Then for any $x \in [a,b]$ we have*

$$0 \le \frac{1}{b-a}\int_a^b f(t)\,dt - f(x) - \left(\frac{a+b}{2}-x\right)f'(x) \le \begin{cases} I_1(x)\\ I_2(x)\\ I_3(x)\end{cases}\tag{8.12}$$

where

$$I_1(x) := \frac{(b-x) f(b) + (x-a) f(a)}{b-a} - f(x) - 2f'(x)\left(\frac{a+b}{2} - x\right),$$

$$I_2(x) := \frac{1}{2}\frac{f'(b)(b-x)^2 - f'(a)(x-a)^2}{b-a} - f'(x)\left(\frac{a+b}{2} - x\right)$$

and

$$I_3(x) := \frac{1}{2}\left[\frac{f(b)(b-x) + f(a)(x-a)}{b-a} - f(x)\right] - f'(x)\left(\frac{a+b}{2} - x\right).$$

Proof. We have the equality

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)\,dt - f(x) - \left(\frac{a+b}{2} - x\right) f'(x) \\ &= \frac{1}{b-a}\int_a^x (t-a)\left[f'(x) - f'(t)\right] dt + \frac{1}{b-a}\int_x^b (b-t)\left[f'(t) - f'(x)\right] dt \end{aligned} \tag{8.13}$$

for any $x \in [a,b]$.

Since f is a differentiable convex function on $\mathring{I}$, then f' is monotonic nondecreasing on $\mathring{I}$ and then

$$\int_a^x (t-a)\left[f'(x) - f'(t)\right] dt \geq 0$$

and

$$\int_x^b (b-t)\left[f'(t) - f'(x)\right] dt \geq 0,$$

which proves the first inequality in (8.12).

We have

$$\begin{aligned} \int_a^x (t-a)\left[f'(x) - f'(t)\right] dt &\leq (x-a)\int_a^x \left[f'(x) - f'(t)\right] dt \\ &= (x-a)\left[f'(x)(x-a) - f(x) + f(a)\right] \end{aligned}$$

and

$$\begin{aligned} \int_x^b (b-t)\left[f'(t) - f'(x)\right] dt &\leq (b-x)\int_x^b \left[f'(t) - f'(x)\right] dt \\ &= (b-x)\left[f(b) - f(x) - f'(x)(b-x)\right]. \end{aligned}$$

Adding these inequalities we get

$$\begin{aligned} &\int_a^x (t-a)\left[f'(x) - f'(t)\right] dt + \int_x^b (b-t)\left[f'(t) - f'(x)\right] dt \\ &\leq (x-a)\left[f'(x)(x-a) - f(x) + f(a)\right] \\ &+ (b-x)\left[f(b) - f(x) - f'(x)(b-x)\right] \\ &= (b-x) f(b) + (x-a) f(a) - (b-a) f(x) \\ &+ f'(x)\left[2x - (a+b)\right](b-a) \end{aligned}$$

and by (8.13) we get the second inequality for $I_1(x)$.

We also have

$$\begin{aligned}\int_a^x (t-a)\left[f'(x)-f'(t)\right]dt &\le \int_a^x (t-a)\left[f'(x)-f'(a)\right]dt\\ &= \frac{1}{2}\left[f'(x)-f'(a)\right](x-a)^2\end{aligned}$$

and

$$\begin{aligned}\int_x^b (b-t)\left[f'(t)-f'(x)\right]dt &\le \int_x^b (b-t)\left[f'(b)-f'(x)\right]dt\\ &= \frac{1}{2}\left[f'(b)-f'(x)\right](b-x)^2.\end{aligned}$$

Adding these inequalities we get

$$\begin{aligned}&\int_a^x (t-a)\left[f'(x)-f'(t)\right]dt+\int_x^b (b-t)\left[f'(t)-f'(x)\right]dt\\ &\le \frac{1}{2}\left[f'(x)-f'(a)\right](x-a)^2+\frac{1}{2}\left[f'(b)-f'(x)\right](b-x)^2\\ &= \frac{1}{2}\left[f'(b)(b-x)^2-f'(a)(x-a)^2+f'(x)(b-a)\left[2x-(a+b)\right]\right]\end{aligned}$$

and by (8.13) we get the second inequality for $I_2(x)$.

