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GENERALIZATION OF OSTROWSKI INEQUALITY FOR CONVEX FUNCTIONS

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Abstract. In this paper we establish some related Ostrowski inequalities for the case of convex functions and general Lebesgue integral on measurable spaces. Midpoint and integral mean inequalities are provided, some particular results related to the famous *Fejér's inequality* are also given.

1. Introduction

In 1938, A. Ostrowski [13] proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b h(t)\,dt$ and the value $h(x)$, $x\in[a,b]$ in the case of differentiable functions on an open interval:

Theorem 1 (Ostrowski, 1938 [13])**.** *Let $h:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $h':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|h'\|_\infty := \sup_{t\in(a,b)}|h'(t)|<\infty$. Then*

$$\left|h(x)-\frac{1}{b-a}\int_a^b h(t)\,dt\right|\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|h'\|_\infty (b-a), \tag{1.1}$$

for all $x\in[a,b]$ and the constant $\frac{1}{4}$ is the best possible.

Let $(\Omega,\mathcal{A},\mu)$ be a measurable space consisting of a set Ω, a $\sigma-$ algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\ge 0$ for μ -a.e.(almost every) $x\in\Omega$, consider the *Lebesgue space*

$$L_w(\Omega,\mu):=\{f:\Omega\to\mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)|\,w(x)\,d\mu(x)<\infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$.

In what follows we assume that $w\ge 0$ μ-a.e. on Ω with $\int_\Omega w d\mu = 1$.

Let $\Phi: I\to\mathbb{C}$ be an absolutely continuous functions on $[a,b]\subset\mathring{I}$, the interior of I. If $f:\Omega\to[a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi\circ f,\ f\in L(\Omega,\mu)$, then the following Ostrowski type inequality for the general Lebesgue integral on measurable spaces holds [6]

$$\left|\int_\Omega(\Phi\circ f)\,wd\mu-\Phi(x)\right|\le\|\Phi'\|_{[a,b],\infty}\int_\Omega|f-x|\,wd\mu \tag{1.2}$$

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for any $x \in [a,b]$.

In particular, we have

$$\left| \int_{\Omega} (\Phi \circ f)\, w d\mu - \Phi\left(\frac{a+b}{2}\right) \right| \leq \|\Phi'\|_{[a,b],\infty} \int_{\Omega} \left| f - \frac{a+b}{2} \right| w d\mu \tag{1.3}$$

and

$$\left| \int_{\Omega} (\Phi \circ f)\, w d\mu - \Phi\left(\int_{\Omega} f w d\mu\right) \right| \leq \|\Phi'\|_{[a,b],\infty} \int_{\Omega} \left| f - \int_{\Omega} f w d\mu \right| w d\mu. \tag{1.4}$$

Motivated by the above results, in this paper we establish some related inequalities for the case of Φ being a convex function. Midpoint and integral mean inequalities are provided, some particular results related to the famous *Fejér's inequality* are also given.

2. General Results

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $\Phi : I \to \mathbb{R}$ is a convex function on I. Then Φ is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $\Phi'_-(x) \leq \Phi'_+(x) \leq \Phi'_-(y) \leq \Phi'_+(y)$ which shows that both Φ'_- and Φ'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $\Phi : I \to \mathbb{R}$, the *subdifferential* of Φ denoted by $\partial\Phi$ is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$\Phi(x) \geq \Phi(a) + (x-a)\,\varphi(a) \text{ for any } x, a \in I. \tag{2.1}$$

It is also well known that if Φ is convex on I, then $\partial\Phi$ is nonempty, Φ'_-, $\Phi'_+ \in \partial\Phi$ and if $\varphi \in \partial\Phi$, then

$$\Phi'_-(x) \leq \varphi(x) \leq \Phi'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a *nondecreasing function*.

If Φ is differentiable and convex on $\mathring{I}$, then $\partial\Phi = \{\Phi'\}$.

The following result holds:

Theorem 2. *Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[m,M]$, $\varphi \in \partial\Phi$ and $f : \Omega \to \mathbb{R}$ satisfying the condition*

$$-\infty < m \leq f \leq M < \infty \tag{2.2}$$

μ-a.e. on Ω and so that $\Phi \circ f,\ \varphi \circ f,\ (\varphi \circ f) f,\ f \in L_w(\Omega, \mu)$. Then we have the inequalities:

$$
\begin{aligned}
0 &\le \int_\Omega (\Phi \circ f)\, w d\mu - \Phi(s) - \varphi(s) \int_\Omega (f - s)\, w d\mu \\
&\le \int_\Omega (\varphi \circ f - \varphi(s))(f - s)\, w d\mu \\
&\le \operatorname*{essup}_\Omega |f - s| \int_\Omega |\varphi \circ f - \varphi(s)|\, w d\mu \\
&\le \begin{cases} \left(\frac{1}{2}(M - m) + \left|s - \frac{m+M}{2}\right|\right) \int_\Omega |\varphi \circ f - \varphi(s)|\, w d\mu \\ \operatorname{essup}_\Omega |f - s| \left(\Phi'_-(M) - \Phi'_+(m)\right) \end{cases} \\
&\le \left(\frac{1}{2}(M - m) + \left|s - \frac{m + M}{2}\right|\right) \left(\Phi'_-(M) - \Phi'_+(m)\right),
\end{aligned}
\tag{2.3}
$$

for any $s \in (m, M)$.

Proof. By the gradient inequality (2.1) we have

$$\Phi(t) - \Phi(s) \ge (t - s)\varphi(s) \tag{2.4}$$

for any $\varphi \in \partial\Phi$ and for any $t \in [m, M]$.

This inequality implies that

$$\Phi(f(x)) \ge \Phi(s) + (f(x) - s)\varphi(s) \tag{2.5}$$

for any $x \in \Omega$.

If we multiply (2.5) by $w \ge 0$ μ-a.e and integrate on Ω, we get the first inequality in (2.3).

By the gradient inequality (2.1) we also have

$$\varphi(t)(t - s) \ge \Phi(t) - \Phi(s)$$

for any $\varphi \in \partial\Phi$ and for any $t \in (m, M)$.

This inequality is equivalent to

$$(\varphi(t) - \varphi(s))(t - s) \ge \Phi(t) - \Phi(s) - \varphi(s)(t - s)$$

for any $\varphi \in \partial\Phi$ and for any $t \in (m, M)$.

This inequality implies that

$$(\varphi(f(x)) - \varphi(s))(f(x) - s) \ge \Phi(f(x)) - \Phi(s) - \varphi(s)(f(x) - s) \tag{2.6}$$

for μ-a.e $x \in \Omega$.

If we multiply (2.6) by $w \ge 0$ μ-a.e and integrate on Ω, we get

$$
\begin{aligned}
&\int_\Omega (\varphi \circ f - \varphi(s))(f - s)\, w d\mu \\
&\quad \ge \int_\Omega (\Phi \circ f)\, w d\mu - \Phi(s) - \varphi(s) \int_\Omega (f - s)\, w d\mu,
\end{aligned}
\tag{2.7}
$$

which proves the second inequality in (2.3).

Now, since φ is monotonic nondecreasing on $[m, M]$, then for any $s \in (m, M)$

$$(\varphi \circ f - \varphi(s))(f - s) \ge 0$$

μ-a.e. on Ω.

We then have

$$\begin{aligned}\int_{\Omega}(\varphi\circ f-\varphi(s))(f-s)\,wd\mu &= \left|\int_{\Omega}(\varphi\circ f-\varphi(s))(f-s)\,wd\mu\right| \\ &\leq \int_{\Omega}|(\varphi\circ f-\varphi(s))(f-s)|\,wd\mu \\ &\leq \operatorname*{essup}_{\Omega}|f-s|\int_{\Omega}|\varphi\circ f-\varphi(s)|\,wd\mu\end{aligned}$$

and the third inequality in (2.3) is proved.

