Received 24/06/15

GENERALIZATION, REFINEMENT AND REVERSES OF THE RIGHT FEJÉR INEQUALITY FOR CONVEX FUNCTIONS

S. S. DRAGOMIR[1,2]

ABSTRACT. In this paper we establish a generalization of the right Fejér inequality for general Lebesgue integral on measurable spaces as well as a positive *lower bound* and some *upper bounds* for the difference

$$\frac{h(a)+h(b)}{2}-\frac{1}{\int_a^b g(x)\,dx}\int_a^b h(x)\,g(x)\,dx,$$

where $h:[a,b]\to\mathbb{R}$ is a convex function and $g:[a,b]\to[0,\infty)$ is an integrable weight. Applications for discrete means are also provided.

1. Introduction

The *Hermite-Hadamard* integral inequality for convex functions $f:[a,b]\to\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2} \tag{HH}$$

is well known in the literature and has many applications for Special Means, in Information Theory and in Probability Theory and Statistics.

For related results, see for instance the research papers [1], [9], [10], [11], [13], [12], [14], [15], [16], the monograph online [7] and the references therein.

In 1906, Fejér [8], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Theorem 1 (Fejér's Inequality). *Consider the integral $\int_a^b h(x)\,g(x)\,dx$, where h is a convex function in the interval (a,b) and g is a positive function in the same interval such that*

$$g(a+t)=g(b-t),\ \ 0\leq t\leq\frac{1}{2}(b-a),$$

i.e., $y=g(x)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b),0\right)$ and is normal to the x-axis. Under those conditions the following inequalities are valid:

$$h\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx\leq\int_a^b h(x)\,g(x)\,dx\leq\frac{h(a)+h(b)}{2}\int_a^b g(x)\,dx. \tag{1.1}$$

If h is concave on (a,b), then the inequalities reverse in (1.1).

Clearly, for $g(x)\equiv 1$ on $[a,b]$ we get (HH).

1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 26D20.

Key words and phrases. Convex functions, Integral inequalities, Jensen's type inequalities, Fejér type inequalities, Lebesgue integral, Hermite-Hadamard type inequalities, Special means.

Motivated by the above result, we establish in this paper a generalization of the right inequality (1.1) for general Lebesgue integral on measurable spaces as well as a positive *lower bound* and some *upper bounds* for the difference

$$\frac{h(a)+h(b)}{2}-\frac{1}{\int_a^b g(x)\,dx}\int_a^b h(x)\,g(x)\,dx,$$

where g and h are as above.

Applications for discrete means are also provided.

2. General Results

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $\Phi : I \to \mathbb{R}$ is a convex function on I. Then Φ is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $\Phi'_-(x) \le \Phi'_+(x) \le \Phi'_-(y) \le \Phi'_+(y)$ which shows that both Φ'_- and Φ'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $\Phi : I \to \mathbb{R}$, the subdifferential of Φ denoted by $\partial\Phi$ is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$\Phi(x) \ge \Phi(a) + (x-a)\,\varphi(a) \ \text{ for any } x, a \in I. \tag{2.1}$$

It is also well known that if Φ is convex on I, then $\partial\Phi$ is nonempty, Φ'_-, $\Phi'_+ \in \partial\Phi$ and if $\varphi \in \partial\Phi$, then

$$\Phi'_-(x) \le \varphi(x) \le \Phi'_+(x) \ \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If Φ is differentiable and convex on $\mathring{I}$, then $\partial\Phi = \{\Phi'\}$.

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a $\sigma-$ algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \ge 0$ for μ -a.e.(almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)|\,w(x)\,d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$.

