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REFINING JENSEN'S INTEGRAL INEQUALITY FOR PARTITIONS OF WEIGHTS

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Abstract. In this paper we establish a refinement and some reverses for Jensen's inequality for the general Lebesgue integral on measurable spaces and partitions of weights. Applications for discrete inequalities and weighted means of positive numbers are also given.

1. Introduction

Let C be a convex subset of the linear space X and f a convex function on C. If $\mathbf{p} = (p_1, \dots, p_n)$ is a *probability sequence,* i.e. $p_1, \dots, p_n \geq 0$ with $\sum_{i=1}^{n} p_i = 1$ and $\mathbf{x} = (x_1, \dots, x_n) \in C^n$, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i), \tag{1.1}$$

is well known in the literature as *Jensen's inequality*.

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as the arithmetic mean-geometric mean inequality, Hölder and Minkowski's inequalities, Ky Fan's inequality etc. can be obtained as particular cases of it.

In 1989, J. Pečarić and the author obtained the following refinement of (1.1) (see [14]):

$$\begin{aligned} f\left(\sum_{i=1}^{n} p_i x_i\right) &\leq \sum_{i_1,\dots,i_{k+1}=1}^{n} p_{i_1} \dots p_{i_{k+1}} f\left(\frac{x_{i_1} + \dots + x_{i_{k+1}}}{k+1}\right) \\ &\leq \sum_{i_1,\dots,i_k=1}^{n} p_{i_1} \dots p_{i_k} f\left(\frac{x_{i_1} + \dots + x_{i_k}}{k}\right) \leq \dots \leq \sum_{i=1}^{n} p_i f(x_i), \end{aligned} \tag{1.2}$$

for $k \geq 1$ and $\mathbf{p}$, $\mathbf{x}$ as above.

If $q_1, \dots, q_k \geq 0$ with $\sum_{j=1}^{k} q_j = 1$, then the following weighted refinement obtained in 1994 by the author also holds (see [5]):

$$\begin{aligned} f\left(\sum_{i=1}^{n} p_i x_i\right) &\leq \sum_{i_1,\dots,i_k=1}^{n} p_{i_1} \dots p_{i_k} f\left(\frac{x_{i_1} + \dots + x_{i_k}}{k}\right) \\ &\leq \sum_{i_1,\dots,i_k=1}^{n} p_{i_1} \dots p_{i_k} f(q_1 x_{i_1} + \dots + q_k x_{i_k}) \leq \sum_{i=1}^{n} p_i f(x_i), \end{aligned} \tag{1.3}$$

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where $1 \leq k \leq n$ and $\mathbf{p}$, $\mathbf{x}$ are as above.

More recently the author obtained a different refinement of Jensen's inequality incorporated in (see [7]):

$$f\left(\sum_{j=1}^{n} p_j x_j\right) \leq \min_{k\in\{1,\dots,n\}} \left[(1-p_k) f\left(\frac{\sum_{j=1}^{n} p_j x_j - p_k x_k}{1-p_k}\right) + p_k f(x_k)\right] \tag{1.4}$$
$$\leq \frac{1}{n}\left[\sum_{k=1}^{n} (1-p_k) f\left(\frac{\sum_{j=1}^{n} p_j x_j - p_k x_k}{1-p_k}\right) + \sum_{k=1}^{n} p_k f(x_k)\right]$$
$$\leq \max_{k\in\{1,\dots,n\}} \left[(1-p_k) f\left(\frac{\sum_{j=1}^{n} p_j x_j - p_k x_k}{1-p_k}\right) + p_k f(x_k)\right]$$
$$\leq \sum_{j=1}^{n} p_j f(x_j),$$

where f, x_k and p_k are as above.

For other refinements and applications related to Ky Fan's inequality, the arithmetic mean-geometric mean inequality, the generalized triangle inequality, the f-Divergence measure etc., see [1], [2]-[12], [13] and [15]-[16]

Motivated by the above results, we investigate in this paper the integral version of Jensen inequality and establish some refinements and reverses of interest for applications.

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ − algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For the μ-integrable positive μ-a.e. weight w consider the Lebesgue space

$$L_w(\Omega,\mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)| w(t) d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t) d\mu(t)$ etc..

For the μ-integrable positive μ-a.e. weight w and a given $n \geq 2$ we consider the set $\mathfrak{P}_n(w)$ all possible n-tuples of μ-integrable positive μ-a.e. weights $\overline{w} = (w_1, ..., w_n)$ with the property that $\sum_{i=1}^{n} w_i = w$. The n-tuple $\overline{w} = (w_1, ..., w_n)$ it is called *a partition* of the weight w. It is clear that $\sum_{i=1}^{n} \int_\Omega w_i d\mu = \int_\Omega w d\mu$ for any $(w_1, ..., w_n) \in \mathfrak{P}_n(w)$ and $\int_\Omega w_i d\mu > 0$.

For a convex function $\Phi : [m, M] \to \mathbb{R}$, a μ-measurable function $f : \Omega \to [m, M]$ such that f, $\Phi \circ f \in L_w(\Omega, \mu)$ we define the functional $\psi(\Phi, f, \cdot) : \mathfrak{P}_n(w) \to \mathbb{R}$ by

$$\psi(\Phi, f, \overline{w}) := \frac{1}{\int_\Omega w d\mu} \sum_{i=1}^{n} \Phi\left(\frac{\int_\Omega f w_i d\mu}{\int_\Omega w_i d\mu}\right) \int_\Omega w_i d\mu. \tag{1.5}$$

In the next section we establish some results concerning this functional that are related to Jensen's integral inequality. Applications for discrete inequalities and weighted means are provided in the third section. In the last section some applications for univariate functions are also given.

2. Main Results

The following basic result holds:

Theorem 1. *Let $\Phi : [m, M] \to \mathbb{R}$ be a convex function, $f : \Omega \to [m, M]$ a μ-measurable function such that f, $\Phi \circ f \in L_w(\Omega, \mu)$. Then for any $\overline{w} \in \mathfrak{P}_n(w)$ we have*

$$\frac{\int_\Omega (\Phi \circ f)\, w d\mu}{\int_\Omega w d\mu} \geq \psi(\Phi, f, \overline{w}) \geq \Phi\left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right), \tag{2.1}$$

where $n \geq 2$.

Proof. From Jensen's integral inequality we have

$$\int_\Omega (\Phi \circ f)\, w_i d\mu \geq \Phi\left(\frac{\int_\Omega f w_i d\mu}{\int_\Omega w_i d\mu}\right) \int_\Omega w_i d\mu \tag{2.2}$$

for any $i \in \{1, ..., n\}$.