Since for any $\varphi\in\partial\Phi$ we have $\Phi_{+}'(m)\leq\varphi(t)\leq\Phi_{-}'(M)$ for any $t\in(m,M)$, then $\Phi_{+}'(m)\leq\varphi(s)\leq\Phi_{-}'(M)$ and

$$|\varphi\circ f-\varphi(s)|\leq\Phi_{-}'(M)-\Phi_{+}'(m) \tag{2.8}$$

μ-a.e. on Ω.

If we multiply (2.8) by $w\geq0$ μ-a.e and integrate on Ω, we get

$$\int_{\Omega}|\varphi\circ f-\varphi(s)|\,wd\mu\leq\Phi_{-}'(M)-\Phi_{+}'(m).$$

Also, for any $s\in(m,M)$

$$|f-s|=\left|f-\frac{m+M}{2}+\frac{m+M}{2}-s\right|\leq\left|f-\frac{m+M}{2}\right|+\left|s-\frac{m+M}{2}\right|,$$

which implies that

$$\begin{aligned}\operatorname*{essup}_{\Omega}|f-s| &\leq \operatorname*{essup}_{\Omega}\left|f-\frac{m+M}{2}\right|+\left|s-\frac{m+M}{2}\right| \\ &\leq \frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\end{aligned}$$

and the last part in (2.3) is proved. □

We have the following result:

Corollary 1. *Let $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a convex function on $[m,M]$, $\varphi\in\partial\Phi$ and $f:\Omega\to\mathbb{R}$ satisfying the condition (2.2) μ-a.e. on Ω and so that $\Phi\circ f$, $\varphi\circ f$, $(\varphi\circ f)f$, $f\in L_{w}(\Omega,\mu)$. Then we have the inequalities:*

$$\begin{aligned}0 &\leq \int_{\Omega}(\Phi\circ f)\,wd\mu-\Phi\left(\int_{\Omega}fwd\mu\right)\leq\int_{\Omega}(\varphi\circ f-\lambda)\left(f-\int_{\Omega}fwd\mu\right)wd\mu \\ &\leq \operatorname*{essup}_{\Omega}\left|f-\int_{\Omega}fwd\mu\right|\int_{\Omega}|\varphi\circ f-\lambda|\,wd\mu \\ &\leq \left[\frac{1}{2}(M-m)+\left|\int_{\Omega}fwd\mu-\frac{m+M}{2}\right|\right]\int_{\Omega}|\varphi\circ f-\lambda|\,wd\mu\end{aligned} \tag{2.9}$$

for any $\lambda\in\mathbb{R}$.

Proof. It follows by Theorem 2 on observing that

$$\begin{aligned}&\int_{\Omega}\left(\varphi\circ f-\varphi\left(\int_{\Omega}fwd\mu\right)\right)\left(f-\int_{\Omega}fwd\mu\right)wd\mu \\ &=\int_{\Omega}(\varphi\circ f)\left(f-\int_{\Omega}fwd\mu\right)wd\mu=\int_{\Omega}(\varphi\circ f-\lambda)\left(f-\int_{\Omega}fwd\mu\right)wd\mu\end{aligned}$$

for any $\lambda\in\mathbb{R}$. □

Remark 1. *From the inequality (2.9) we have*

$$0 \leq \int_{\Omega} (\Phi \circ f) w d\mu - \Phi\left(\int_{\Omega} f w d\mu\right) \leq \int_{\Omega} (\varphi \circ f)\left(f - \int_{\Omega} f w d\mu\right) w d\mu \tag{2.10}$$
$$\leq \operatorname{essup}_{\Omega}\left|f - \int_{\Omega} f w d\mu\right| \int_{\Omega} |\varphi \circ f| w d\mu$$
$$\leq \left[\frac{1}{2}(M-m) + \left|\int_{\Omega} f w d\mu - \frac{m+M}{2}\right|\right] \int_{\Omega} |\varphi \circ f| w d\mu.$$

Since for any $\varphi \in \partial\Phi$ we have $\Phi'_{+}(m) \leq \varphi(t) \leq \Phi'_{-}(M)$ for any $t \in (m, M)$, then from (2.9) we have

$$0 \leq \int_{\Omega} (\Phi \circ f) w d\mu - \Phi\left(\int_{\Omega} f w d\mu\right) \tag{2.11}$$
$$\leq \int_{\Omega}\left(\varphi \circ f - \frac{\Phi'_{-}(M) + \Phi'_{+}(m)}{2}\right)\left(f - \int_{\Omega} f w d\mu\right) w d\mu$$
$$\leq \operatorname{essup}_{\Omega}\left|f - \int_{\Omega} f w d\mu\right| \int_{\Omega}\left|\varphi \circ f - \frac{\Phi'_{-}(M) + \Phi'_{+}(m)}{2}\right| w d\mu$$
$$\leq \begin{cases} \left[\frac{1}{2}(M-m) + \left|\int_{\Omega} f w d\mu - \frac{m+M}{2}\right|\right] \int_{\Omega}\left|\varphi \circ f - \frac{\Phi'_{-}(M)+\Phi'_{+}(m)}{2}\right| w d\mu \\ \\ \frac{1}{2}\operatorname{essup}_{\Omega}\left|f - \int_{\Omega} f w d\mu\right| \left(\Phi'_{-}(M) - \Phi'_{+}(m)\right) \end{cases}$$
$$\leq \frac{1}{2}\left[\frac{1}{2}(M-m) + \left|\int_{\Omega} f w d\mu - \frac{m+M}{2}\right|\right]\left(\Phi'_{-}(M) - \Phi'_{+}(m)\right).$$

We have:

Theorem 3. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[m, M]$, $\varphi \in \partial\Phi$ and $f : \Omega \to \mathbb{R}$ satisfying the condition (2.2) μ-a.e. on Ω and so that $\Phi \circ f$, $(\varphi \circ f)^q$, $f^p \in L_w(\Omega, \mu)$, for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then we have the inequalities:*

$$0 \leq \int_{\Omega} (\Phi \circ f) w d\mu - \Phi(s) - \varphi(s) \int_{\Omega} (f - s) w d\mu \tag{2.12}$$
$$\leq \int_{\Omega} (\varphi \circ f - \varphi(s))(f - s) w d\mu$$
$$\leq \left(\int_{\Omega} |f - s|^p w d\mu\right)^{1/p} \left(\int_{\Omega} |\varphi \circ f - \varphi(s)|^q w d\mu\right)^{1/q}$$
$$\leq \begin{cases} \left[\left(\int_{\Omega}\left|f - \frac{m+M}{2}\right|^p w d\mu\right)^{1/p} + \left|\frac{m+M}{2} - s\right|\right] \\ \times \left(\int_{\Omega} |\varphi \circ f - \varphi(s)|^q w d\mu\right)^{1/q} \\ \\ \left(\Phi'_{-}(M) - \Phi'_{+}(m)\right)\left(\int_{\Omega} |f - s|^p w d\mu\right)^{1/p} \end{cases}$$
$$\leq \left[\left(\int_{\Omega}\left|f - \frac{m+M}{2}\right|^p w d\mu\right)^{1/p} + \left|\frac{m+M}{2} - s\right|\right]\left(\Phi'_{-}(M) - \Phi'_{+}(m)\right)$$
$$\leq \left[\frac{1}{2}(M-m) + \left|\frac{m+M}{2} - s\right|\right]\left(\Phi'_{-}(M) - \Phi'_{+}(m)\right)$$

for any $s \in (m, M)$.