Further, we use the Čebyšev inequality for asynchronous functions (functions of opposite monotonicity), namely

$$\frac{1}{d-c}\int_c^d g(t)\,h(t)\,dt \le \frac{1}{d-c}\int_c^d g(t)\,dt\cdot\frac{1}{d-c}\int_c^d h(t)\,dt.$$

Therefore

$$\begin{aligned}&\frac{1}{x-a}\int_a^x (t-a)\left[f'(x)-f'(t)\right]dt\\ &\le \frac{1}{x-a}\int_a^x (t-a)\,dt\cdot\frac{1}{x-a}\int_a^x \left[f'(x)-f'(t)\right]dt\\ &= \frac{(x-a)^2}{2(x-a)}\cdot\frac{f'(x)(x-a)-f(x)+f(a)}{x-a}\\ &= \frac{1}{2}\left[f'(x)(x-a)-f(x)+f(a)\right]\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{b-x}\int_x^b (b-t)\left[f'(t)-f'(x)\right]dt\\ &\le \frac{1}{b-x}\int_x^b (b-t)\,dt\cdot\frac{1}{b-x}\int_x^b \left[f'(t)-f'(x)\right]dt\\ &= \frac{(b-x)^2}{2(b-x)}\cdot\frac{f(b)-f(x)-f'(x)(b-x)}{b-x}\\ &= \frac{1}{2}\left[f(b)-f(x)-f'(x)(b-x)\right].\end{aligned}$$

Adding these inequalities, we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^x (t-a)\left[f'(x)-f'(t)\right]dt+\frac{1}{b-a}\int_x^b (b-t)\left[f'(t)-f'(x)\right]dt\\
&\leq \frac{1}{2}\frac{\left[f'(x)(x-a)-f(x)+f(a)\right](x-a)}{b-a}\\
&+\frac{1}{2}\frac{\left[f(b)-f(x)-f'(x)(b-x)\right](b-x)}{b-a}\\
&=\frac{1}{2(b-a)}\left[\left[f'(x)(x-a)-f(x)+f(a)\right](x-a)\right]\\
&+\frac{1}{2(b-a)}\left[\left[f(b)-f(x)-f'(x)(b-x)\right](b-x)\right]\\
&=\frac{1}{2}\left[\frac{f(b)(b-x)+f(a)(x-a)}{b-a}-f(x)\right]+f'(x)\left(x-\frac{a+b}{2}\right)
\end{aligned}$$

which proves the inequality for $I_3(x)$. □

Remark 16. *From the first inequality in (8.12) we have*

$$\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-f'(x)\left(\frac{a+b}{2}-x\right)\tag{8.14}$$

for any $x\in[a,b]$.

From the second inequality in (8.12) we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt-f(x)\leq\frac{1}{2}\cdot\frac{f'(b)(b-x)^2-f'(a)(x-a)^2}{b-a}\tag{8.15}$$

for any $x\in[a,b]$.

From the third inequality in (8.12) we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{1}{2}\left[\frac{f(b)(b-x)+f(a)(x-a)}{b-a}+f(x)\right]\tag{8.16}$$

for any $x\in[a,b]$.

8.4. **Inequalities for Absolutely Continuous Derivatives.** The case of absolutely continuous derivatives is as follows:

Theorem 20 (Dragomir, 2013 [13])**.** *Let* $f:I\to\mathbb{C}$ *be a differentiable function on* $\mathring{I}$ *and* $[a,b]\subset\mathring{I}$*. If the derivative* f' *is absolutely continuous on* $[a,b]$*, then for any* $x\in[a,b]$

$$\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\tag{8.17}$$

$$\leq \frac{1}{b-a} \times \begin{cases} \frac{1}{6}(x-a)^3 \|f''\|_{[a,x],\infty}, \\ \frac{q}{(q+1)(q+2)}(x-a)^{1/q+2} \|f''\|_{[a,x],p}, \\ \frac{1}{2}(x-a)^2 \|f''\|_{[a,x],1}, \end{cases}$$