If we sum the inequality (2.2) over i from 1 to n we get

$$\sum_{i=1}^{n} \int_\Omega (\Phi \circ f)\, w_i d\mu \geq \sum_{i=1}^{n} \Phi\left(\frac{\int_\Omega f w_i d\mu}{\int_\Omega w_i d\mu}\right) \int_\Omega w_i d\mu \tag{2.3}$$

and since

$$\sum_{i=1}^{n} \int_\Omega (\Phi \circ f)\, w_i d\mu = \int_\Omega (\Phi \circ f)\left(\sum_{i=1}^{n} w_i\right) d\mu = \int_\Omega (\Phi \circ f)\, w d\mu$$

then from (2.3) we get the first part of (2.1).

Let

$$p_i = \int_\Omega w_i d\mu > 0 \;\; z_i = \frac{\int_\Omega f w_i d\mu}{\int_\Omega w_i d\mu} \in [m, M],\;\; i \in \{1, ..., n\}.$$

Then

$$P_n := \sum_{i=1}^{n} p_i = \int_\Omega w d\mu,$$

and

$$\sum_{i=1}^{n} p_i z_i = \sum_{i=1}^{n} \int_\Omega f w_i d\mu = \int_\Omega f w d\mu.$$

From Jensen's discrete inequality

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i \Phi(z_i) \geq \Phi\left(\frac{\sum_{i=1}^{n} p_i z_i}{P_n}\right)$$

we have

$$\frac{1}{\int_\Omega w d\mu} \sum_{i=1}^{n} \Phi\left(\frac{\int_\Omega f w_i d\mu}{\int_\Omega w_i d\mu}\right) \int_\Omega w_i d\mu \geq \Phi\left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right)$$

and the second inequality in (2.1) is also proved. □

Remark 1. *The double inequality (2.1) is equivalent to*

$$\frac{\int_\Omega (\Phi \circ f)\, w d\mu}{\int_\Omega w d\mu} \geq \sup_{\overline{w} \in \mathfrak{P}_n(w)} \psi(\Phi, f, \overline{w}) \tag{2.4}$$

and

$$\inf_{\overline{w} \in \mathfrak{P}_n(w)} \psi(\Phi, f, \overline{w}) \geq \Phi\left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right), \tag{2.5}$$

where $n \geq 2$.

We use the following lemma [9].

Lemma 1. *Let $\Phi : I \to \mathbb{R}$ be a convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, $\mathring{I}$ is the interior of I. If $g : \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty < m \leq g(x) \leq M < \infty \text{ for } \mu\text{-a.e. } x \in \Omega$$

and such that $g, \Phi \circ g \in L_p(\Omega, \mu)$, where $p \geq 0$ μ-a.e. on Ω with $\int_\Omega p d\mu = 1$, then

$$\begin{aligned} (2.6) \qquad 0 &\leq \int_\Omega p(\Phi \circ f) d\mu - \Phi\left(\int_\Omega pf d\mu\right) \\ &\leq \frac{\left(M - \int_\Omega pf d\mu\right)\left(\int_\Omega pf d\mu - m\right)}{M - m} \sup_{t \in (m,M)} \Psi_\Phi(t; m, M) \\ &\leq \left(M - \int_\Omega pf d\mu\right)\left(\int_\Omega pf d\mu - m\right) \frac{\Phi'_-(M) - \Phi'_+(m)}{M - m} \\ &\leq \frac{1}{4}(M - m)\left[\Phi'_-(M) - \Phi'_+(m)\right], \end{aligned}$$

where $\Psi_\Phi(\cdot; m, M) : (m, M) \to \mathbb{R}$ is defined by

$$\Psi_\Phi(t; m, M) = \frac{\Phi(M) - \Phi(t)}{M - t} - \frac{\Phi(t) - \Phi(m)}{t - m}.$$

We have the following reverse of the first inequality in (2.1).

Theorem 2. *Let $\Phi : [m, M] \to \mathbb{R}$ be a convex function, $f : \Omega \to [m, M]$ a μ-measurable function such that f, $\Phi \circ f \in L_w(\Omega, \mu)$. Then for any $\overline{w} \in \mathfrak{P}_n(w)$, $n \geq 2$ we have*

$$\begin{aligned} (2.7) \qquad 0 &\leq \frac{\int_\Omega (\Phi \circ f) w d\mu}{\int_\Omega w d\mu} - \psi(\Phi, f, \overline{w}) \\ &\leq \frac{1}{M - m} \sup_{t \in (m,M)} \Psi_\Phi(t; m, M) \\ &\times \frac{1}{\int_\Omega w d\mu} \sum_{i=1}^n \left(\int_\Omega w_i d\mu\right)\left(M - \frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu}\right)\left(\frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu} - m\right) \\ &\leq \frac{1}{M - m} \sup_{t \in (m,M)} \Psi_\Phi(t; m, M)\left(M - \frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right)\left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu} - m\right). \end{aligned}$$

Proof. From the second inequality in (2.6) for $g = f$ and $p = \frac{w_i}{\int_\Omega w_i d\mu}$, $i \in \{1, ..., n\}$ we have

$$\begin{aligned} (2.8) \qquad 0 &\leq \frac{1}{\int_\Omega w_i d\mu} \int_\Omega w_i (\Phi \circ f) d\mu - \Phi\left(\frac{1}{\int_\Omega w_i d\mu} \int_\Omega w_i f d\mu\right) \\ &\leq \frac{1}{M - m}\left(M - \frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu}\right)\left(\frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu} - m\right) \sup_{t \in (m,M)} \Psi_\Phi(t; m, M) \end{aligned}$$

for any $i \in \{1, ..., n\}$.

If we multiply by $\int_\Omega w_i d\mu > 0$ and sum over i from 1 to n we get

$$
\begin{aligned}
(2.9)\qquad 0 &\le \sum_{i=1}^{n} \int_\Omega w_i \left(\Phi \circ f\right) d\mu - \sum_{i=1}^{n} \Phi\left(\frac{1}{\int_\Omega w_i d\mu} \int_\Omega w_i f d\mu\right) \int_\Omega w_i d\mu \\
&\le \frac{1}{M-m} \sup_{t\in(m,M)} \Psi_\Phi\left(t; m, M\right) \\
&\times \sum_{i=1}^{n} \left(\int_\Omega w_i d\mu\right) \left(M - \frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu}\right) \left(\frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu} - m\right),
\end{aligned}
$$

which proves the second inequality in (2.7).