Proof. By Hölder's integral inequality we have for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\begin{aligned}\int_\Omega (\varphi \circ f - \varphi(s))(f - s)\, w d\mu &\leq \int_\Omega |(\varphi \circ f - \varphi(s))(f - s)|\, w d\mu \\ &\leq \left(\int_\Omega |f - s|^p\, w d\mu\right)^{1/p} \left(\int_\Omega |\varphi \circ f - \varphi(s)|^q\, w d\mu\right)^{1/q}\end{aligned}$$

for any $s \in (m, M)$, which proves the third inequality in (2.12).

Also, by Minkowski's inequality we have

$$\begin{aligned}\left(\int_\Omega |f - s|^p\, w d\mu\right)^{1/p} &= \left(\int_\Omega \left|f - \frac{m+M}{2} + \frac{m+M}{2} - s\right|^p w d\mu\right)^{1/p} \\ &\leq \left(\int_\Omega \left|f - \frac{m+M}{2}\right|^p w d\mu\right)^{1/p} + \left(\int_\Omega \left|\frac{m+M}{2} - s\right|^p w d\mu\right)^{1/p} \\ &= \left(\int_\Omega \left|f - \frac{m+M}{2}\right|^p w d\mu\right)^{1/p} + \left|\frac{m+M}{2} - s\right| \\ &\leq \frac{1}{2}(M - m) + \left|\frac{m+M}{2} - s\right|\end{aligned}$$

and

$$\begin{aligned}\left(\int_\Omega |\varphi \circ f - \varphi(s)|^q\, w d\mu\right)^{1/q} &\leq \left(\left(\Phi'_-(M) - \Phi'_+(m)\right)^q \int_\Omega w d\mu\right)^{1/q} \\ &= \Phi'_-(M) - \Phi'_+(m)\end{aligned}$$

for any $s \in (m, M)$ that proves the last part of (2.12). $\square$

Remark 2. *In a similar way, the inequality (2.12) can be extended for $p = 1$ and $q = \infty$ to obtain the result:*

$$\begin{aligned}(2.13)\qquad 0 &\leq \int_\Omega (\Phi \circ f)\, w d\mu - \Phi(s) - \varphi(s) \int_\Omega (f - s)\, w d\mu \\ &\leq \int_\Omega (\varphi \circ f - \varphi(s))(f - s)\, w d\mu \\ &\leq \operatorname*{essup}_\Omega |\varphi \circ f - \varphi(s)| \int_\Omega |f - s|\, w d\mu \\ &\leq \begin{cases} \left[\int_\Omega \left|f - \frac{m+M}{2}\right| w d\mu + \left|s - \frac{m+M}{2}\right|\right] \operatorname{essup}_\Omega |\varphi \circ f - \varphi(s)| \\ \left(\Phi'_-(M) - \Phi'_+(m)\right) \int_\Omega |f - s|\, w d\mu \end{cases} \\ &\leq \left[\int_\Omega \left|f - \frac{m+M}{2}\right| w d\mu + \left|s - \frac{m+M}{2}\right|\right] \left(\Phi'_-(M) - \Phi'_+(m)\right) \\ &\leq \left[\frac{1}{2}(M - m) + \left|s - \frac{m+M}{2}\right|\right] \left(\Phi'_-(M) - \Phi'_+(m)\right)\end{aligned}$$

for any $s \in (m, M)$ provided that $\varphi \circ f$ is essentially bounded on Ω.

Corollary 2. *Under the assumptions of Theorem 3 and if we take* $s=\int_{\Omega}fwd\mu$, *then we have the simpler inequalities*

$$0\le\int_{\Omega}(\Phi\circ f)\,wd\mu-\Phi\left(\int_{\Omega}fwd\mu\right)\le\int_{\Omega}(\varphi\circ f-\lambda)\left(f-\int_{\Omega}fwd\mu\right)wd\mu \tag{2.14}$$
$$\le\left(\int_{\Omega}\left|f-\int_{\Omega}fwd\mu\right|^{p}wd\mu\right)^{1/p}\left(\int_{\Omega}|\varphi\circ f-\lambda|^{q}\,wd\mu\right)^{1/q}$$

for any $\lambda\in\mathbb{R}$.

In particular we have

$$0\le\int_{\Omega}(\Phi\circ f)\,wd\mu-\Phi\left(\int_{\Omega}fwd\mu\right)\le\int_{\Omega}(\varphi\circ f)\left(f-\int_{\Omega}fwd\mu\right)wd\mu \tag{2.15}$$
$$\le\left(\int_{\Omega}\left|f-\int_{\Omega}fwd\mu\right|^{p}wd\mu\right)^{1/p}\left(\int_{\Omega}|\varphi\circ f|^{q}\,wd\mu\right)^{1/q}$$

and

$$0\le\int_{\Omega}(\Phi\circ f)\,wd\mu-\Phi\left(\int_{\Omega}fwd\mu\right) \tag{2.16}$$
$$\le\int_{\Omega}\left(\varphi\circ f-\frac{\Phi'_{-}(M)+\Phi'_{+}(m)}{2}\right)\left(f-\int_{\Omega}fwd\mu\right)wd\mu$$
$$\le\left(\int_{\Omega}\left|f-\int_{\Omega}fwd\mu\right|^{p}wd\mu\right)^{1/p}$$
$$\times\left(\int_{\Omega}\left|\varphi\circ f-\frac{\Phi'_{-}(M)+\Phi'_{+}(m)}{2}\right|^{q}wd\mu\right)^{1/q}$$
$$\le\begin{cases}(M-m)\left(\int_{\Omega}\left|\varphi\circ f-\frac{\Phi'_{-}(M)+\Phi'_{+}(m)}{2}\right|^{q}wd\mu\right)^{1/q}\\[2ex] \frac{1}{2}\left(\int_{\Omega}\left|f-\frac{m+M}{2}\right|^{p}wd\mu\right)^{1/p}\left(\Phi'_{-}(M)-\Phi'_{+}(m)\right)\end{cases}$$
$$\le\frac{1}{2}(M-m)\left(\Phi'_{-}(M)-\Phi'_{+}(m)\right).$$

If f satisfies the condition (2.2), then by Schwarz and Grüss' inequality [6] we have

$$\int_{\Omega}\left|f-\int_{\Omega}fwd\mu\right|wd\mu\le\left(\int_{\Omega}f^{2}wd\mu-\left(\int_{\Omega}fwd\mu\right)^{2}\right)^{1/2}\le\frac{1}{2}(M-m).$$

We observe that, under the assumptions of Remark 2, we have

$$0\le\int_{\Omega}(\Phi\circ f)\,wd\mu-\Phi\left(\int_{\Omega}fwd\mu\right)\le\int_{\Omega}(\varphi\circ f-\lambda)\left(f-\int_{\Omega}fwd\mu\right)wd\mu \tag{2.17}$$
$$\le\operatorname*{essup}_{\Omega}|\varphi\circ f-\lambda|\int_{\Omega}\left|f-\int_{\Omega}fwd\mu\right|wd\mu$$
$$\le\operatorname*{essup}_{\Omega}|\varphi\circ f-\lambda|\left(\int_{\Omega}f^{2}wd\mu-\left(\int_{\Omega}fwd\mu\right)^{2}\right)^{1/2}$$
$$\le\frac{1}{2}(M-m)\operatorname*{essup}_{\Omega}|\varphi\circ f-\lambda|$$

for any $\lambda \in \mathbb{R}$.