The following result holds:

Theorem 2. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[m, M]$ and $f : \Omega \to \mathbb{R}$ satisfying the condition*

$$-\infty < m \le f \le M < \infty \tag{2.2}$$

μ-a.e. on Ω and so that $\Phi \circ f,\ f \in L_w(\Omega,\mu)$. Then we have

$$
\begin{aligned}
0 &\leq \frac{\Phi(m)+\Phi(M)}{2}+\frac{\Phi(M)-\Phi(m)}{M-m}\int_{\Omega}\left(f-\frac{m+M}{2}\right)wd\mu \\
&\quad -\int_{\Omega}(\Phi\circ f)\,wd\mu \\
&\leq \frac{\Phi'_{-}(M)-\Phi'_{+}(m)}{M-m}\int_{\Omega}(M-f)(f-m)\,wd\mu \\
&\leq \frac{\Phi'_{-}(M)-\Phi'_{+}(m)}{M-m}\left(M-\int_{\Omega}fwd\mu\right)\left(\int_{\Omega}fwd\mu-m\right) \\
&\leq \frac{1}{4}(M-m)\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right].
\end{aligned}
\tag{2.3}
$$

Proof. By the convexity of Φ we have

$$
\begin{aligned}
\Phi(t) &= \Phi\left(\frac{M-t}{M-m}m+\frac{t-m}{M-m}M\right) \\
&\leq \frac{M-t}{M-m}\Phi(m)+\frac{t-m}{M-m}\Phi(M) \\
&= \frac{\Phi(m)+\Phi(M)}{2}+\left(\frac{M-t}{M-m}-\frac{1}{2}\right)\Phi(m)+\left(\frac{t-m}{M-m}-\frac{1}{2}\right)\Phi(M) \\
&= \frac{\Phi(m)+\Phi(M)}{2}-\Phi(m)\left(\frac{t-\frac{m+M}{2}}{M-m}\right)+\Phi(M)\left(\frac{t-\frac{m+M}{2}}{M-m}\right) \\
&= \frac{\Phi(m)+\Phi(M)}{2}+\frac{\Phi(M)-\Phi(m)}{M-m}\left(t-\frac{m+M}{2}\right)
\end{aligned}
$$

for any $t \in [m, M]$.

This inequality implies that

$$
\Phi(f(x)) \leq \frac{\Phi(m)+\Phi(M)}{2}+\frac{\Phi(M)-\Phi(m)}{M-m}\left(f(x)-\frac{m+M}{2}\right)
\tag{2.4}
$$

for any $x \in \Omega$.

If we multiply (2.4) by $w \geq 0$ μ-a.e and integrate on Ω, we get the first inequality in (2.3).

We will prove now that

$$
\begin{aligned}
&\frac{M-t}{M-m}\Phi(m)+\frac{t-m}{M-m}\Phi(M)-\Phi(t) \\
&\leq \frac{\Phi'_{-}(M)-\Phi'_{+}(m)}{M-m}(M-t)(t-m) \\
&\leq \frac{1}{4}(M-m)\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right],
\end{aligned}
\tag{2.5}
$$

for any $t \in [m, M]$, and if $\Phi'_{-}(M)$ and $\Phi'_{+}(m)$ are finite, the inequalities in (2.5) are sharp.

By the convexity of Φ we have $\Phi(t)-\Phi(M) \geq \Phi'_{-}(M)(t-M)$ for any $t \in (m, M)$. If we multiply this inequality with $t-m \geq 0$, we deduce

$$
(t-m)\Phi(t)-(t-m)\Phi(M) \geq \Phi'_{-}(M)(t-M)(t-m), \quad t \in (m, M).
\tag{2.6}
$$

Similarly, we get

$$(M-t)\,\Phi(t)-(M-t)\,\Phi(m)\geq\Phi'_{+}(m)\,(t-m)\,(M-t)\,,\quad t\in(m,M)\,. \tag{2.7}$$

Adding (2.6) to (2.7) and dividing by $M-m$, we deduce

$$\Phi(t)-\frac{(t-m)\,\Phi(M)+(M-t)\,\Phi(m)}{M-m}\geq\frac{(M-t)\,(t-m)}{M-m}\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right],$$

for any $t\in(m,M)$, which proves the first inequality in (2.5) for $t\in(m,M)$.

The second inequality in (2.5) is obvious by the elementary fact that

$$\alpha\beta\leq\left(\frac{\alpha+\beta}{2}\right)^{2}\text{ for real }\alpha,\beta. \tag{2.8}$$

If $t=m$ or $t=M$, the inequality also holds.