Now, observe that the function $\Psi : [m, M] \to [0, \infty)$, $\Psi(t) = (M-t)(t-m)$ is concave and by Jensen's inequality for concave functions with

$$
p_i = \int_\Omega w_i d\mu > 0 \;\; z_i = \frac{\int_\Omega f w_i d\mu}{\int_\Omega w_i d\mu} \in [m, M]\,,\;\; i \in \{1, ..., n\}
$$

we have

$$
\begin{aligned}
&\frac{1}{\int_\Omega w d\mu} \sum_{i=1}^{n} \left(\int_\Omega w_i d\mu\right) \left(M - \frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu}\right) \left(\frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu} - m\right) \\
&\le \left(M - \frac{1}{\int_\Omega w d\mu} \sum_{i=1}^{n} \left(\int_\Omega w_i d\mu\right) \frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu}\right) \\
&\times \left(\frac{1}{\int_\Omega w d\mu} \sum_{i=1}^{n} \left(\int_\Omega w_i d\mu\right) \frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu} - m\right) \\
&= \left(M - \frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right) \left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu} - m\right),
\end{aligned}
$$

which proves the last part of (2.7). □

Remark 2. *Since, as shown in* [9],

$$
\sup_{t\in(m,M)} \Psi_\Phi\left(t; m, M\right) \le \Phi'_-\left(M\right) - \Phi'_+\left(m\right)
$$

then we have the following simpler inequality

$$
\begin{aligned}
(2.10)\qquad 0 &\le \frac{\int_\Omega \left(\Phi \circ f\right) w d\mu}{\int_\Omega w d\mu} - \psi\left(\Phi, f, \overline{w}\right) \\
&\le \frac{\Phi'_-\left(M\right) - \Phi'_+\left(m\right)}{M-m} \\
&\times \sum_{i=1}^{n} \frac{\int_\Omega w_i d\mu}{\int_\Omega w d\mu} \left(M - \frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu}\right) \left(\frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu} - m\right) \\
&\le \frac{\Phi'_-\left(M\right) - \Phi'_+\left(m\right)}{M-m} \left(M - \frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right) \left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu} - m\right).
\end{aligned}
$$

If we use Lemma 1 for the discrete measure, we can state the following result:

Lemma 2. *Let $\Phi : [a, b] \to \mathbb{R}$ be a convex function on the interval of real numbers $[a, b]$, $z_i \in [a, b]$, $p_i \geq 0$, $i \in \{1, ..., n\}$ and $\sum_{i=1}^{n} p_i = 1$. Then*

$$
\begin{aligned}
(2.11)\qquad 0 &\leq \sum_{i=1}^{n} p_i \Phi(z_i) - \Phi\left(\sum_{i=1}^{n} p_i z_i\right) \\
&\leq \frac{\left(b - \sum_{i=1}^{n} p_i z_i\right)\left(\sum_{i=1}^{n} p_i z_i - a\right)}{b-a} \sup_{t\in(a,b)} \Psi_{\Phi}(t; a, b) \\
&\leq \left(b - \sum_{i=1}^{n} p_i z_i\right)\left(\sum_{i=1}^{n} p_i z_i - a\right) \frac{\Phi'_{-}(b) - \Phi'_{+}(a)}{b-a} \\
&\leq \frac{1}{4}(b-a)\left[\Phi'_{-}(b) - \Phi'_{+}(a)\right].
\end{aligned}
$$

The following reverse of the second inequality in (2.1) holds:

Theorem 3. *Let $\Phi : [m, M] \to \mathbb{R}$ be a convex function, $f : \Omega \to [m, M]$ a μ-measurable function such that f, $\Phi \circ f \in L_w(\Omega, \mu)$. Then for any $\overline{w} \in \mathfrak{P}_n(w)$, $n \geq 2$ we have*

$$
\begin{aligned}
(2.12)\qquad 0 &\leq \psi(\Phi, f, \overline{w}) - \Phi\left(\frac{\int_{\Omega} fwd\mu}{\int_{\Omega} wd\mu}\right) \\
&\leq \frac{\left(L(\overline{w}) - \frac{\int_{\Omega} fwd\mu}{\int_{\Omega} wd\mu}\right)\left(\frac{\int_{\Omega} fwd\mu}{\int_{\Omega} wd\mu} - l(\overline{w})\right)}{L(\overline{w}) - l(\overline{w})} \sup_{t\in(l(\overline{w}),L(\overline{w}))} \Psi_{\Phi}(t; l(\overline{w}), L(\overline{w})) \\
&\leq \frac{1}{4}\left(L(\overline{w}) - l(\overline{w})\right) \sup_{t\in(l(\overline{w}),L(\overline{w}))} \Psi_{\Phi}(t; l(\overline{w}), L(\overline{w})),
\end{aligned}
$$

where

$$
(2.13)\qquad l(\overline{w}) := \min_{i\in\{1,...,n\}} \left\{\frac{\int_{\Omega} fw_i d\mu}{\int_{\Omega} w_i d\mu}\right\},\ L(\overline{w}) := \max_{i\in\{1,...,n\}} \left\{\frac{\int_{\Omega} fw_i d\mu}{\int_{\Omega} w_i d\mu}\right\}.
$$

Proof. If we write the first two inequalities in (2.11) for

$$
p_i = \frac{\int_{\Omega} w_i d\mu}{\int_{\Omega} wd\mu} > 0\ z_i = \frac{\int_{\Omega} fw_i d\mu}{\int_{\Omega} w_i d\mu},\ i \in \{1, ..., n\}
$$

and for $a = l(\overline{w}), b = L(\overline{w})$ as above we have

$$
\begin{aligned}
(2.14)\quad 0 &\leq \frac{1}{\int_{\Omega} wd\mu} \sum_{i=1}^{n} \left(\int_{\Omega} w_i d\mu\right) \Phi\left(\frac{\int_{\Omega} fw_i d\mu}{\int_{\Omega} w_i d\mu}\right) - \Phi\left(\frac{\sum_{i=1}^{n} \int_{\Omega} fw_i d\mu}{\int_{\Omega} wd\mu}\right) \\
&\leq \frac{\sup_{t\in(l(\overline{w}),L(\overline{w}))} \Psi_{\Phi}(t; l(\overline{w}), L(\overline{w}))}{L(\overline{w}) - l(\overline{w})} \\
&\times \left(L(\overline{w}) - \sum_{i=1}^{n} \frac{\int_{\Omega} w_i d\mu}{\int_{\Omega} wd\mu} \cdot \frac{\int_{\Omega} fw_i d\mu}{\int_{\Omega} w_i d\mu}\right)\left(\sum_{i=1}^{n} \frac{\int_{\Omega} w_i d\mu}{\int_{\Omega} wd\mu} \cdot \frac{\int_{\Omega} fw_i d\mu}{\int_{\Omega} w_i d\mu} - l(\overline{w})\right) \\
&= \frac{\left(L(\overline{w}) - \frac{\int_{\Omega} fwd\mu}{\int_{\Omega} wd\mu}\right)\left(\frac{\int_{\Omega} fwd\mu}{\int_{\Omega} wd\mu} - l(\overline{w})\right)}{L(\overline{w}) - l(\overline{w})} \sup_{t\in(l(\overline{w}),L(\overline{w}))} \Psi_{\Phi}(t; l(\overline{w}), L(\overline{w})),
\end{aligned}
$$

which proves the second inequality in (2.12).