In particular, we have

$$
\begin{aligned}
(2.18)\quad 0 &\leq \int_{\Omega} (\Phi \circ f)\, w d\mu - \Phi\left(\int_{\Omega} f w d\mu\right) \leq \int_{\Omega} (\varphi \circ f)\left(f - \int_{\Omega} f w d\mu\right) w d\mu \\
&\leq \operatorname*{essup}_{\Omega} |\varphi \circ f| \int_{\Omega} \left| f - \int_{\Omega} f w d\mu \right| w d\mu \\
&\leq \operatorname*{essup}_{\Omega} |\varphi \circ f| \left(\int_{\Omega} f^2 w d\mu - \left(\int_{\Omega} f w d\mu\right)^2\right)^{1/2} \\
&\leq \frac{1}{2}(M - m) \operatorname*{essup}_{\Omega} |\varphi \circ f|
\end{aligned}
$$

and

$$
\begin{aligned}
(2.19)\quad 0 &\leq \int_{\Omega} (\Phi \circ f)\, w d\mu - \Phi\left(\int_{\Omega} f w d\mu\right) \\
&\leq \int_{\Omega} \left(\varphi \circ f - \frac{\Phi'_{-}(M) + \Phi'_{+}(m)}{2}\right)\left(f - \int_{\Omega} f w d\mu\right) w d\mu \\
&\leq \operatorname*{essup}_{\Omega} \left|\varphi \circ f - \frac{\Phi'_{-}(M) + \Phi'_{+}(m)}{2}\right| \int_{\Omega} \left| f - \int_{\Omega} f w d\mu \right| w d\mu \\
&\leq \operatorname*{essup}_{\Omega} \left|\varphi \circ f - \frac{\Phi'_{-}(M) + \Phi'_{+}(m)}{2}\right| \left(\int_{\Omega} f^2 w d\mu - \left(\int_{\Omega} f w d\mu\right)^2\right)^{1/2} \\
&\leq \begin{cases} \frac{1}{2}\left(\Phi'_{-}(M) - \Phi'_{+}(m)\right)\left(\int_{\Omega} f^2 w d\mu - \left(\int_{\Omega} f w d\mu\right)^2\right)^{1/2} \\ \\ \frac{1}{2}(M - m)\operatorname{essup}_{\Omega}\left|\varphi \circ f - \frac{\Phi'_{-}(M) + \Phi'_{+}(m)}{2}\right| \end{cases} \\
&\leq \frac{1}{4}(M - m)\left(\Phi'_{-}(M) - \Phi'_{+}(m)\right).
\end{aligned}
$$

We observe that, if Φ is differentiable on (m, M) then we can replace in all inequalities above φ by Φ'. We omit the details.

3. Midpoint Inequalities

If we take in (2.3) $s = \frac{m+M}{2}$ then we get

$$
\begin{aligned}
(3.1)\quad 0 &\leq \int_{\Omega} (\Phi \circ f)\, w d\mu - \Phi\left(\frac{m+M}{2}\right) - \varphi\left(\frac{m+M}{2}\right)\int_{\Omega}\left(f - \frac{m+M}{2}\right) w d\mu \\
&\leq \int_{\Omega} \left(\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right)\left(f - \frac{m+M}{2}\right) w d\mu \\
&\leq \operatorname*{essup}_{\Omega} \left| f - \frac{m+M}{2} \right| \int_{\Omega} \left|\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right| w d\mu \\
&\leq \begin{cases} \frac{1}{2}(M - m)\int_{\Omega}\left|\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right| w d\mu \\ \\ \operatorname{essup}_{\Omega}\left|f - \frac{m+M}{2}\right|\left(\Phi'_{-}(M) - \Phi'_{+}(m)\right) \end{cases} \\
&\leq \frac{1}{2}(M - m)\left(\Phi'_{-}(M) - \Phi'_{+}(m)\right).
\end{aligned}
$$

If we take in (2.12) $s=\frac{m+M}{2}$, then we get

$$
\begin{aligned}
(3.2)\quad 0 &\le \int_\Omega (\Phi\circ f)\, wd\mu - \Phi\left(\frac{m+M}{2}\right) - \varphi\left(\frac{m+M}{2}\right)\int_\Omega \left(f-\frac{m+M}{2}\right) wd\mu \\
&\le \int_\Omega \left(\varphi\circ f - \varphi\left(\frac{m+M}{2}\right)\right)\left(f-\frac{m+M}{2}\right) wd\mu \\
&\le \left(\int_\Omega \left|f-\frac{m+M}{2}\right|^p wd\mu\right)^{1/p}\left(\int_\Omega \left|\varphi\circ f-\varphi\left(\frac{m+M}{2}\right)\right|^q wd\mu\right)^{1/q} \\
&\le \begin{cases} \frac{1}{2}(M-m)\left(\int_\Omega \left|\varphi\circ f-\varphi\left(\frac{m+M}{2}\right)\right|^q wd\mu\right)^{1/q} \\ \\ \left(\Phi'_-(M)-\Phi'_+(m)\right)\left(\int_\Omega \left|f-\frac{m+M}{2}\right|^p wd\mu\right)^{1/p} \end{cases} \\
&\le \frac{1}{2}(M-m)\left(\Phi'_-(M)-\Phi'_+(m)\right).
\end{aligned}
$$

If we take in (2.13) $s=\frac{m+M}{2}$ then, we get

$$
\begin{aligned}
(3.3)\quad 0 &\le \int_\Omega (\Phi\circ f)\, wd\mu - \Phi\left(\frac{m+M}{2}\right) - \varphi\left(\frac{m+M}{2}\right)\int_\Omega \left(f-\frac{m+M}{2}\right) wd\mu \\
&\le \int_\Omega \left(\varphi\circ f - \varphi\left(\frac{m+M}{2}\right)\right)\left(f-\frac{m+M}{2}\right) wd\mu \\
&\le \operatorname*{essup}_\Omega \left|\varphi\circ f-\varphi\left(\frac{m+M}{2}\right)\right| \int_\Omega \left|f-\frac{m+M}{2}\right| wd\mu \\
&\le \begin{cases} \frac{1}{2}(M-m)\operatorname{essup}_\Omega \left|\varphi\circ f-\varphi\left(\frac{m+M}{2}\right)\right| \\ \left(\Phi'_-(M)-\Phi'_+(m)\right)\int_\Omega \left|f-\frac{m+M}{2}\right| wd\mu \end{cases} \\
&\le \frac{1}{2}(M-m)\left(\Phi'_-(M)-\Phi'_+(m)\right).
\end{aligned}
$$

Now, if f and w are such that

$$
(3.4)\qquad \int_\Omega \left(f-\frac{m+M}{2}\right) wd\mu = 0,
$$

then

$$
\begin{aligned}
&\int_\Omega \left(\varphi\circ f - \varphi\left(\frac{m+M}{2}\right)\right)\left(f-\frac{m+M}{2}\right) wd\mu \\
&= \int_\Omega (\varphi\circ f-\gamma)\left(f-\frac{m+M}{2}\right) wd\mu
\end{aligned}
$$

for any $\gamma\in\mathbb{R}$.

By making use of the first part of (3.1)-(3.3) we get

$$
\begin{aligned}
(3.5)\quad 0 &\le \int_\Omega (\Phi\circ f)\, wd\mu - \Phi\left(\frac{m+M}{2}\right) \le \int_\Omega (\varphi\circ f-\gamma)\left(f-\frac{m+M}{2}\right) wd\mu \\
&\le \operatorname*{essup}_\Omega |\varphi\circ f-\gamma| \int_\Omega \left|f-\frac{m+M}{2}\right| wd\mu \le \frac{1}{2}(M-m)\operatorname*{essup}_\Omega |\varphi\circ f-\gamma|,
\end{aligned}
$$

$$0 \le \int_\Omega (\Phi \circ f)\, w d\mu - \Phi\left(\frac{m+M}{2}\right) \le \int_\Omega (\varphi \circ f - \gamma)\left(f - \frac{m+M}{2}\right) w d\mu \tag{3.6}$$
$$\le \left(\int_\Omega |\varphi \circ f - \gamma|^p\right)^{1/p} \left(\int_\Omega \left|f - \frac{m+M}{2}\right|^q w d\mu\right)^{1/q}$$
$$\le \frac{1}{2}(M-m)\left(\int_\Omega |\varphi \circ f - \gamma|^p\right)^{1/p}$$

and

$$0 \le \int_\Omega (\Phi \circ f)\, w d\mu - \Phi\left(\frac{m+M}{2}\right) \le \int_\Omega (\varphi \circ f - \gamma)\left(f - \frac{m+M}{2}\right) w d\mu \tag{3.7}$$
$$\le \operatorname*{essup}_\Omega \left|f - \frac{m+M}{2}\right| \int_\Omega |\varphi \circ f - \gamma|\, w d\mu \le \frac{1}{2}(M-m)\int_\Omega |\varphi \circ f - \gamma|,$$

for any $\gamma \in \mathbb{R}$.

In particular, we have

$$0 \le \int_\Omega (\Phi \circ f)\, w d\mu - \Phi\left(\frac{m+M}{2}\right) \le \int_\Omega (\varphi \circ f)\left(f - \frac{m+M}{2}\right) w d\mu \tag{3.8}$$
$$\le \operatorname*{essup}_\Omega |\varphi \circ f| \int_\Omega \left|f - \frac{m+M}{2}\right| w d\mu \le \frac{1}{2}(M-m)\operatorname*{essup}_\Omega |\varphi \circ f|,$$

$$0 \le \int_\Omega (\Phi \circ f)\, w d\mu - \Phi\left(\frac{m+M}{2}\right) \le \int_\Omega (\varphi \circ f)\left(f - \frac{m+M}{2}\right) w d\mu \tag{3.9}$$
$$\le \left(\int_\Omega |\varphi \circ f|^p\, w d\mu\right)^{1/p} \left(\int_\Omega \left|f - \frac{m+M}{2}\right|^q w d\mu\right)^{1/q}$$
$$\le \frac{1}{2}(M-m)\left(\int_\Omega |\varphi \circ f|^p\, w d\mu\right)^{1/p}$$

and

$$0 \le \int_\Omega (\Phi \circ f)\, w d\mu - \Phi\left(\frac{m+M}{2}\right) \le \int_\Omega (\varphi \circ f)\left(f - \frac{m+M}{2}\right) w d\mu \tag{3.10}$$
$$\le \operatorname*{essup}_\Omega \left|f - \frac{m+M}{2}\right| \int_\Omega |\varphi \circ f|\, w d\mu \le \frac{1}{2}(M-m)\int_\Omega |\varphi \circ f|\, w d\mu.$$

We also have

$$0 \le \int_\Omega (\Phi \circ f)\, w d\mu - \Phi\left(\frac{m+M}{2}\right) \tag{3.11}$$
$$\le \int_\Omega \left(\varphi \circ f - \frac{\Phi'_-(M) + \Phi'_+(m)}{2}\right)\left(f - \frac{m+M}{2}\right) w d\mu$$
$$\le \operatorname*{essup}_\Omega \left|f - \frac{m+M}{2}\right| \int_\Omega \left|\varphi \circ f - \frac{\Phi'_-(M) + \Phi'_+(m)}{2}\right| w d\mu$$
$$\le \begin{cases} \frac{1}{2}(M-m)\int_\Omega \left|\varphi \circ f - \frac{\Phi'_-(M)+\Phi'_+(m)}{2}\right| w d\mu \\ \\ \frac{1}{2}\left(\Phi'_-(M) - \Phi'_+(m)\right)\operatorname{essup}_\Omega \left|f - \frac{m+M}{2}\right| \end{cases}$$
$$\le \frac{1}{4}(M-m)\left(\Phi'_-(M) - \Phi'_+(m)\right),$$

$$
\begin{aligned}
0 &\le \int_{\Omega} (\Phi \circ f)\, w d\mu - \Phi\left(\frac{m+M}{2}\right) \\
&\le \int_{\Omega} \left(\varphi \circ f - \frac{\Phi'_{-}(M) + \Phi'_{+}(m)}{2}\right)\left(f - \frac{m+M}{2}\right) w d\mu \\
&\le \left(\int_{\Omega} \left|\varphi \circ f - \frac{\Phi'_{-}(M) + \Phi'_{+}(m)}{2}\right|^{p}\right)^{1/p} \left(\int_{\Omega} \left|f - \frac{m+M}{2}\right|^{q} w d\mu\right)^{1/q} \\
&\le \begin{cases} \frac{1}{2}(M-m)\left(\int_{\Omega} \left|\varphi \circ f - \frac{\Phi'_{-}(M)+\Phi'_{+}(m)}{2}\right|^{p} w d\mu\right)^{1/p} \\ \\ \frac{1}{2}\left(\Phi'_{-}(M) - \Phi'_{+}(m)\right)\left(\int_{\Omega} \left|f - \frac{m+M}{2}\right|^{q} w d\mu\right)^{1/q} \end{cases} \\
&\le \frac{1}{4}(M-m)\left(\Phi'_{-}(M) - \Phi'_{+}(m)\right)
\end{aligned} \tag{3.12}
$$

and

$$
\begin{aligned}
0 &\le \int_{\Omega} (\Phi \circ f)\, w d\mu - \Phi\left(\frac{m+M}{2}\right) \\
&\le \int_{\Omega} \left(\varphi \circ f - \frac{\Phi'_{-}(M) + \Phi'_{+}(m)}{2}\right)\left(f - \frac{m+M}{2}\right) w d\mu \\
&\le \operatorname*{essup}_{\Omega} \left|f - \frac{m+M}{2}\right| \int_{\Omega} \left|\varphi \circ f - \frac{\Phi'_{-}(M) + \Phi'_{+}(m)}{2}\right| w d\mu \\
&\le \begin{cases} \frac{1}{2}(M-m)\int_{\Omega} \left|\varphi \circ f - \frac{\Phi'_{-}(M)+\Phi'_{+}(m)}{2}\right| w d\mu \\ \\ \frac{1}{2}\left(\Phi'_{-}(M) - \Phi'_{+}(m)\right) \operatorname{essup}_{\Omega} \left|f - \frac{m+M}{2}\right| \end{cases} \\
&\le \frac{1}{4}(M-m)\left(\Phi'_{-}(M) - \Phi'_{+}(m)\right).
\end{aligned} \tag{3.13}
$$

4. Inequalities for Integral Means

We have the following result as well:

Theorem 4. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[m, M]$, $\varphi \in \partial\Phi$ and $f : \Omega \to \mathbb{R}$ satisfying the condition (2.2) μ-a.e. on Ω and so that $\Phi \circ f$, $\varphi \circ f$, $(\varphi \circ f) f$, $f \in L_w(\Omega, \mu)$. Then we have the inequalities:*

$$
\begin{aligned}
0 &\le \frac{1}{2}\left(\int_{\Omega} (\Phi \circ f)\, w d\mu + \frac{\Phi(M) M - \Phi(m) m}{M-m}\right) \\
&\quad - \frac{1}{M-m}\int_{m}^{M} \Phi(s)\, ds - \frac{1}{2}\frac{\Phi(M) - \Phi(m)}{M-m}\int_{\Omega} f w d\mu \\
&\le B(m, M, \varphi, f)
\end{aligned} \tag{4.1}
$$

where

$$
\begin{aligned}
& B(m, M, \varphi, f) \\
&:= \frac{1}{2}\frac{1}{M-m}\int_{\Omega}\left(\int_{m}^{M} ((\varphi \circ f)(x) - \varphi(s))(f(x) - s)\, ds\right) w(x)\, d\mu(x).
\end{aligned} \tag{4.2}
$$

We have the bounds

$$
\begin{aligned}
(4.3)\qquad & B(m,M,\varphi,f)\\
&\le \frac{1}{8}(M-m)\int_\Omega\left(\operatorname*{essup}_{s\in[m,M]}\left|(\varphi\circ f)(x)-\varphi(s)\right|w(x)\right)d\mu(x)\\
&+\frac{1}{2}(M-m)\\
&\times\int_\Omega\left(\operatorname*{essup}_{s\in[m,M]}\left|(\varphi\circ f)(x)-\varphi(s)\right|\left(\frac{f(x)-\frac{m+M}{2}}{M-m}\right)^2\right)w(x)\,d\mu(x),
\end{aligned}
$$

$$
\begin{aligned}
(4.4)\qquad & B(m,M,\varphi,f)\\
&\le \frac{1}{2}\frac{1}{(M-m)(q+1)^{1/q}}\int_\Omega\left[\left(\int_m^M\left|(\varphi\circ f)(x)-\varphi(s)\right|^p ds\right)^{1/p}\right.\\
&\times\left.\left((M-f(x))^{q+1}+(f(x)-m)^{q+1}\right)^{1/q}\right]w(x)\,d\mu(x)
\end{aligned}
$$

and

$$
\begin{aligned}
(4.5)\qquad & B(m,M,\varphi,f)\\
&\le \frac{1}{4}\int_\Omega\left(\int_m^M\left|(\varphi\circ f)(x)-\varphi(s)\right|ds\right)w(x)\,d\mu(x)\\
&+\frac{1}{2}\frac{1}{M-m}\\
&\times\int_\Omega\left(\left|f(x)-\frac{m+M}{2}\right|\int_m^M\left|(\varphi\circ f)(x)-\varphi(s)\right|ds\right)w(x)\,d\mu(x).
\end{aligned}
$$

Proof. If we take the integral mean in the first two inequalities in (2.3) we have

$$
\begin{aligned}
(4.6)\quad 0 &\le \int_\Omega(\Phi\circ f)\,wd\mu-\frac{1}{M-m}\int_m^M\Phi(s)\,ds-\frac{1}{M-m}\int_m^M\varphi(s)\,ds\int_\Omega fwd\mu\\
&+\frac{1}{M-m}\int_m^M\varphi(s)\,sds\\
&\le\frac{1}{M-m}\int_m^M\left(\int_\Omega(\varphi\circ f-\varphi(s))(f-s)\,wd\mu\right)ds\\
&=\frac{1}{M-m}\int_\Omega\left(\int_m^M(\varphi\circ f-\varphi(s))(f-s)\,ds\right)wd\mu.
\end{aligned}
$$

We also have

$$
\frac{1}{M-m}\int_m^M\varphi(s)\,ds=\frac{\Phi(M)-\Phi(m)}{M-m},
$$

$$\frac{1}{M-m}\int_{m}^{M}\varphi(s)\,sds=\frac{1}{M-m}\left[\Phi(s)\,s|_{m}^{M}-\int_{m}^{M}\Phi(s)\,ds\right]$$
$$=\frac{1}{M-m}\left[\Phi(M)\,M-\Phi(m)\,m-\int_{m}^{M}\Phi(s)\,ds\right]$$

and by (4.6) we get

$$\begin{aligned}(4.7)\qquad 0&\le\int_{\Omega}(\Phi\circ f)\,wd\mu-\frac{1}{M-m}\int_{m}^{M}\Phi(s)\,ds-\frac{\Phi(M)-\Phi(m)}{M-m}\int_{\Omega}fwd\mu\\&+\frac{\Phi(M)\,M-\Phi(m)\,m}{M-m}-\frac{1}{M-m}\int_{m}^{M}\Phi(s)\,ds\\&\le\frac{1}{M-m}\int_{\Omega}\left(\int_{m}^{M}(\varphi\circ f-\varphi(s))\,(f-s)\,ds\right)wd\mu,\end{aligned}$$

which is equivalent to (4.1).

Since Φ is convex, then for any $\varphi\in\partial\Phi$ we have $(\varphi\circ f-\varphi(s))\,(f-s)\ge 0$ μ-a.e. on Ω then

$$\begin{aligned}(4.8)\qquad B(m,M,\varphi,f)&=\frac{1}{2}\frac{1}{M-m}\int_{\Omega}\left(\int_{m}^{M}(\varphi\circ f-\varphi(s))\,(f-s)\,ds\right)wd\mu\\&=\frac{1}{2}\frac{1}{M-m}\left|\int_{\Omega}\left(\int_{m}^{M}(\varphi\circ f-\varphi(s))\,(f-s)\,ds\right)wd\mu\right|\\&\le\frac{1}{2}\frac{1}{M-m}\int_{\Omega}\left(\int_{m}^{M}|\varphi\circ f-\varphi(s)|\,|f-s|\,ds\right)wd\mu\\&=:C(m,M,\varphi,f).\end{aligned}$$

We have for each $x\in\Omega$ that

$$\begin{aligned}(4.9)\qquad &\int_{m}^{M}|(\varphi\circ f)(x)-\varphi(s)|\,|f(x)-s|\,ds\\&\le\operatorname*{essup}_{s\in[m,M]}|(\varphi\circ f)(x)-\varphi(s)|\int_{m}^{M}|f(x)-s|\,ds\\&=\operatorname*{essup}_{s\in[m,M]}|(\varphi\circ f)(x)-\varphi(s)|\left[\int_{m}^{f(x)}(f(x)-s)\,ds+\int_{f(x)}^{M}(s-f(x))\,ds\right]\\&=\operatorname*{essup}_{s\in[m,M]}|(\varphi\circ f)(x)-\varphi(s)|\,\frac{(f(x)-m)^{2}+(M-f(x))^{2}}{2}\\&=\operatorname*{essup}_{s\in[m,M]}|(\varphi\circ f)(x)-\varphi(s)|\left[\frac{1}{4}(M-m)^{2}+\left(f(x)-\frac{m+M}{2}\right)^{2}\right]\\&=\frac{1}{4}(M-m)^{2}\operatorname*{essup}_{s\in[m,M]}|(\varphi\circ f)(x)-\varphi(s)|\\&+\left(f(x)-\frac{m+M}{2}\right)^{2}\operatorname*{essup}_{s\in[m,M]}|(\varphi\circ f)(x)-\varphi(s)|.\end{aligned}$$

Then

$$
\begin{aligned}
(4.10)\qquad & C(m,M,\varphi,f) \\
& \leq \frac{1}{8}(M-m)\int_{\Omega}\operatorname*{essup}_{s\in[m,M]}\left|(\varphi\circ f)(x)-\varphi(s)\right| w(x)\,d\mu(x) \\
& +\frac{1}{2}(M-m) \\
& \times\int_{\Omega}\left(\frac{f(x)-\frac{m+M}{2}}{M-m}\right)^{2}\operatorname*{essup}_{s\in[m,M]}\left|(\varphi\circ f)(x)-\varphi(s)\right| w(x)\,d\mu(x)
\end{aligned}
$$

and the inequality (4.3) is proved.

Using Hölder's inequality for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ we have

$$
\begin{aligned}
& \int_m^M \left|(\varphi\circ f)(x)-\varphi(s)\right|\left|f(x)-s\right| ds \\
& \leq \left(\int_m^M \left|(\varphi\circ f)(x)-\varphi(s)\right|^p ds\right)^{1/p}\left(\int_m^M \left|f(x)-s\right|^q ds\right)^{1/q} \\
& = \left(\int_m^M \left|(\varphi\circ f)(x)-\varphi(s)\right|^p ds\right)^{1/p}\left(\frac{(M-f(x))^{q+1}+(f(x)-m)^{q+1}}{q+1}\right)^{1/q},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& C(m,M,\varphi,f) \\
& \leq \frac{1}{2}\frac{1}{(M-m)(q+1)^{1/q}}\int_{\Omega}\left[\left(\int_m^M \left|(\varphi\circ f)(x)-\varphi(s)\right|^p ds\right)^{1/p}\right. \\
& \left.\times\left((M-f(x))^{q+1}+(f(x)-m)^{q+1}\right)^{1/q}\right] w(x)\,d\mu(x)
\end{aligned}
$$

and the inequality (4.4) is proved.

We also have

$$
\begin{aligned}
& \int_m^M \left|(\varphi\circ f)(x)-\varphi(s)\right|\left|f(x)-s\right| ds \\
& \leq \sup_{s\in[m,M]}\left|f(x)-s\right|\int_m^M \left|(\varphi\circ f)(x)-\varphi(s)\right| ds \\
& = \max\left\{M-f(x),f(x)-m\right\}\int_m^M \left|(\varphi\circ f)(x)-\varphi(s)\right| ds \\
& = \left[\frac{1}{2}(M-m)+\left|f(x)-\frac{m+M}{2}\right|\right]\int_m^M \left|(\varphi\circ f)(x)-\varphi(s)\right| ds \\
& = \frac{1}{2}(M-m)\int_m^M \left|(\varphi\circ f)(x)-\varphi(s)\right| ds \\
& +\left|f(x)-\frac{m+M}{2}\right|\int_m^M \left|(\varphi\circ f)(x)-\varphi(s)\right| ds
\end{aligned}
$$

for each $x\in\Omega$.

Then

$$
\begin{aligned}
& C(m, M, \varphi, f) \\
& \leq \frac{1}{4} \int_{\Omega}\left(\int_{m}^{M}|(\varphi \circ f)(x)-\varphi(s)| d s\right) w(x) d \mu(x) \\
& +\frac{1}{2} \frac{1}{M-m} \\
& \times \int_{\Omega}\left(\left|f(x)-\frac{m+M}{2}\right| \int_{m}^{M}|(\varphi \circ f)(x)-\varphi(s)| d s\right) w(x) d \mu(x)
\end{aligned}
$$

and the inequality (4.5) is proved. □

Remark 3. *We observe that*

$$
\begin{aligned}
& \frac{1}{2}\left(\int_{\Omega}(\Phi \circ f) w d \mu+\frac{\Phi(M) M-\Phi(m) m}{M-m}\right) \\
& -\frac{1}{M-m} \int_{m}^{M} \Phi(s) d s-\frac{1}{2} \frac{\Phi(M)-\Phi(m)}{M-m} \int_{\Omega} f w d \mu \\
& =\frac{1}{2}\left(\int_{\Omega}(\Phi \circ f) w d \mu+\frac{\Phi(M) M-\Phi(m) m}{M-m}\right) \\
& -\frac{1}{M-m} \int_{m}^{M} \Phi(s) d s-\frac{1}{2} \frac{\Phi(M)-\Phi(m)}{M-m} \int_{\Omega}\left(f-\frac{m+M}{2}\right) w d \mu \\
& -\frac{1}{4} \frac{(\Phi(M)-\Phi(m))(m+M)}{M-m}
\end{aligned} \tag{4.11}
$$

and

$$
\begin{aligned}
& \frac{1}{2} \frac{\Phi(M) M-\Phi(m) m}{M-m}-\frac{1}{4} \frac{(\Phi(M)-\Phi(m))(m+M)}{M-m} \\
& =\frac{1}{4(M-m)}[2 \Phi(M) M-2 \Phi(m) m-(\Phi(M)-\Phi(m))(m+M)] \\
& =\frac{1}{4(M-m)}[\Phi(M) M-\Phi(m) m-\Phi(M) m+\Phi(m) M] \\
& =\frac{1}{4(M-m)}[(\Phi(M)+\Phi(m))(M-m)]=\frac{\Phi(M)+\Phi(m)}{4}
\end{aligned}
$$

and by (4.11) we get

$$
\begin{aligned}
& \frac{1}{2}\left(\int_{\Omega}(\Phi \circ f) w d \mu+\frac{\Phi(M) M-\Phi(m) m}{M-m}\right) \\
& -\frac{1}{M-m} \int_{m}^{M} \Phi(s) d s-\frac{1}{2} \frac{\Phi(M)-\Phi(m)}{M-m} \int_{\Omega} f w d \mu \\
& =\frac{1}{2}\left(\int_{\Omega}(\Phi \circ f) w d \mu+\frac{\Phi(M)+\Phi(m)}{2}\right) \\
& -\frac{1}{M-m} \int_{m}^{M} \Phi(s) d s-\frac{1}{2} \frac{\Phi(M)-\Phi(m)}{M-m} \int_{\Omega}\left(f-\frac{m+M}{2}\right) w d \mu .
\end{aligned} \tag{4.12}
$$

Therefore, by (4.1) we have

$$0 \leq \frac{1}{2}\left(\int_{\Omega}(\Phi\circ f)\,w d\mu+\frac{\Phi(M)+\Phi(m)}{2}\right) - \frac{1}{M-m}\int_{m}^{M}\Phi(s)\,ds-\frac{1}{2}\frac{\Phi(M)-\Phi(m)}{M-m}\int_{\Omega}\left(f-\frac{m+M}{2}\right)w d\mu \leq B(m,M,\varphi,f) \tag{4.13}$$

with $B(m,M,\varphi,f)$ defined by (4.2).

Corollary 3. *With the assumptions of Theorem 4 and if the condition (3.4) is satisfied, then we have the simpler inequality*

$$0 \leq \frac{1}{2}\left(\int_{\Omega}(\Phi\circ f)\,w d\mu+\frac{\Phi(M)+\Phi(m)}{2}\right)-\frac{1}{M-m}\int_{m}^{M}\Phi(s)\,ds \leq B(m,M,\varphi,f). \tag{4.14}$$

5. Applications for Functions of a Real Variable

Let $h:[a,b]\to\mathbb{R}$ be a convex function and $g:[a,b]\to[0,\infty)$ an integrable weight with $\int_a^b g(x)\,dx=1$. Then by taking $\Omega=[a,b]$, $\Phi=h$, $f(x)=x$, $x\in[a,b]$ and $w=g$ in the inequalities above we can get some inequalities of interest for convex functions of a real variable.

From the inequality (2.3) we have for each $\varphi\in\partial h$, the subdifferential of h on $[a,b]$, that

$$\begin{aligned} 0 &\leq \int_a^b h(x)\,g(x)\,dx-h(s)-\varphi(s)\int_a^b(x-s)\,g(x)\,dx \\ &\leq \int_a^b(\varphi(x)-\varphi(s))(x-s)\,g(x)\,dx \\ &\leq \max\{s-a,b-s\}\int_a^b|\varphi(x)-\varphi(s)|\,g(x)\,dx \\ &= \left(\frac{1}{2}(b-a)+\left|s-\frac{a+b}{2}\right|\right)\int_a^b|\varphi(x)-\varphi(s)|\,g(x)\,dx \end{aligned} \tag{5.1}$$

for any $s\in[a,b]$.

From (2.9) we have

$$\begin{aligned} 0 &\leq \int_a^b h(x)\,g(x)\,dx-\Phi\left(\int_a^b xg(x)\,dx\right) \\ &\leq \int_a^b(\varphi(x)-\lambda)\left(x-\int_a^b yg(y)\,dy\right)g(x)\,dx \\ &\leq \left[\frac{1}{2}(b-a)+\left|\int_a^b xg(x)\,dx-\frac{a+b}{2}\right|\right]\int_{\Omega}|\varphi(x)-\lambda|\,g(x)\,dx \end{aligned} \tag{5.2}$$

for any $\lambda\in\mathbb{R}$.

In particular, we have

$$0 \le \int_a^b h(x) g(x) dx - \Phi\left(\int_a^b x g(x) dx\right) \le \int_a^b \varphi(x)\left(x - \int_a^b y g(y) dy\right) g(x) dx \le \left[\frac{1}{2}(b-a) + \left|\int_a^b x g(x) dx - \frac{a+b}{2}\right|\right] \int_\Omega |\varphi(x)| g(x) dx \tag{5.3}$$

and

$$\begin{aligned} 0 &\le \int_a^b h(x) g(x) dx - \Phi\left(\int_a^b x g(x) dx\right) \\ &\le \int_a^b \left(\varphi(x) - \frac{h'_-(b) + h'_+(a)}{2}\right)\left(x - \int_a^b y g(y) dy\right) g(x) dx \\ &\le \left[\frac{1}{2}(b-a) + \left|\int_a^b x g(x) dx - \frac{a+b}{2}\right|\right] \\ &\times \int_\Omega \left|\varphi(x) - \frac{h'_-(b) + h'_+(a)}{2}\right| g(x) dx \\ &\le \frac{1}{2}\left[\frac{1}{2}(b-a) + \left|\int_a^b x g(x) dx - \frac{a+b}{2}\right|\right]\left(h'_-(b) - h'_+(a)\right). \end{aligned} \tag{5.4}$$

From (3.1) we also have

$$\begin{aligned} 0 &\le \int_a^b h(x) g(x) dx - h\left(\frac{a+b}{2}\right) - \varphi\left(\frac{a+b}{2}\right)\int_a^b \left(x - \frac{a+b}{2}\right) g(x) dx \\ &\le \int_a^b \left(\varphi(x) - \varphi\left(\frac{a+b}{2}\right)\right)\left(x - \frac{a+b}{2}\right) g(x) dx \\ &\le \frac{1}{2}(b-a)\int_a^b \left|\varphi(x) - \varphi\left(\frac{a+b}{2}\right)\right| g(x) dx. \end{aligned} \tag{5.5}$$

Assume that g is a positive function in $[a,b]$ and such that

$$g(a+t) = g(b-t), \quad 0 \le t \le \frac{1}{2}(b-a),$$

i.e., $y = g(x)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b), 0\right)$ and is normal to the x-axis. Under those conditions and if $\int_a^b g(x) dx = 1$, then the following inequalities are valid:

$$h\left(\frac{a+b}{2}\right) \le \int_a^b h(x) g(x) dx \le \frac{h(a) + h(b)}{2}. \tag{5.6}$$

This result is well known in the literature as *Fejér's inequality.*

We observe that if g is symmetric on $[a,b]$, then

$$\int_a^b \left(x - \frac{a+b}{2}\right) g(x) dx = 0. \tag{5.7}$$

However the converse is not true.

From (3.5), if we assume that the condition (5.7) is satisfied, then we have

$$0 \leq \int_a^b h(x) g(x) dx - h\left(\frac{a+b}{2}\right) \leq \int_a^b (\varphi(x) - \gamma)\left(x - \frac{a+b}{2}\right) g(x) dx \tag{5.8}$$
$$\leq \frac{1}{2}(b-a) \int_a^b |\varphi(x) - \gamma| g(x) dx,$$

for any $\lambda \in \mathbb{R}$.

The inequality (5.8) provides both a generalization for weights satisfying the condition (5.7) of the left Fejér's inequality as well as a reverse of this inequality.

In particular we have

$$0 \leq \int_a^b h(x) g(x) dx - h\left(\frac{a+b}{2}\right) \leq \int_a^b \varphi(x)\left(x - \frac{a+b}{2}\right) g(x) dx \tag{5.9}$$
$$\leq \frac{1}{2}(b-a) \int_a^b |\varphi(x)| g(x) dx,$$

and

$$\begin{aligned} 0 &\leq \int_a^b h(x) g(x) dx - h\left(\frac{a+b}{2}\right) \\ &\leq \int_a^b \left(\varphi(x) - \frac{h'_-(b) + h'_+(a)}{2}\right)\left(x - \frac{a+b}{2}\right) g(x) dx \\ &\leq \frac{1}{2}(b-a) \int_a^b \left|\varphi(x) - \frac{h'_-(b) + h'_+(a)}{2}\right| g(x) dx, \\ &\leq \frac{1}{4}(b-a)\left(h'_-(b) - h'_+(a)\right). \end{aligned} \tag{5.10}$$

From (5.10) we also have the dual inequality

$$\begin{aligned} 0 &\leq \int_a^b h(x) g(x) dx - h\left(\frac{a+b}{2}\right) \\ &\leq \int_a^b \left(\varphi(x) - \frac{h'_-(b) + h'_+(a)}{2}\right)\left(x - \frac{a+b}{2}\right) g(x) dx \\ &\leq \operatorname*{essup}_{s\in[a,b]} \left|\varphi(x) - \frac{h'_-(b) + h'_+(a)}{2}\right| \int_a^b \left|x - \frac{a+b}{2}\right| g(x) dx, \\ &\leq \frac{1}{4}(b-a)\left(h'_-(b) - h'_+(a)\right). \end{aligned} \tag{5.11}$$

Moreover, if $g(x) \leq K$ for a.e. $x \in [a,b]$, then we also have

$$\begin{aligned} 0 &\leq \int_a^b h(x) g(x) dx - h\left(\frac{a+b}{2}\right) \\ &\leq \int_a^b \left(\varphi(x) - \frac{h'_-(b) + h'_+(a)}{2}\right)\left(x - \frac{a+b}{2}\right) g(x) dx \\ &\leq \operatorname*{essup}_{s\in[a,b]} \left|\varphi(x) - \frac{h'_-(b) + h'_+(a)}{2}\right| \int_a^b \left|x - \frac{a+b}{2}\right| g(x) dx \\ &\leq \frac{1}{4}(b-a)^2 K\left(h'_-(b) - h'_+(a)\right). \end{aligned} \tag{5.12}$$

The other inequalities from Section 2 and 3 have similar forms for functions of a real variable. The details are not provided here.

From the inequality (4.1) we have

$$0 \le \frac{1}{2}\left(\int_a^b h(x)\, g(x)\, dx + \frac{h(b)\, b - h(a)\, a}{b-a}\right) - \frac{1}{b-a}\int_a^b h(s)\, ds - \frac{1}{2}\frac{h(b)-h(a)}{b-a}\int_a^b x g(x)\, dx \le \frac{1}{2}\frac{1}{b-a}\int_a^b \left(\int_a^b (\varphi(x)-\varphi(s))(x-s)\, ds\right) g(x)\, dx. \tag{5.13}$$

If g satisfies the condition (5.7), then from (4.14) we have

$$0 \le \frac{1}{2}\left(\int_a^b h(x)\, g(x)\, dx + \frac{h(b)+h(a)}{2}\right) - \frac{1}{b-a}\int_a^b h(s)\, ds \le \frac{1}{2}\frac{1}{b-a}\int_a^b \left(\int_a^b (\varphi(x)-\varphi(s))(x-s)\, ds\right) g(x)\, dx. \tag{5.14}$$

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