Now, assume that (2.5) holds with D and E greater than zero, i.e.,

$$\begin{aligned}\Phi_{\Phi}(t)&\leq D\cdot\frac{(M-t)\,(t-m)}{M-m}\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right]\\&\leq E\,(M-m)\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right]\end{aligned}$$

for any $t\in[m,M]$. If we choose $t=\frac{m+M}{2}$, then we get

$$\begin{aligned}\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)&\leq\frac{1}{4}D\,(M-m)\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right]\\&\leq E\,(M-m)\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right].\end{aligned} \tag{2.9}$$

Consider $\Phi:[m,M]\to\mathbb{R}$, $\Phi(t)=\left|t-\frac{m+M}{2}\right|$. Then Φ is convex, $\Phi(m)=\Phi(M)=\frac{M-m}{2}$, $\Phi\left(\frac{m+M}{2}\right)=0$, $\Phi'_{-}(M)=1$, $\Phi'_{+}(m)=-1$ and by (2.9) we deduce

$$\frac{M-m}{2}\leq\frac{1}{2}D\,(M-m)\leq 2E\,(M-m)\,,$$

which implies that $D\geq 1$ and $E\geq\frac{1}{4}$.

From (2.5) we have

$$\begin{aligned}&\frac{M-f(x)}{M-m}\Phi(m)+\frac{f(x)-m}{M-m}\Phi(M)-\Phi(f(x))\\&\leq\frac{\Phi'_{-}(M)-\Phi'_{+}(m)}{M-m}\,(M-f(x))\,(f(x)-m)\end{aligned} \tag{2.10}$$

for μ -a.e. $x\in\Omega$.

If we multiply (2.10) by $w\geq 0$ μ-a.e and integrate on Ω, we get the second inequality in (2.3).

Observe that the function $g:[m,M]\to\mathbb{R}$, $g(t)=(M-t)\,(t-m)$ is a concave function on $[m,M]$. Then by Jensen's inequality for concave functions we have

$$\int_{\Omega}(M-f)\,(f-m)\,wd\mu\leq\left(M-\int_{\Omega}fwd\mu\right)\left(\int_{\Omega}fwd\mu-m\right),$$

which proves the third inequality in (2.3).

The last part follows by (2.8). □

Corollary 1. *With the assumptions of Theorem 2 and if*

$$\int_{\Omega}\left(f-\frac{m+M}{2}\right)wd\mu=0, \tag{2.11}$$

then we have

$$0 \leq \frac{\Phi(m)+\Phi(M)}{2} - \int_\Omega (\Phi \circ f)\, w d\mu \leq \frac{\Phi'_-(M)-\Phi'_+(m)}{M-m} \int_\Omega (M-f)(f-m)\, w d\mu \leq \frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right]. \tag{2.12}$$

Remark 1. *Let $h:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a convex function and $g:[a,b]\to[0,\infty)$ a Lebesgue integrable function on $[a,b]$. Then from Theorem 2 we have*

$$\begin{aligned} 0 &\leq \frac{h(a)+h(b)}{2} + \frac{h(b)-h(a)}{b-a}\cdot\frac{1}{\int_a^b g(x)\,dx}\int_a^b \left(x-\frac{a+b}{2}\right) g(x)\,dx \\ &\quad - \frac{1}{\int_a^b g(x)\,dx}\int_a^b h(x)\,g(x)\,dx \\ &\leq \frac{h'_-(b)-h'_+(a)}{b-a}\frac{1}{\int_a^b g(x)\,dx}\int_a^b (b-x)(x-a)\,g(x)\,dx \\ &\leq \frac{h'_-(b)-h'_+(a)}{b-a} \\ &\quad\times\left(b-\frac{1}{\int_a^b g(x)\,dx}\int_a^b xg(x)\,dx\right)\left(\frac{1}{\int_a^b g(x)\,dx}\int_a^b xg(x)\,dx - a\right) \\ &\leq \frac{1}{4}(b-a)\left[h'_-(b)-h'_+(a)\right]. \end{aligned} \tag{2.13}$$

If we assume that $g:[a,b]\to\mathbb{R}$ is integrable and symmetric on the interval $[a,b]$, then

$$\int_a^b \left(x-\frac{a+b}{2}\right) g(x)\,dx = 0. \tag{2.14}$$

The converse is obviously not true.