The last part in (2.12) follows by the elementary inequality

$$\alpha\beta \le \left(\frac{\alpha+\beta}{2}\right)^2, \ \alpha,\beta \in \mathbb{R}.$$

□

Remark 3. *Since*

$$\Psi_{\Phi}\left(t; l\left(\overline{w}\right), L\left(\overline{w}\right)\right) \le \Phi'_{-}\left(L\left(\overline{w}\right)\right) - \Phi'_{+}\left(l\left(\overline{w}\right)\right),$$

then from (2.12) we have the simpler inequalities

$$\begin{aligned}
(2.15)\qquad 0 &\le \psi\left(\Phi, f, \overline{w}\right) - \Phi\left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right) \\
&\le \frac{\Phi'_{-}\left(L\left(\overline{w}\right)\right) - \Phi'_{+}\left(l\left(\overline{w}\right)\right)}{L\left(\overline{w}\right) - l\left(\overline{w}\right)}\left(L\left(\overline{w}\right) - \frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right)\left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu} - l\left(\overline{w}\right)\right) \\
&\le \frac{1}{4}\left(L\left(\overline{w}\right) - l\left(\overline{w}\right)\right)\left[\Phi'_{-}\left(L\left(\overline{w}\right)\right) - \Phi'_{+}\left(l\left(\overline{w}\right)\right)\right].
\end{aligned}$$

The following reverse of Jensen inequality also holds [9]:

Lemma 3. *Let $\Phi : I \to \mathbb{R}$ be a convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, $\mathring{I}$ is the interior of I. If $g : \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty < m \le g(x) \le M < \infty \text{ for } \mu\text{-a.e. } x \in \Omega$$

and such that $g, \Phi \circ g \in L_p(\Omega, \mu)$, where $p \ge 0$ μ-a.e. on Ω with $\int_\Omega p d\mu = 1$, then

(2.16)

$$\begin{aligned}
0 &\le \int_\Omega p\left(\Phi \circ f\right) d\mu - \Phi\left(\int_\Omega p f d\mu\right) \\
&\le \left[\frac{\Phi(m) + \Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right]\left(1 + \frac{2}{M-m}\left|\int_\Omega p f d\mu - \frac{m+M}{2}\right|\right) \\
&\le 2\left[\frac{\Phi(m) + \Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right].
\end{aligned}$$

We have the following result:

Theorem 4. *Let $\Phi : [m, M] \to \mathbb{R}$ be a convex function, $f : \Omega \to [m, M]$ a μ-measurable function such that $f, \Phi \circ f \in L_w(\Omega, \mu)$. Then for any $\overline{w} \in \mathfrak{P}_n(w)$, $n \ge 2$ we have*

$$\begin{aligned}
(2.17)\qquad 0 &\le \frac{\int_\Omega \left(\Phi \circ f\right) w d\mu}{\int_\Omega w d\mu} - \psi\left(\Phi, f, \overline{w}\right) \\
&\le \left[\frac{\Phi(m) + \Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right] \\
&\quad \times \left(1 + \frac{2}{M-m}\sum_{i=1}^{n}\left|\frac{1}{\int_\Omega w d\mu}\int_\Omega w_i\left(f - \frac{m+M}{2}\right) d\mu\right|\right)
\end{aligned}$$

$$\leq \left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(1+\frac{2}{M-m}\frac{1}{\int_\Omega w d\mu}\int_\Omega w\left|f-\frac{m+M}{2}\right|d\mu\right).$$

Proof. From the second inequality in (2.16) for $g=f$ and $p=\frac{w_i}{\int_\Omega w_i d\mu}$, $i\in\{1,...,n\}$ we have

$$0\leq \frac{1}{\int_\Omega w_i d\mu}\int_\Omega w_i(\Phi\circ f)d\mu-\Phi\left(\frac{1}{\int_\Omega w_i d\mu}\int_\Omega w_i f d\mu\right)$$
$$\leq\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(1+\frac{2}{M-m}\left|\frac{1}{\int_\Omega w_i d\mu}\int_\Omega w_i f d\mu-\frac{m+M}{2}\right|\right) \tag{2.18}$$

for any $i\in\{1,...,n\}$.

If we multiply the inequality (2.18) by $\int_\Omega w_i d\mu>0$ and sum over i from 1 to n we get

$$0\leq\sum_{i=1}^n\int_\Omega w_i(\Phi\circ f)d\mu-\sum_{i=1}^n\Phi\left(\frac{1}{\int_\Omega w_i d\mu}\int_\Omega w_i f d\mu\right)\int_\Omega w_i d\mu$$
$$\leq\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\sum_{i=1}^n\int_\Omega w_i d\mu+\frac{2}{M-m}\sum_{i=1}^n\left(\int_\Omega w_i d\mu\right)\left|\frac{1}{\int_\Omega w_i d\mu}\int_\Omega w_i f d\mu-\frac{m+M}{2}\right|\right)$$
$$=\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\int_\Omega w d\mu+\frac{2}{M-m}\sum_{i=1}^n\left|\int_\Omega w_i\left(f-\frac{m+M}{2}\right)d\mu\right|\right)$$

any by dividing with $\int_\Omega w d\mu=\sum_{i=1}^n\int_\Omega w_i d\mu$, we get the second inequality in (2.17).