$$+ \frac{1}{b-a} \times \begin{cases} \frac{1}{6}(b-x)^3 \|f''\|_{[x,b],\infty}, \\ \frac{q}{(q+1)(q+2)}(b-x)^{1/q+2} \|f''\|_{[x,b],p}, \\ \frac{1}{2}(b-x)^2 \|f''\|_{[x,b],1}, \end{cases}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Taking the modulus in (8.1) we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt + \left(\frac{a+b}{2} - x\right) f'(x) \right| \tag{8.18}$$
$$\leq \frac{1}{b-a}\int_a^x (t-a)\left|f'(t) - f'(x)\right| dt + \frac{1}{b-a}\int_x^b (b-t)\left|f'(t) - f'(x)\right| dt$$
$$= \frac{1}{b-a}\int_a^x (t-a)\left|\int_x^t f''(s)\,ds\right| + \frac{1}{b-a}\int_x^b (b-t)\left|\int_x^t f''(s)\,ds\right|$$
$$\leq \frac{1}{b-a}\int_a^x (t-a)\int_t^x \left|f''(s)\right| ds + \frac{1}{b-a}\int_x^b (b-t)\int_x^t \left|f''(s)\right| ds.$$

Using Hölder's integral inequality we have for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_a^x (t-a)\int_t^x \left|f''(s)\right| ds \leq \begin{cases} \int_a^x (t-a)(x-t)\|f''\|_{[t,x],\infty}\,dt \\ \int_a^x (t-a)(x-t)^{1/q}\|f''\|_{[t,x],p}\,dt \\ \int_a^x (t-a)\|f''\|_{[t,x],1}\,dt \end{cases}$$

$$\leq \begin{cases} \|f''\|_{[a,x],\infty}\int_a^x (t-a)(x-t)\,dt \\ \|f''\|_{[a,x],p}\int_a^x (t-a)(x-t)^{1/q}\,dt \\ \|f''\|_{[a,x],1}\int_a^x (t-a)\,dt \end{cases}$$
$$= \begin{cases} \frac{1}{6}(x-a)^3\|f''\|_{[a,x],\infty} \\ \frac{q}{(q+1)(q+2)}(x-a)^{1/q+2}\|f''\|_{[a,x],p} \\ \frac{1}{2}(x-a)^2\|f''\|_{[a,x],1} \end{cases}$$

and, similarly

$$\int_{x}^{b}(b-t)\int_{x}^{t}|f''(s)|\,ds\leq\begin{cases}\frac{1}{6}(b-x)^{3}\|f''\|_{[x,b],\infty}\\[2ex]\frac{q}{(q+1)(q+2)}(b-x)^{1/q+2}\|f''\|_{[x,b],p}\\[2ex]\frac{1}{2}(b-x)^{2}\|f''\|_{[x,b],1}\,.\end{cases}$$

Utilizing the inequality (8.18) we get the desired result (8.17). □

Remark 17. *Since*

$$\begin{aligned}&\frac{1}{6}(x-a)^{3}\|f''\|_{[a,x],\infty}+\frac{1}{6}(b-x)^{3}\|f''\|_{[x,b],\infty}\\&\leq\frac{1}{6}\left[(x-a)^{3}+(b-x)^{3}\right]\max\left\{\|f''\|_{[a,x],\infty},\|f''\|_{[x,b],\infty}\right\}\\&=\frac{1}{6}(b-a)\left[(x-a)^{2}-(x-a)(b-x)+(b-x)^{2}\right]\|f''\|_{[a,b],\infty}\,,\end{aligned}$$

then by (8.17) we get

$$\begin{aligned}&\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|\\&\leq\frac{1}{6}\left[\left(\frac{x-a}{b-a}\right)^{2}-\left(\frac{x-a}{b-a}\right)\left(\frac{b-x}{b-a}\right)+\left(\frac{b-x}{b-a}\right)^{2}\right]\\&\times(b-a)^{2}\|f''\|_{[a,b],\infty}\,,\end{aligned}\tag{8.19}$$

for any $x\in[a,b]$.