Now, if $g:[a,b]\to[0,\infty)$ is integrable and satisfies the condition (2.14), then from (2.13) we get

$$\begin{aligned} 0 &\leq \frac{h(a)+h(b)}{2} - \frac{1}{\int_a^b g(x)\,dx}\int_a^b h(x)\,g(x)\,dx \\ &\leq \frac{h'_-(b)-h'_+(a)}{b-a}\frac{1}{\int_a^b g(x)\,dx}\int_a^b (b-x)(x-a)\,g(x)\,dx \\ &\leq \frac{1}{4}(b-a)\left[h'_-(b)-h'_+(a)\right]. \end{aligned} \tag{2.15}$$

The above inequality (2.15) provides both a generalization and a reverse for the right Fejér inequality (1.1) as announced in the introduction.

Example 1. *The first two inequalities in (2.15) can be written as*

$$0 \leq \frac{h(a)+h(b)}{2}\int_a^b g(x)\,dx - \int_a^b h(x)\,g(x)\,dx \leq \frac{h'_-(b)-h'_+(a)}{b-a}\int_a^b (b-x)(x-a)\,g(x)\,dx. \tag{2.16}$$

If in this inequality we make $g:[a,b]\to[0,\infty)$, $g(x)=(b-x)(x-a)$ *and since*

$$\int_a^b (b-x)(x-a)\,dx = \frac{1}{6}(b-a)^3, \quad \int_a^b (b-x)^2(x-a)^2\,dx = \frac{1}{30}(b-a)^5,$$

hence

$$0 \leq \frac{h(a)+h(b)}{12}(b-a)^3 - \int_a^b (b-x)(x-a)\,h(x)\,dx \leq \frac{1}{30}(b-a)^4\left[h'_-(b)-h'_+(a)\right], \tag{2.17}$$

for any convex function $h:[a,b]\to\mathbb{R}$.

We have the following result as well:

Theorem 3. *Let* $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ *be a convex function on* $[m,M]$ *and* $f:\Omega\to\mathbb{R}$ *satisfying the condition (2.2)* μ*-a.e. on* Ω *and so that* $\Phi\circ f,\ f\in L_w(\Omega,\mu)$. *Then we have*

$$\begin{aligned} 0 &\leq \left(1-\frac{2}{M-m}\int_\Omega\left|t-\frac{M+m}{2}\right|wd\mu\right) \times\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right] \\ &\leq \frac{\Phi(m)+\Phi(M)}{2}+\frac{\Phi(M)-\Phi(m)}{M-m}\int_\Omega\left(f-\frac{m+M}{2}\right)wd\mu - \int_\Omega(\Phi\circ f)\,wd\mu \\ &\leq \left(1+\frac{2}{M-m}\int_\Omega\left|t-\frac{M+m}{2}\right|wd\mu\right) \times\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]. \end{aligned} \tag{2.18}$$

Proof. First of all, we recall the following result obtained by the author in [5] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} 0 &\leq n\min_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^n\Phi(x_i)-\Phi\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right] \\ &\leq \frac{1}{P_n}\sum_{i=1}^n p_i\Phi(x_i)-\Phi\left(\frac{1}{P_n}\sum_{i=1}^n p_ix_i\right) \\ &\leq n\max_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^n\Phi(x_i)-\Phi\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right], \end{aligned} \tag{2.19}$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i\in\{1,\dots,n\}}$ are vectors and $\{p_i\}_{i\in\{1,\dots,n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^n p_i > 0$.

For $n = 2$ we deduce from (2.19) that

$$\begin{aligned}0 &\le 2\min\{\lambda, 1-\lambda\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] \\ &\le \lambda\Phi(x) + (1-\lambda)\Phi(y) - \Phi(\lambda x + (1-\lambda) y) \\ &\le 2\max\{\lambda, 1-\lambda\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right]\end{aligned} \tag{2.20}$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

If we replace in (2.20) $C = [x,y] = [m,M]$ and $\lambda = \frac{M-t}{M-m}$, then we get:

$$\begin{aligned}0 &\le 2\min\left\{\frac{M-t}{M-m}, \frac{t-m}{M-m}\right\}\left[\frac{\Phi(m)+\Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right] \\ &\le \frac{M-t}{M-m}\Phi(m) + \frac{t-m}{M-m}\Phi(M) - \Phi(t) \\ &\le 2\max\left\{\frac{M-t}{M-m}, \frac{t-m}{M-m}\right\}\left[\frac{\Phi(m)+\Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right].\end{aligned} \tag{2.21}$$