By the properties of modulus we have

$$\sum_{i=1}^n\left|\frac{1}{\int_\Omega w d\mu}\int_\Omega w_i\left(f-\frac{m+M}{2}\right)d\mu\right|\leq\frac{1}{\int_\Omega w d\mu}\sum_{i=1}^n\int_\Omega w_i\left|f-\frac{m+M}{2}\right|d\mu$$
$$=\frac{1}{\int_\Omega w d\mu}\int_\Omega w\left|f-\frac{m+M}{2}\right|d\mu$$

and the last part of (2.17) is proved. □

If we use Lemma 1 for the discrete measure, we can state the following result:

Lemma 4. *Let $\Phi : [a,b] \to \mathbb{R}$ be a convex function on the interval of real numbers $[a,b]$, $z_i \in [a,b]$, $p_i \geq 0$, $i \in \{1,...,n\}$ and $\sum_{i=1}^{n} p_i = 1$. Then*

$$0 \leq \sum_{i=1}^{n} p_i \Phi(z_i) - \Phi\left(\sum_{i=1}^{n} p_i z_i\right) \leq \left[\frac{\Phi(a)+\Phi(b)}{2} - \Phi\left(\frac{a+b}{2}\right)\right]\left(1 + \frac{2}{b-a}\left|\sum_{i=1}^{n} p_i z_i - \frac{a+b}{2}\right|\right) \leq 2\left[\frac{\Phi(a)+\Phi(b)}{2} - \Phi\left(\frac{a+b}{2}\right)\right]. \tag{2.19}$$

Using this lemma we can state and prove the following result as well:

Theorem 5. *Let $\Phi : [m,M] \to \mathbb{R}$ be a convex function, $f : \Omega \to [m,M]$ a μ-measurable function such that f, $\Phi \circ f \in L_w(\Omega,\mu)$. Then for any $\overline{w} \in \mathfrak{P}_n(w)$, $n \geq 2$ we have*

$$\begin{aligned} 0 &\leq \psi(\Phi, f, \overline{w}) - \Phi\left(\frac{\int_\Omega fwd\mu}{\int_\Omega wd\mu}\right) \\ &\leq \left[\frac{\Phi(l(\overline{w})) + \Phi(L(\overline{w}))}{2} - \Phi\left(\frac{l(\overline{w}) + L(\overline{w})}{2}\right)\right] \\ &\times \left(1 + \frac{2}{L(\overline{w}) - l(\overline{w})}\left|\frac{1}{\int_\Omega wd\mu}\int_\Omega wfd\mu - \frac{l(\overline{w}) + L(\overline{w})}{2}\right|\right) \\ &\leq 2\left[\frac{\Phi(l(\overline{w})) + \Phi(L(\overline{w}))}{2} - \Phi\left(\frac{l(\overline{w}) + L(\overline{w})}{2}\right)\right], \end{aligned} \tag{2.20}$$

where $l(\overline{w})$, $L(\overline{w})$ are defined by (2.13).

Proof. If we write the first two inequalities in (2.11) for

$$p_i = \frac{\int_\Omega w_i d\mu}{\int_\Omega wd\mu} > 0 \ z_i = \frac{\int_\Omega fw_i d\mu}{\int_\Omega w_i d\mu}, \ i \in \{1,...,n\}$$

and for $a = l(\overline{w})$, $b = L(\overline{w})$ as above we have

$$\begin{aligned} 0 &\leq \frac{1}{\int_\Omega wd\mu}\sum_{i=1}^{n}\left(\int_\Omega w_i d\mu\right)\Phi\left(\frac{\int_\Omega fw_i d\mu}{\int_\Omega w_i d\mu}\right) \\ &- \Phi\left(\sum_{i=1}^{n}\frac{\int_\Omega w_i d\mu}{\int_\Omega wd\mu}\cdot\frac{\int_\Omega fw_i d\mu}{\int_\Omega w_i d\mu}\right) \\ &\leq \left[\frac{\Phi(l(\overline{w})) + \Phi(L(\overline{w}))}{2} - \Phi\left(\frac{l(\overline{w}) + L(\overline{w})}{2}\right)\right] \\ &\times \left(1 + \frac{2}{L(\overline{w}) - l(\overline{w})}\left|\sum_{i=1}^{n}\frac{\int_\Omega w_i d\mu}{\int_\Omega wd\mu}\frac{\int_\Omega fw_i d\mu}{\int_\Omega w_i d\mu} - \frac{l(\overline{w}) + L(\overline{w})}{2}\right|\right) \\ &= \left[\frac{\Phi(l(\overline{w})) + \Phi(L(\overline{w}))}{2} - \Phi\left(\frac{l(\overline{w}) + L(\overline{w})}{2}\right)\right] \\ &\times \left(1 + \frac{2}{L(\overline{w}) - l(\overline{w})}\left|\frac{1}{\int_\Omega wd\mu}\int_\Omega\left(\sum_{i=1}^{n} w_i\right)fd\mu - \frac{l(\overline{w}) + L(\overline{w})}{2}\right|\right) \end{aligned}$$

that proves the required inequalities in (2.20). □

3. Discrete Case and Some Applications

Let $\Phi : [a, b] \to \mathbb{R}$ be a convex function on the interval of real numbers $[a, b]$, $x_k \in [a, b]$, $w_k > 0$, $k \in \{1, ..., m\}$. Let $w_{ki} > 0$ for $k \in \{1, ..., m\}$, $i \in \{1, ..., n\}$ with $m, n \geq 2$ and $\sum_{i=1}^{n} w_{ki} = w_k$ for any $k \in \{1, ..., m\}$.

We consider the functional associated with the matrix $W := \{w_{ki}\}_{k\in\{1,...,m\},i\in\{1,...,n\}}$

$$\psi(\Phi, \overline{x}, W) := \frac{1}{\sum_{k=1}^{m} w_k} \sum_{i=1}^{n} \Phi\left(\frac{\sum_{k=1}^{m} x_k w_{ki}}{\sum_{k=1}^{m} w_{ki}}\right) \sum_{k=1}^{m} w_{ki}, \tag{3.1}$$

where $\overline{x} = (x_1, ..., x_m) \in [a, b]^m$.

Using the results from the previous section we have

$$\frac{\sum_{k=1}^{m} \Phi(x_k) w_k}{\sum_{k=1}^{m} w_k} \geq \psi(\Phi, \overline{x}, W) \geq \Phi\left(\frac{\sum_{k=1}^{m} x_k w_k}{\sum_{k=1}^{m} w_k}\right), \tag{3.2}$$