Since

$$\begin{aligned}&(x-a)^{1/q+2}\|f''\|_{[a,x],p}+(b-x)^{1/q+2}\|f''\|_{[x,b],p}\\&\leq\left[(x-a)^{2q+1}+(b-x)^{2q+1}\right]^{1/q}\left[\|f''\|_{[a,x],p}^{p}+\|f''\|_{[x,b],p}\right]^{1/p}\\&=\left[(x-a)^{2q+1}+(b-x)^{2q+1}\right]^{1/q}\|f''\|_{[a,b],p}\,,\end{aligned}$$

then by (8.17) we get

$$\begin{aligned}&\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|\\&\leq\frac{q}{(q+1)(q+2)}\left[\left(\frac{x-a}{b-a}\right)^{2q+1}+\left(\frac{b-x}{b-a}\right)^{2q+1}\right]^{1/q}\\&\times(b-a)^{1+1/q}\|f''\|_{[a,b],p}\,,\end{aligned}\tag{8.20}$$

for any $x\in[a,b]$.

Since

$$\begin{aligned}
&(x-a)^2 \|f''\|_{[a,x],1} + (b-x)^2 \|f''\|_{[x,b],1} \\
&\le \max\left\{(x-a)^2, (b-x)^2\right\} \left[\|f''\|_{[a,x],1} + \|f''\|_{[x,b],1}\right] \\
&= \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^2 \|f''\|_{[a,b],1},
\end{aligned}$$

then by (8.17) we get

$$\begin{aligned}
&\left|f(x) + \left(\frac{a+b}{2} - x\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\le \frac{1}{2}\left[\frac{1}{2} + \left|\frac{x - \frac{a+b}{2}}{b-a}\right|\right]^2 (b-a) \|f''\|_{[a,b],1}
\end{aligned}$$

for any $x \in [a,b]$.

9. More Perturbed Ostrowski Type Inequalities for Absolutely Continuous Functions

9.1. **Inequalities for Derivatives of Bounded Variation.** Assume that the function $f : I \to \mathbb{C}$ is differentiable on the interior of I, denoted $\mathring{I}$, and $[a,b] \subset \mathring{I}$. Then, we have the equality

$$\begin{aligned}
(9.1)\quad & f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \\
&= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t) - f'(a)\right] dt + \frac{1}{b-a}\int_x^b (b-t)\left[f'(b) - f'(t)\right] dt,
\end{aligned}$$

for any $x \in [a,b]$.

In particular, for $x = \frac{a+b}{2}$, we have

$$\begin{aligned}
(9.2)\quad & f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)\left[f'(b) - f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \\
&= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t) - f'(a)\right] dt \\
&\quad + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (b-t)\left[f'(b) - f'(t)\right] dt.
\end{aligned}$$

Theorem 21 (Dragomir, 2013 [14]). *Let* $f : I \to \mathbb{C}$ *be a differentiable function on* $\mathring{I}$ *and* $[a,b] \subset \mathring{I}$. *If the derivative* $f' : \mathring{I} \to \mathbb{C}$ *is of bounded variation on* $[a,b]$, *then* *for any* $x \in [a,b]$

$$\left| f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \tag{9.3}$$
$$\leq \frac{1}{b-a} \left[\int_a^x (t-a) \bigvee_a^t (f')\,dt + \int_x^b (b-t) \bigvee_t^b (f')\,dt \right]$$
$$\leq \frac{1}{b-a} \begin{cases} \frac{1}{2}(x-a)^2 \bigvee_a^x (f'), \\ \frac{1}{(q+1)^{1/q}} (x-a)^{1+1/q} \left(\int_a^x \left(\bigvee_a^t (f') \right)^p dt \right)^{1/p}, \\ (x-a) \int_a^x \left(\bigvee_a^t (f') \right) dt \end{cases}$$
$$+ \frac{1}{b-a} \begin{cases} \frac{1}{2}(b-x)^2 \bigvee_x^b (f'), \\ \frac{1}{(q+1)^{1/q}} (b-x)^{1+1/q} \left(\int_x^b \left(\bigvee_t^b (f') \right)^p dt \right)^{1/p}, \\ (b-x) \int_x^b \left(\bigvee_t^b (f') \right) dt. \end{cases}$$

where $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1.$

Proof. Taking the modulus in (9.1) we have

$$\left| f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \tag{9.4}$$
$$\leq \frac{1}{b-a} \int_a^x (t-a) \left| f'(t) - f'(a) \right| dt + \frac{1}{b-a} \int_x^b (b-t) \left| f'(b) - f'(t) \right| dt,$$

for any $x \in [a,b]$.