The inequality (2.21) implies that

$$\begin{aligned}0 &\le 2\min\left\{\frac{M-f(x)}{M-m}, \frac{f(x)-m}{M-m}\right\}\left[\frac{\Phi(m)+\Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right] \\ &\le \frac{M-f(x)}{M-m}\Phi(m) + \frac{f(x)-m}{M-m}\Phi(M) - \Phi(f(x)) \\ &\le 2\max\left\{\frac{M-f(x)}{M-m}, \frac{f(x)-m}{M-m}\right\}\left[\frac{\Phi(m)+\Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right]\end{aligned} \tag{2.22}$$

for any $x \in \Omega$.

If we multiply (2.22) by $w \ge 0$ μ-a.e and integrate on Ω, we get

$$\begin{aligned}0 &\le 2\left[\frac{\Phi(m)+\Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right] \\ &\times \int_\Omega \min\left\{\frac{M-f}{M-m}, \frac{f-m}{M-m}\right\} w d\mu \\ &\le \frac{M-\int_\Omega f w d\mu}{M-m}\Phi(m) + \frac{\int_\Omega f w d\mu - m}{M-m}\Phi(M) - \int_\Omega (\Phi\circ f) w d\mu \\ &\le 2\left[\frac{\Phi(m)+\Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right] \\ &\times \int_\Omega \max\left\{\frac{M-f}{M-m}, \frac{f-m}{M-m}\right\} w d\mu.\end{aligned} \tag{2.23}$$

Using the elementary facts

$$\min\{\alpha,\beta\} = \frac{1}{2}(\alpha+\beta-|\alpha-\beta|), \quad \max\{\alpha,\beta\} = \frac{1}{2}(\alpha+\beta+|\alpha-\beta|),$$

where $\alpha, \beta \in \mathbb{R}$, then we have

$$\int_\Omega \min\left\{\frac{M-f}{M-m}, \frac{f-m}{M-m}\right\} w d\mu = \frac{1}{2} - \frac{1}{M-m}\int_\Omega \left|f - \frac{M+m}{2}\right| w d\mu \tag{2.24}$$

and

$$\int_\Omega \max\left\{\frac{M-f}{M-m},\frac{f-m}{M-m}\right\}wd\mu=\frac{1}{2}-\frac{1}{M-m}\int_\Omega\left|f-\frac{M+m}{2}\right|wd\mu. \tag{2.25}$$

Making use of (2.23)-(2.25) we deduce the desired result (2.18). □

Corollary 2. *With the assumptions of Theorem 3 and if the condition (2.11) is satisfied, then*

$$\begin{aligned}0 &\le \left(1-\frac{2}{M-m}\int_\Omega\left|t-\frac{M+m}{2}\right|wd\mu\right)\\ &\quad\times\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]\\ &\le \frac{\Phi(m)+\Phi(M)}{2}-\int_\Omega(\Phi\circ f)\,wd\mu\\ &\le \left(1+\frac{2}{M-m}\int_\Omega\left|t-\frac{M+m}{2}\right|wd\mu\right)\\ &\quad\times\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right].\end{aligned} \tag{2.26}$$

Remark 2. *Let $h:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a convex function and $g:[a,b]\to[0,\infty)$ a Lebesgue integrable function on $[a,b]$. Then from Theorem 3 we have*

$$\begin{aligned}0 &\le \left(1-\frac{2}{b-a}\frac{1}{\int_a^b g(x)\,dx}\int_a^b\left|x-\frac{a+b}{2}\right|g(x)\,dx\right)\\ &\quad\times\left[\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right]\\ &\le \frac{h(a)+h(b)}{2}+\frac{h(b)-h(a)}{b-a}\cdot\frac{1}{\int_a^b g(x)\,dx}\int_a^b\left(x-\frac{a+b}{2}\right)g(x)\,dx\\ &\quad-\frac{1}{\int_a^b g(x)\,dx}\int_a^b h(x)\,g(x)\,dx\\ &\le \left(1+\frac{2}{b-a}\frac{1}{\int_a^b g(x)\,dx}\int_a^b\left|x-\frac{a+b}{2}\right|g(x)\,dx\right)\\ &\quad\times\left[\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right].\end{aligned} \tag{2.27}$$