$$\begin{aligned} 0 &\leq \frac{\sum_{k=1}^{m} \Phi(x_k) w_k}{\sum_{k=1}^{m} w_k} - \psi(\Phi, \overline{x}, W) \\ &\leq \frac{\Phi'_-(b) - \Phi'_+(a)}{b - a} \\ &\times \frac{1}{\sum_{k=1}^{m} w_k} \sum_{i=1}^{n} \sum_{k=1}^{m} w_{ki} \left(b - \frac{\sum_{k=1}^{m} x_k w_{ki}}{\sum_{k=1}^{m} w_{ki}}\right)\left(\frac{\sum_{k=1}^{m} x_k w_{ki}}{\sum_{k=1}^{m} w_{ki}} - a\right) \\ &\leq \frac{\Phi'_-(b) - \Phi'_+(a)}{b - a}\left(b - \frac{\sum_{k=1}^{m} x_k w_k}{\sum_{k=1}^{m} w_k}\right)\left(\frac{\sum_{k=1}^{m} x_k w_k}{\sum_{k=1}^{m} w_k} - a\right), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} 0 &\leq \frac{\sum_{k=1}^{m} \Phi(x_k) w_k}{\sum_{k=1}^{m} w_k} - \psi(\Phi, \overline{x}, W) \\ &\leq \left[\frac{\Phi(a) + \Phi(b)}{2} - \Phi\left(\frac{a+b}{2}\right)\right] \\ &\times \left(1 + \frac{2}{b-a} \sum_{i=1}^{n} \left|\frac{1}{\sum_{k=1}^{m} w_k} \sum_{k=1}^{m} w_{ki}\left(x_k - \frac{a+b}{2}\right)\right|\right) \\ &\leq \left[\frac{\Phi(a) + \Phi(b)}{2} - \Phi\left(\frac{a+b}{2}\right)\right] \\ &\times \left(1 + \frac{2}{b-a} \frac{1}{\sum_{k=1}^{m} w_k} \sum_{k=1}^{m} w_k \left|x_k - \frac{a+b}{2}\right|\right). \end{aligned} \tag{3.4}$$

Define

$$l(W) := \min_{i\in\{1,...,n\}} \left\{\frac{\sum_{k=1}^{m} w_{ki} x_k}{\sum_{k=1}^{m} w_{ki}}\right\}, \quad L(W) := \max_{i\in\{1,...,n\}} \left\{\frac{\sum_{k=1}^{m} w_{ki} x_k}{\sum_{k=1}^{m} w_{ki}}\right\}. \tag{3.5}$$

Then we also have

$$
\begin{aligned}
0 &\le \psi(\Phi,\overline{x},W) - \Phi\left(\frac{\sum_{k=1}^m x_k w_k}{\sum_{k=1}^m w_k}\right) \\
&\le \frac{\Phi'_-(L(W)) - \Phi'_+(l(W))}{L(W)-l(W)} \\
&\times \left(L(W) - \frac{\sum_{k=1}^m x_k w_k}{\sum_{k=1}^m w_k}\right)\left(\frac{\sum_{k=1}^m x_k w_k}{\sum_{k=1}^m w_k} - l(W)\right) \\
&\le \frac{1}{4}(L(W)-l(W))\left[\Phi'_-(L(W)) - \Phi'_+(l(W))\right]
\end{aligned} \tag{3.6}
$$

and

$$
\begin{aligned}
0 &\le \psi(\Phi,\overline{x},W) - \Phi\left(\frac{\sum_{k=1}^m x_k w_k}{\sum_{k=1}^m w_k}\right) \\
&\le \left[\frac{\Phi(l(W)) + \Phi(L(W))}{2} - \Phi\left(\frac{l(W)+L(W)}{2}\right)\right] \\
&\times \left(1 + \frac{2}{L(W)-l(W)}\left|\frac{\sum_{k=1}^m x_k w_k}{\sum_{k=1}^m w_k} - \frac{l(W)+L(W)}{2}\right|\right) \\
&\le 2\left[\frac{\Phi(l(W)) + \Phi(L(W))}{2} - \Phi\left(\frac{l(W)+L(W)}{2}\right)\right].
\end{aligned} \tag{3.7}
$$

We consider the convex function $\Phi : (0,\infty) \to (0,\infty)$, $\Phi(t) = t^p$ with $p \in (-\infty,0)\cup(1,\infty)$. Then

$$
\begin{aligned}
\psi(\Phi,\overline{x},W) &:= \frac{1}{\sum_{k=1}^m w_k}\sum_{i=1}^n \left(\frac{\sum_{k=1}^m x_k w_{ki}}{\sum_{k=1}^m w_{ki}}\right)^p \sum_{k=1}^m w_{ki} \\
&= \frac{1}{\sum_{k=1}^m w_k}\sum_{i=1}^n \left(\sum_{k=1}^m x_k w_{ki}\right)^p \left(\sum_{k=1}^m w_{ki}\right)^{1-p}
\end{aligned} \tag{3.8}
$$

where $x_k > 0$, $w_k > 0$, $w_{ki} > 0$ for $k \in \{1,...,m\}$, $i \in \{1,...,n\}$ with m, $n \ge 2$ and $\sum_{i=1}^n w_{ki} = w_k$ for any $k \in \{1,...,m\}$.

From (3.2) we have

$$
\begin{aligned}
\frac{\sum_{k=1}^m x_k^p w_k}{\sum_{k=1}^m w_k} &\ge \frac{1}{\sum_{k=1}^m w_k}\sum_{i=1}^n \left(\sum_{k=1}^m x_k w_{ki}\right)^p \left(\sum_{k=1}^m w_{ki}\right)^{1-p} \\
&\ge \left(\frac{\sum_{k=1}^m x_k w_k}{\sum_{k=1}^m w_k}\right)^p.
\end{aligned} \tag{3.9}
$$

If we set $a=\min\{x_k\}_{k\in\{1,\dots,m\}}$ and $b=\max\{x_k\}_{k\in\{1,\dots,m\}}$ then from (3.3) and (3.4) we have

$$
\begin{aligned}
0 &\le \frac{\sum_{k=1}^m x_k^p w_k}{\sum_{k=1}^m w_k} - \frac{1}{\sum_{k=1}^m w_k}\sum_{i=1}^n \left(\sum_{k=1}^m x_k w_{ki}\right)^p \left(\sum_{k=1}^m w_{ki}\right)^{1-p} \\
&\le p\frac{b^{p-1}-a^{p-1}}{b-a} \\
&\times \frac{1}{\sum_{k=1}^m w_k}\sum_{i=1}^n\sum_{k=1}^m w_{ki}\left(b-\frac{\sum_{k=1}^m x_k w_{ki}}{\sum_{k=1}^m w_{ki}}\right)\left(\frac{\sum_{k=1}^m x_k w_{ki}}{\sum_{k=1}^m w_{ki}}-a\right) \\
&\le p\frac{b^{p-1}-a^{p-1}}{b-a}\left(b-\frac{\sum_{k=1}^m x_k w_k}{\sum_{k=1}^m w_k}\right)\left(\frac{\sum_{k=1}^m x_k w_k}{\sum_{k=1}^m w_k}-a\right),
\end{aligned}
\tag{3.10}
$$

and

$$
\begin{aligned}
0 &\le \frac{\sum_{k=1}^m x_k^p w_k}{\sum_{k=1}^m w_k} - \frac{1}{\sum_{k=1}^m w_k}\sum_{i=1}^n \left(\sum_{k=1}^m x_k w_{ki}\right)^p \left(\sum_{k=1}^m w_{ki}\right)^{1-p} \\
&\le \left[\frac{a^p+b^p}{2}-\left(\frac{a+b}{2}\right)^p\right] \\
&\times \left(1+\frac{2}{b-a}\sum_{i=1}^n\left|\frac{1}{\sum_{k=1}^m w_k}\sum_{k=1}^m w_{ki}\left(x_k-\frac{a+b}{2}\right)\right|\right) \\
&\le \left[\frac{a^p+b^p}{2}-\left(\frac{a+b}{2}\right)^p\right] \\
&\times \left(1+\frac{2}{b-a}\frac{1}{\sum_{k=1}^m w_k}\sum_{k=1}^m w_k\left|x_k-\frac{a+b}{2}\right|\right).
\end{aligned}
\tag{3.11}
$$

If $l(W)$ and $L(W)$ are defined as in (3.5), then from (3.6) and (3.7) we also have

$$
\begin{aligned}
0 &\le \frac{1}{\sum_{k=1}^m w_k}\sum_{i=1}^n \left(\sum_{k=1}^m x_k w_{ki}\right)^p \left(\sum_{k=1}^m w_{ki}\right)^{1-p} - \left(\frac{\sum_{k=1}^m x_k w_k}{\sum_{k=1}^m w_k}\right)^p \\
&\le p\frac{(L(W))^{p-1}-(l(W))^{p-1}}{L(W)-l(W)} \\
&\times \left(L(W)-\frac{\sum_{k=1}^m x_k w_k}{\sum_{k=1}^m w_k}\right)\left(\frac{\sum_{k=1}^m x_k w_k}{\sum_{k=1}^m w_k}-l(W)\right) \\
&\le \frac{1}{4}(L(W)-l(W))\left[(L(W))^{p-1}-(l(W))^{p-1}\right]
\end{aligned}
\tag{3.12}
$$

and

$$0 \leq \frac{1}{\sum_{k=1}^{m} w_k} \sum_{i=1}^{n} \left(\sum_{k=1}^{m} x_k w_{ki} \right)^p \left(\sum_{k=1}^{m} w_{ki} \right)^{1-p} - \left(\frac{\sum_{k=1}^{m} x_k w_k}{\sum_{k=1}^{m} w_k} \right)^p \tag{3.13}$$
$$\leq \left[\frac{(l(W))^p + (L(W))^p}{2} - \left(\frac{l(W) + L(W)}{2} \right)^p \right]$$
$$\times \left(1 + \frac{2}{L(W) - l(W)} \left| \frac{\sum_{k=1}^{m} x_k w_k}{\sum_{k=1}^{m} w_k} - \frac{l(W) + L(W)}{2} \right| \right)$$
$$\leq 2 \left[\frac{(l(W))^p + (L(W))^p}{2} - \left(\frac{l(W) + L(W)}{2} \right)^p \right].$$

Further on, consider the convex function $\Phi : (0, \infty) \to (0, \infty)$, $\Phi(t) = -\ln t$ then from (3.2) for $x_k > 0$, $w_k > 0$, $w_{ki} > 0$ for $k \in \{1, ..., m\}$, $i \in \{1, ..., n\}$ with m, $n \geq 2$ and $\sum_{i=1}^{n} w_{ki} = w_k$ for any $k \in \{1, ..., m\}$ we get

$$\left(\prod_{k=1}^{m} x_k^{w_k} \right)^{\frac{1}{\sum_{k=1}^{m} w_k}} \leq \left(\prod_{i=1}^{n} \left(\frac{\sum_{k=1}^{m} x_k w_{ki}}{\sum_{k=1}^{m} w_{ki}} \right)^{\sum_{k=1}^{m} w_{ki}} \right)^{\frac{1}{\sum_{k=1}^{m} w_k}} \tag{3.14}$$
$$\leq \frac{\sum_{k=1}^{m} x_k w_k}{\sum_{k=1}^{m} w_k}.$$

By (3.3) and (3.4) we have

$$1 \leq \frac{\left(\prod_{i=1}^{n} \left(\frac{\sum_{k=1}^{m} x_k w_{ki}}{\sum_{k=1}^{m} w_{ki}} \right)^{\sum_{k=1}^{m} w_{ki}} \right)^{\frac{1}{\sum_{k=1}^{m} w_k}}}{\left(\prod_{k=1}^{m} x_k^{w_k} \right)^{\frac{1}{\sum_{k=1}^{m} w_k}}} \tag{3.15}$$
$$\leq \exp \left[\frac{1}{ba \sum_{k=1}^{m} w_k} \sum_{i=1}^{n} \sum_{k=1}^{m} w_{ki} \left(b - \frac{\sum_{k=1}^{m} x_k w_{ki}}{\sum_{k=1}^{m} w_{ki}} \right) \left(\frac{\sum_{k=1}^{m} x_k w_{ki}}{\sum_{k=1}^{m} w_{ki}} - a \right) \right]$$
$$\leq \exp \left[\frac{1}{ab} \left(b - \frac{\sum_{k=1}^{m} x_k w_k}{\sum_{k=1}^{m} w_k} \right) \left(\frac{\sum_{k=1}^{m} x_k w_k}{\sum_{k=1}^{m} w_k} - a \right) \right]$$

and

$$1 \leq \frac{\left(\prod_{i=1}^{n} \left(\frac{\sum_{k=1}^{m} x_k w_{ki}}{\sum_{k=1}^{m} w_{ki}} \right)^{\sum_{k=1}^{m} w_{ki}} \right)^{\frac{1}{\sum_{k=1}^{m} w_k}}}{\left(\prod_{k=1}^{m} x_k^{w_k} \right)^{\frac{1}{\sum_{k=1}^{m} w_k}}} \tag{3.16}$$
$$\leq \left(\frac{a+b}{2\sqrt{ab}} \right)^{1 + \frac{2}{b-a} \sum_{i=1}^{n} \left| \frac{1}{\sum_{k=1}^{m} w_k} \sum_{k=1}^{m} w_{ki} \left(x_k - \frac{a+b}{2} \right) \right|}$$
$$\leq \left(\frac{a+b}{2\sqrt{ab}} \right)^{1 + \frac{2}{b-a} \frac{1}{\sum_{k=1}^{m} w_k} \sum_{k=1}^{m} w_k \left| x_k - \frac{a+b}{2} \right|}.$$

From (3.6) and (3.7) we also have

$$1 \leq \frac{\frac{1}{\sum_{k=1}^{m} w_k} \sum_{k=1}^{m} x_k w_k}{\left(\prod_{i=1}^{n} \left(\frac{\sum_{k=1}^{m} x_k w_{ki}}{\sum_{k=1}^{m} w_{ki}} \right)^{\sum_{k=1}^{m} w_{ki}} \right)^{\frac{1}{\sum_{k=1}^{m} w_k}}} \tag{3.17}$$

$$\le \exp\left[\frac{1}{L(W)\,l(W)}\left(L(W)-\frac{\sum_{k=1}^{m}x_k w_k}{\sum_{k=1}^{m}w_k}\right)\left(\frac{\sum_{k=1}^{m}x_k w_k}{\sum_{k=1}^{m}w_k}-l(W)\right)\right]$$
$$\le \frac{1}{4}\frac{(L(W)-l(W))^2}{L(W)\,l(W)}$$

and

$$1 \le \frac{\frac{1}{\sum_{k=1}^{m}w_k}\sum_{k=1}^{m}x_k w_k}{\left(\prod_{i=1}^{n}\left(\frac{\sum_{k=1}^{m}x_k w_{ki}}{\sum_{k=1}^{m}w_{ki}}\right)^{\sum_{k=1}^{m}w_{ki}}\right)^{\frac{1}{\sum_{k=1}^{m}w_k}}} \tag{3.18}$$
$$\le \left(\frac{l(W)+L(W)}{2\sqrt{l(W)\,L(W)}}\right)^{1+\frac{2}{L(W)-l(W)}\left|\frac{\sum_{k=1}^{m}x_k w_k}{\sum_{k=1}^{m}w_k}-\frac{l(W)+L(W)}{2}\right|}$$
$$\le \left(\frac{l(W)+L(W)}{2\sqrt{l(W)\,L(W)}}\right)^{2}.$$

4. Applications for Univariate Functions

Let $\Phi:[m,M]\to\mathbb{R}$ be a convex function and $f:[0,\pi]\to[m,M]$ an integrable function. Since $\sin^2 t+\cos^2 t=1$ for any $t\in[0,\pi]$ then $\overline{w}=\left(\sin^2,\cos^2\right)$ is a partition of the unity. We then have

$$\psi(\Phi,f,\overline{w})$$
$$:=\frac{1}{\pi}\left[\Phi\left(\frac{\int_0^\pi f(t)\sin^2 t dt}{\int_0^\pi \sin^2 t dt}\right)\int_0^\pi \sin^2 t dt+\Phi\left(\frac{\int_0^\pi f(t)\cos^2 t dt}{\int_0^\pi \cos^2 t dt}\right)\int_0^\pi \cos^2 t dt\right]$$
$$=\frac{1}{2}\left[\Phi\left(\frac{2}{\pi}\int_0^\pi f(t)\sin^2 t dt\right)+\Phi\left(\frac{2}{\pi}\int_0^\pi f(t)\cos^2 t dt\right)\right].$$

By the inequality (2.1) we have

$$\frac{\int_0^\pi(\Phi\circ f)(t)\,dt}{\pi}\ge\frac{1}{2}\left[\Phi\left(\frac{2}{\pi}\int_0^\pi f(t)\sin^2 t dt\right)+\Phi\left(\frac{2}{\pi}\int_0^\pi f(t)\cos^2 t dt\right)\right] \tag{4.1}$$
$$\ge\Phi\left(\frac{\int_0^\pi f(t)\,dt}{\pi}\right),$$

while from (2.10) we have

$$0\le\frac{\int_0^\pi(\Phi\circ f)(t)\,dt}{\pi} \tag{4.2}$$
$$-\frac{1}{2}\left[\Phi\left(\frac{2}{\pi}\int_0^\pi f(t)\sin^2 t dt\right)+\Phi\left(\frac{2}{\pi}\int_0^\pi f(t)\cos^2 t dt\right)\right]$$

$$\leq \frac{1}{2}\frac{\Phi'_{-}(M)-\Phi'_{+}(m)}{M-m}$$
$$\left[\left(M-\frac{2}{\pi}\int_0^{\pi} f(t)\sin^2 t dt\right)\left(\frac{2}{\pi}\int_0^{\pi} f(t)\sin^2 t dt-m\right)\right.$$
$$\left.+\left(M-\frac{2}{\pi}\int_0^{\pi} f(t)\cos^2 t dt\right)\left(\frac{2}{\pi}\int_0^{\pi} f(t)\cos^2 t dt-m\right)\right]$$
$$\leq \frac{\Phi'_{-}(M)-\Phi'_{+}(m)}{M-m}\left(M-\frac{\int_0^{\pi} f(t)\,dt}{\pi}\right)\left(\frac{\int_0^{\pi} f(t)\,dt}{\pi}-m\right).$$

Now, let

$$\begin{aligned} l(\overline{w}) \quad &: = \frac{2}{\pi}\min\left\{\int_0^{\pi} f(t)\sin^2 t dt, \int_0^{\pi} f(t)\cos^2 t dt\right\} \\ &= \frac{1}{\pi}\left[\int_0^{\pi} f(t)\,dt-\left|\int_0^{\pi} f(t)\cos 2t dt\right|\right] \end{aligned}$$

and

$$\begin{aligned} L(\overline{w}) \quad &: = \frac{2}{\pi}\max\left\{\int_0^{\pi} f(t)\sin^2 t dt, \int_0^{\pi} f(t)\cos^2 t dt\right\} \\ &= \frac{1}{\pi}\left[\int_0^{\pi} f(t)\,dt+\left|\int_0^{\pi} f(t)\cos 2t dt\right|\right] \end{aligned}$$

then by (2.15) we have

$$\begin{aligned} (4.3)\qquad 0 &\leq \frac{1}{2}\left[\Phi\left(\frac{2}{\pi}\int_0^{\pi} f(t)\sin^2 t dt\right)+\Phi\left(\frac{2}{\pi}\int_0^{\pi} f(t)\cos^2 t dt\right)\right] \\ &-\Phi\left(\frac{\int_0^{\pi} f(t)\,dt}{\pi}\right) \\ &\leq \frac{1}{2\pi}\left|\int_0^{\pi} f(t)\cos 2t dt\right| \\ &\times\left[\Phi'_{-}\left(\frac{1}{\pi}\left[\int_0^{\pi} f(t)\,dt+\left|\int_0^{\pi} f(t)\cos 2t dt\right|\right]\right)\right. \\ &\left.-\Phi'_{+}\left(\frac{1}{\pi}\left[\int_0^{\pi} f(t)\,dt-\left|\int_0^{\pi} f(t)\cos 2t dt\right|\right]\right)\right]. \end{aligned}$$

Similar inequalities may be obtained if someone would also use the inequalities (2.17) and (2.20). The details are omitted.

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