Since the derivative $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,b]$, then

$$|f'(t) - f'(a)| \leq \bigvee_a^t (f') \text{ for any } t \in [a,x]$$

and

$$|f'(b) - f'(t)| \leq \bigvee_t^b (f') \text{ for any } t \in [x,b].$$

Therefore

$$\int_a^x (t-a)\left|f'(t)-f'(a)\right|dt \le \int_a^x (t-a)\bigvee_a^t (f')\,dt$$

and

$$\int_x^b (b-t)\left|f'(b)-f'(t)\right|dt \le \int_x^b (b-t)\bigvee_t^b (f')\,dt$$

for any $x \in [a,b]$.

Adding these two inequalities and dividing by $b-a$ we get the first inequality in (9.3).

Using Hölder's integral inequality we have

$$\int_a^x (t-a)\bigvee_a^t (f')\,dt \le \begin{cases} \bigvee_a^x (f') \int_a^x (t-a)\,dt, \\ \left(\int_a^x (t-a)^q\,dt\right)^{1/q} \left(\int_a^x \left(\bigvee_a^t (f')\right)^p dt\right)^{1/p}, \\ (x-a)\int_a^x \left(\bigvee_a^t (f')\right) dt, \end{cases}$$

$$= \begin{cases} \frac{1}{2}(x-a)^2 \bigvee_a^x (f'), \\ \frac{1}{(q+1)^{1/q}}(x-a)^{1+1/q} \left(\int_a^x \left(\bigvee_a^t (f')\right)^p dt\right)^{1/p}, \\ (x-a)\int_a^x \left(\bigvee_a^t (f')\right) dt \end{cases}$$

and

$$\int_x^b (b-t)\bigvee_t^b (f')\,dt \le \begin{cases} \frac{1}{2}(b-x)^2 \bigvee_x^b (f'), \\ \frac{1}{(q+1)^{1/q}}(b-x)^{1+1/q} \left(\int_x^b \left(\bigvee_x^b (f')\right)^p dt\right)^{1/p}, \\ (b-x)\int_x^b \left(\bigvee_x^b (f')\right) dt. \end{cases}$$

□

Remark 18. *From the first branch in (9.3) we have the sequence of inequalities*

$$\begin{aligned}(9.5)\quad &\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{b-a}\left[\int_a^x (t-a)\bigvee_a^t (f')\,dt+\int_x^b (b-t)\bigvee_t^b (f')\,dt\right] \\ &\leq \frac{1}{2}(b-a)\left[\left(\frac{x-a}{b-a}\right)^2\bigvee_a^x (f')+\left(\frac{b-x}{b-a}\right)^2\bigvee_x^b (f')\right]\end{aligned}$$

$$\begin{aligned}&\leq \frac{1}{2}(b-a) \\ &\times\begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{2}\bigvee_a^b (f')+\frac{1}{2}\left|\bigvee_a^x (f')-\bigvee_x^b (f')\right|\right], \\ \left[\left(\frac{x-a}{b-a}\right)^{2p}+\left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p}\left[\left[\bigvee_a^x (f')\right]^q+\left[\bigvee_x^b (f')\right]^q\right]^{1/q} \\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b (f'),\end{cases}\end{aligned}$$

for any $x\in[a,b]$.

From the second branch in (9.3) we have

$$\begin{aligned}(9.6)\quad &\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{b-a}\left[\int_a^x (t-a)\bigvee_a^t (f')\,dt+\int_x^b (b-t)\bigvee_t^b (f')\,dt\right] \\ &\leq \frac{1}{(q+1)^{1/q}}\left\{\left(\frac{x-a}{b-a}\right)^{1+1/q}\left(\int_a^x\left(\bigvee_a^t (f')\right)^p dt\right)^{1/p}\right. \\ &\left.+\left(\frac{b-x}{b-a}\right)^{1+1/q}\left(\int_x^b\left(\bigvee_t^b (f')\right)^p dt\right)^{1/p}\right\}(b-a)^{1/q}\end{aligned}$$

$$\leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/p}$$
$$\times\left[\int_a^x\left(\bigvee_a^t(f')\right)^p dt+\int_x^b\left(\bigvee_t^b(f')\right)^p dt\right]^{1/p}(b-a)^{1/q}$$
$$\leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/p}$$
$$\times\left[(x-a)\left(\bigvee_a^x(f')\right)^p+(b-x)\left(\bigvee_x^b(f')\right)^p\right]^{1/p}(b-a)^{1/q}$$

for any $x\in[a,b]$ and $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

From the third branch in (9.3) we have

$$\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{9.7}$$
$$\leq \frac{1}{b-a}\left[\int_a^x (t-a)\bigvee_a^t(f')\,dt+\int_x^b (b-t)\bigvee_t^b(f')\,dt\right]$$
$$\leq \left(\frac{x-a}{b-a}\right)\int_a^x\left(\bigvee_a^t(f')\right)dt+\left(\frac{b-x}{b-a}\right)\int_x^b\left(\bigvee_t^b(f')\right)dt$$
$$\leq\begin{cases}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\int_a^x\left(\bigvee_a^t(f')\right)dt+\int_x^b\left(\bigvee_t^b(f')\right)dt\right]\\[2ex] \left[\left(\frac{x-a}{b-a}\right)^q+\left(\frac{b-x}{b-a}\right)^q\right]^{1/q}\\ \times\left[\left[\int_a^x\left(\bigvee_a^t(f')\right)dt\right]^p+\left[\int_x^b\left(\bigvee_t^b(f')\right)dt\right]^p\right]^{1/p}\\[2ex] \max\left\{\int_a^x\left(\bigvee_a^t(f')\right)dt,\int_x^b\left(\bigvee_t^b(f')\right)dt\right\}\end{cases}$$

for any $x\in[a,b]$ and $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Remark 19. *We observe that, if we take $x=\frac{a+b}{2}$ in (9.5) then we get the perturbed midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)\left[f'(b)-f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{9.8}$$
$$\leq \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}(t-a)\bigvee_a^t(f')\,dt+\int_{\frac{a+b}{2}}^b(b-t)\bigvee_t^b(f')\,dt\right]$$
$$\leq \frac{1}{8}(b-a)\bigvee_a^b(f').$$

9.2. **Inequalities for Lipschitzian Derivatives.** We start with the following result.

Theorem 22 (Dragomir, 2013 [14]). *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. Let $x \in (a,b)$. If $\alpha_i > -1$ and $L_{\alpha_i} > 0$ with $i = 1,2$ are such that*

$$|f'(t) - f'(a)| \le L_{\alpha_1} (t-a)^{\alpha_1} \text{ for any } t \in [a,x) \tag{9.9}$$

and

$$|f'(b) - f'(t)| \le L_{\alpha_2} (b-t)^{\alpha_2} \text{ for any } t \in (x,b], \tag{9.10}$$

then we have

$$\begin{aligned} &\left| f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ &\le \frac{1}{b-a} \left[\frac{L_{\alpha_1}}{\alpha_1 + 2} (x-a)^{\alpha_1+2} + \frac{L_{\alpha_2}}{\alpha_2+2} (b-x)^{\alpha_2+2} \right]. \end{aligned} \tag{9.11}$$

Proof. Using the conditions (9.9) and (9.10) we have

$$\begin{aligned} &\left| f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ &\le \frac{1}{b-a} \int_a^x (t-a) |f'(t) - f'(a)|\, dt + \frac{1}{b-a} \int_x^b (b-t) |f'(b) - f'(t)|\, dt \\ &\le \frac{1}{b-a} L_{\alpha_1} \int_a^x (t-a)^{\alpha_1+1} dt + \frac{1}{b-a} L_{\alpha_2} \int_x^b (b-t)^{\alpha_2+1} dt \\ &= \frac{1}{b-a} L_{\alpha_1} \frac{(x-a)^{\alpha_1+2}}{\alpha_1+2} + \frac{1}{b-a} L_{\alpha_2} \frac{(b-x)^{\alpha_2+2}}{\alpha_2+2} \\ &= \frac{1}{b-a} \left[\frac{L_{\alpha_1}}{\alpha_1+2} (x-a)^{\alpha_1+2} + \frac{L_{\alpha_2}}{\alpha_2+2} (b-x)^{\alpha_2+2} \right] \end{aligned}$$

and the inequality (9.11) is obtained. □

Corollary 24. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative is f' of r-H-Hölder type on $[a,b]$, i.e. we have the condition*

$$|f'(t) - f'(s)| \le H |t-s|^r$$

for any $t,s \in [a,b]$, where $r \in (0,1]$ and $H > 0$ are given, then

$$\begin{aligned} &\left| f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ &\le \frac{H}{r+2} \left[\left(\frac{x-a}{b-a} \right)^{r+2} + \left(\frac{b-x}{b-a} \right)^{r+2} \right] (b-a)^{r+1}, \end{aligned} \tag{9.12}$$

for any $x \in [a,b]$.

In particular, if f' is Lipschitzian with the constant $L > 0$, then

$$\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{3}L\left[\left(\frac{x-a}{b-a}\right)^3+\left(\frac{b-x}{b-a}\right)^3\right](b-a)^2, \tag{9.13}$$

for any $x \in [a,b]$.

Remark 20. *With the assumptions of Corollary 24 we have the midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)\left[f'(b)-f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{H}{2^{r+1}(r+2)}(b-a)^{r+1}. \tag{9.14}$$

If if f' is Lipschitzian with the constant $L > 0$, then

$$\left|f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)\left[f'(b)-f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{12}L(b-a)^2. \tag{9.15}$$

9.3. **Inequalities for Differentiable Functions with the Property** (S)**.** Let $f : I \to \mathbb{C}$ be a differentiable convex function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. Then f' is monotonic nondecreasing and by the equality (9.1) we have

$$f(x)+\frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \geq 0 \tag{9.16}$$

or, equivalently

$$\frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right] \geq \frac{1}{b-a}\int_a^b f(t)\,dt - f(x) \tag{9.17}$$

for any $x \in [a,b]$.

We observe that the inequalities (9.16) and (9.17) remain valid for the larger class of differentiable functions f that that satisfy the *property* (S) on the interval $[a,b]$, namely

$$f'(a) \leq f'(t) \leq f'(b) \tag{S}$$

for any $t \in [a,b]$.

Theorem 23 (Dragomir, 2013 [14])**.** *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$.*

(i) Let $x \in [a,b]$. If f satisfies the property (S) on the interval $[a,x]$ and $[x,b]$, then

$$f'(x)\left(\frac{a+b}{2}-x\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt - f(x). \tag{9.18}$$

(ii) If f satisfies the property (S) on the interval $[a,b]$, then for any $x \in [a,b]$

$$\frac{f(a)(x-a)+f(b)(b-x)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]. \tag{9.19}$$

Proof. (i) Since f satisfies the property (S) on the interval $[a,x]$ and $[x,b]$, then

$$\begin{aligned}
&f(x)+\frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\\
&=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'(a)\right]dt+\frac{1}{b-a}\int_x^b (b-t)\left[f'(b)-f'(t)\right]dt\\
&\leq \frac{1}{b-a}\int_a^x (t-a)\left[f'(x)-f'(a)\right]dt+\frac{1}{b-a}\int_x^b (b-t)\left[f'(b)-f'(x)\right]dt\\
&=\frac{f'(x)-f'(a)}{b-a}\int_a^x (t-a)\,dt+\frac{f'(b)-f'(x)}{b-a}\int_x^b (b-t)\,dt\\
&=\frac{f'(x)-f'(a)}{b-a}\cdot\frac{(x-a)^2}{2}+\frac{f'(b)-f'(x)}{b-a}\cdot\frac{(b-x)^2}{2}\\
&=\frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]-f'(x)\left(\frac{a+b}{2}-x\right),
\end{aligned}$$

which proves the inequality (9.18).