Now, if $g:[a,b]\to[0,\infty)$ is integrable and satisfies the condition (2.14), then from (2.27) we get

$$
\begin{aligned}
0 &\le \left(1-\frac{2}{b-a}\frac{1}{\int_a^b g(x)\,dx}\int_a^b\left|x-\frac{a+b}{2}\right|g(x)\,dx\right)\\
&\quad\times\left[\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right]\\
&\le \frac{h(a)+h(b)}{2}-\frac{1}{\int_a^b g(x)\,dx}\int_a^b h(x)\,g(x)\,dx\\
&\le \left(1+\frac{2}{b-a}\frac{1}{\int_a^b g(x)\,dx}\int_a^b\left|x-\frac{a+b}{2}\right|g(x)\,dx\right)\\
&\quad\times\left[\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right].
\end{aligned}
\tag{2.28}
$$

The above inequality (2.28) provides a generalization, a refinement and a new reverse for the right Fejér inequality (1.1) as claimed in the introduction.

Example 2. *The inequality (2.28) may be written as*

$$
\begin{aligned}
0 &\le \left(\int_a^b g(x)\,dx-\frac{2}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|g(x)\,dx\right)\\
&\quad\times\left[\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right]\\
&\le \frac{h(a)+h(b)}{2}\int_a^b g(x)\,dx-\int_a^b h(x)\,g(x)\,dx\\
&\le \left(\int_a^b g(x)\,dx+\frac{2}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|g(x)\,dx\right)\\
&\quad\times\left[\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right].
\end{aligned}
\tag{2.29}
$$

If in (2.29) we take $g(x)=\left|x-\frac{a+b}{2}\right|$, $x\in[a,b]$, and since

$$\int_a^b\left|x-\frac{a+b}{2}\right|dx=\frac{1}{4}(b-a)^2,\quad \int_a^b\left(x-\frac{a+b}{2}\right)^2dx=\frac{1}{12}(b-a)^2,$$

hence

$$0 \leq \frac{1}{12}(b-a)^2\left[\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right] \tag{2.30}$$

$$\leq \frac{h(a)+h(b)}{8}(b-a)^2-\int_a^b\left|x-\frac{a+b}{2}\right|h(x)\,dx$$

$$\leq \frac{5}{12}(b-a)^2\left[\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right],$$

for any convex function $h:[a,b]\to\mathbb{R}$.

3. Applications for Discrete Inequalities

Let $\mathbf{x}=(x_1,...,x_n)$ be an n-tuple with $x_i\in\mathbb{R}$, $i\in\{1,...,n\}$ and $\mathbf{p}=(p_1,...,p_n)$ a probability distribution, i.e. $p_i\geq 0$, $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$.

If $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ is a convex function on $[m,M]$, then for any $\mathbf{x}=(x_1,...,x_n)$ with $x_i\in(m,M)\subset\mathbb{R}$, $i\in\{1,...,n\}$ and any probability distribution $\mathbf{p}=(p_1,...,p_n)$ we have from (2.3) and (2.18) for the discrete measure that

$$0\leq\frac{\Phi(m)+\Phi(M)}{2}+\frac{\Phi(M)-\Phi(m)}{M-m}\sum_{i=1}^n p_i\left(x_i-\frac{m+M}{2}\right)-\sum_{i=1}^n p_i\Phi(x_i) \tag{3.1}$$

$$\leq\frac{\Phi'_-(M)-\Phi'_+(m)}{M-m}\sum_{i=1}^n p_i(M-x_i)(x_i-m)$$

$$\leq\frac{\Phi'_-(M)-\Phi'_+(m)}{M-m}\left(M-\sum_{i=1}^n p_ix_i\right)\left(\sum_{i=1}^n p_ix_i-m\right)$$

$$\leq\frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right],$$

$$0\leq\left(1-\frac{2}{M-m}\sum_{i=1}^n p_i\left|x_i-\frac{m+M}{2}\right|\right)\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right] \tag{3.2}$$

$$\leq\frac{\Phi(m)+\Phi(M)}{2}+\frac{\Phi(M)-\Phi(m)}{M-m}\sum_{i=1}^n p_i\left(x_i-\frac{m+M}{2}\right)-\sum_{i=1}^n p_i\Phi(x_i)$$

$$\leq\left(1+\frac{2}{M-m}\sum_{i=1}^n p_i\left|x_i-\frac{m+M}{2}\right|\right)\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]$$

and the corresponding inequalities if $\sum_{i=1}^n p_i\left(x_i-\frac{m+M}{2}\right)=0$.