(ii) If f satisfies the property (S) on the interval $[a,b]$, then for any $x \in [a,b]$

$$\begin{aligned}
&\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'(a)\right]dt+\frac{1}{b-a}\int_x^b (b-t)\left[f'(b)-f'(t)\right]dt\\
&\leq \frac{x-a}{b-a}\int_a^x \left[f'(t)-f'(a)\right]dt+\frac{b-x}{b-a}\int_x^b \left[f'(b)-f'(t)\right]dt\\
&=\frac{1}{b-a}(x-a)\left[f(x)-f(a)-f'(a)(x-a)\right]\\
&+\frac{1}{b-a}(b-x)\left[f'(b)(b-x)-f(b)+f(x)\right]\\
&=\frac{1}{b-a}\left[f(x)(x-a)-f(a)(x-a)-f'(a)(x-a)^2\right]\\
&+\frac{1}{b-a}\left[f'(b)(b-x)^2-f(b)(b-x)+f(x)(b-x)\right]\\
&=\frac{f'(b)(b-x)^2-f'(a)(x-a)^2}{b-a}+f(x)-\frac{f(a)(x-a)+f(b)(b-x)}{b-a},
\end{aligned}$$

which proves the inequality (9.19). □

Remark 21. *The inequality (9.18) was obtained for the case of convex functions in* [5] *while (9.19) was established for convex functions in* [6] *with different proofs.*

Further, we use the Čebyšev inequality for synchronous functions (functions with same monotonicity), namely

$$\frac{1}{d-c}\int_c^d g(t)\,h(t)\,dt \geq \frac{1}{d-c}\int_c^d g(t)\,dt\cdot\frac{1}{d-c}\int_c^d h(t)\,dt. \tag{9.20}$$

Theorem 24 (Dragomir, 2013 [14])**.** *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. Let $x \in [a,b]$. If f is convex on the interval $[a,x]$ and $[x,b]$, then*

$$\frac{1}{2}\left[f(x) + \frac{f(a)(x-a) + f(b)(b-x)}{b-a}\right] \geq \frac{1}{b-a}\int_a^b f(t)\,dt. \tag{9.21}$$

Proof. We have

$$\begin{aligned} & f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t) - f'(a)\right]dt + \frac{1}{b-a}\int_x^b (b-t)\left[f'(b) - f'(t)\right]dt \end{aligned} \tag{9.22}$$

for any $x \in [a,b]$.

Since f' is monotonic nondecreasing on $[a,x]$, then by Čebyšev inequality (9.12) we have

$$\begin{aligned} \int_a^x (t-a)\left[f'(t) - f'(a)\right]dt &\geq \frac{1}{x-a}\int_a^x (t-a)\,dt \cdot \int_a^x \left[f'(t) - f'(a)\right]dt \\ &= \frac{1}{2}\left[f(x)(x-a) - f(a)(x-a) - f'(a)(x-a)^2\right] \end{aligned}$$

and, by the same inequality,

$$\begin{aligned} \int_x^b (b-t)\left[f'(b) - f'(t)\right]dt &\geq \frac{1}{b-x}\int_x^b (b-t)\,dt \cdot \int_x^b \left[f'(b) - f'(t)\right]dt \\ &= \frac{1}{2}\left[f'(b)(b-x)^2 - f(b)(b-x) + f(x)(b-x)\right]. \end{aligned}$$

If we add these two inequalities, then we get

$$\begin{aligned} & \int_a^x (t-a)\left[f'(t) - f'(a)\right]dt + \int_x^b (b-t)\left[f'(b) - f'(t)\right]dt \\ & \geq \frac{1}{2}\left[f(x)(x-a) - f(a)(x-a) - f'(a)(x-a)^2\right] \\ & + \frac{1}{2}\left[f'(b)(b-x)^2 - f(b)(b-x) + f(x)(b-x)\right] \\ & = \frac{1}{2}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] + \frac{1}{2}f(x)(b-a) \\ & - \frac{1}{2}\left[f(a)(x-a) + f(b)(b-x)\right]. \end{aligned}$$

Dividing by $b-a$ and utilizing the equality (9.22) we deduce the inequality (9.21). □

Remark 22. *If the function is convex on the whole interval $[a,b]$, then the inequality (9.21) is true for any $x \in [a,b]$.*

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