If we write the inequalities (3.1) and (3.2) for the convex power function $\Phi(t) = t^p, p \in (-\infty, 0) \cup (1, \infty)$, we have

$$
\begin{aligned}
(3.3) \qquad 0 &\leq \frac{m^p + M^p}{2} + \frac{M^p - m^p}{M - m} \sum_{i=1}^{n} p_i \left(x_i - \frac{m + M}{2} \right) - \sum_{i=1}^{n} p_i x_i^p \\
&\leq p \frac{M^{p-1} - m^{p-1}}{M - m} \sum_{i=1}^{n} p_i (M - x_i)(x_i - m) \\
&\leq p \frac{M^{p-1} - m^{p-1}}{M - m} \left(M - \sum_{i=1}^{n} p_i x_i \right) \left(\sum_{i=1}^{n} p_i x_i - m \right) \\
&\leq \frac{1}{4} p (M - m) \left(M^{p-1} - m^{p-1} \right)
\end{aligned}
$$

and

$$
\begin{aligned}
(3.4) \qquad 0 &\leq \left(1 - \frac{2}{M - m} \sum_{i=1}^{n} p_i \left| x_i - \frac{m + M}{2} \right| \right) \left[\frac{m^p + M^p}{2} - \left(\frac{m + M}{2} \right)^p \right] \\
&\leq \frac{m^p + M^p}{2} + \frac{M^p - m^p}{M - m} \sum_{i=1}^{n} p_i \left(x_i - \frac{m + M}{2} \right) - \sum_{i=1}^{n} p_i x_i^p \\
&\leq \left(1 + \frac{2}{M - m} \sum_{i=1}^{n} p_i \left| x_i - \frac{m + M}{2} \right| \right) \left[\frac{m^p + M^p}{2} - \left(\frac{m + M}{2} \right)^p \right].
\end{aligned}
$$

If we write the inequalities (3.1) and (3.2) for the convex function $\Phi(t) = -\ln t$, then we get

$$
\begin{aligned}
(3.5) \qquad 0 &\leq \sum_{i=1}^{n} p_i \ln x_i - \ln G(m, M) - \frac{1}{L(m, M)} \sum_{i=1}^{n} p_i \left(x_i - \frac{m + M}{2} \right) \\
&\leq \frac{1}{G^2(m, M)} \sum_{i=1}^{n} p_i (M - x_i)(x_i - m) \\
&\leq \frac{1}{G^2(m, M)} \left(M - \sum_{i=1}^{n} p_i x_i \right) \left(\sum_{i=1}^{n} p_i x_i - m \right) \leq \frac{1}{4} \frac{(M - m)^2}{mM},
\end{aligned}
$$

and

$$
\begin{aligned}
(3.6) \qquad 0 &\leq \left(1 - \frac{2}{M - m} \sum_{i=1}^{n} p_i \left| x_i - \frac{m + M}{2} \right| \right) \ln \left(\frac{A(m, M)}{G(m, M)} \right) \\
&\leq \sum_{i=1}^{n} p_i \ln x_i - \ln G(m, M) - \frac{1}{L(m, M)} \sum_{i=1}^{n} p_i \left(x_i - \frac{m + M}{2} \right) \\
&\leq \left(1 + \frac{2}{M - m} \sum_{i=1}^{n} p_i \left| x_i - \frac{m + M}{2} \right| \right) \ln \left(\frac{A(m, M)}{G(m, M)} \right),
\end{aligned}
$$

where $A(m, M) := \frac{m+M}{2}$ is *the arithmetic mean*, $G(m, M) := \sqrt{mM}$ is the *geometric mean* and

$$L(m, M) := \frac{M - m}{\ln M - \ln m}$$

is the *logarithmic mean*.

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[1]Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa