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BOUNDS FOR THE GENERALIZED (Φ, f)-MEAN DIFFERENCE

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Abstract. In this paper we establish some bounds for the (Φ, f)-*mean difference* introduced in the general settings of measurable spaces and Lebesgue integral, which is a two functions generalization of *Gini mean difference* that has been widely used by economists and sociologists to measure economic inequality.

1. Introduction

Let $(\Omega, \mathcal{A}, \nu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure ν on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a ν-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geq 0$ for ν-a.e. (almost every) $x\in\Omega$ and $\int_\Omega w(x)\,d\nu(x)=1$, consider the *Lebesgue space*

$$L_w(\Omega,\nu) := \{f:\Omega\to\mathbb{R},\ f \text{ is } \nu\text{-measurable and } \int_\Omega w(x)\,|f(x)|\,d\nu(x)<\infty\}.$$

Let I be an interval of real numbers and $\Phi: I\to\mathbb{R}$ a Lebesgue measurable function on I. For $f:\Omega\to I$ a ν-measurable function with $\Phi\circ f\in L_w(\Omega,\nu)$ we define the *generalized* (Φ, f)-*mean difference* $R_G(\Phi, f; w)$ by

$$R_G(\Phi, f; w) := \frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|(\Phi\circ f)(x)-(\Phi\circ f)(y)|\,d\nu(x)\,d\nu(y) \tag{1.1}$$

and the *generalized* (Φ, f)-*mean deviation* $M_D(\Phi, f; w)$ by

$$M_D(\Phi, f; w) := \int_\Omega w(x)\,|(\Phi\circ f)(x)-E(\Phi, f; w)|\,d\nu(x), \tag{1.2}$$

where

$$E(\Phi, f; w) := \int_\Omega (\Phi\circ f)(y)\,w(y)\,d\nu(y)$$

the *generalized* (Φ, f)-expectation.

If $\Phi = e$, where $e(t)=t$, $t\in\mathbb{R}$ is the *identity mapping*, then we can consider the particular cases of interest, the *generalized* f-*mean difference*

$$R_G(f; w) := R_G(e, f; w) = \frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|f(x)-f(y)|\,d\nu(x)\,d\nu(y) \tag{1.3}$$

and the *generalized* f-*mean deviation*

$$M_D(f; w) := M_D(e, f; w) = \int_\Omega w(x)\,|f(x)-E(f; w)|\,d\nu(x), \tag{1.4}$$

where $E(f; w) := \int_\Omega f(y)\,w(y)\,d\nu(y)$ is the *generalized* f-expectation.

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If $\Omega = [-\infty, \infty]$ and $f = e$ then we have the usual *mean difference*

$$R_G(w) := R_G(f;w) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x)\,w(y)\,|x-y|\,dxdy \tag{1.5}$$

and the *mean deviation*

$$M_D(w) := M_D(f;w) = \int_{\Omega} w(x)\,|x-E(w)|\,dx, \tag{1.6}$$

where $w : \mathbb{R} \to [0,\infty)$ is a *density function*, this means that w is integrable on $\mathbb{R}$ and $\int_{-\infty}^{\infty} w(t)\,dt = 1$, and

$$E(w) := \int_{-\infty}^{\infty} xw(x)\,dx \tag{1.7}$$

denote the *expectation of* w provided that the integral exists and is finite.

The mean difference $R_G(w)$ was proposed by Gini in 1912 [21], after whom it is usually named, but was discussed by Helmert and other German writers in the 1870's (cf. H. A. David [13], see also [26, p. 48]). It has a certain theoretical attraction, being dependent on the spread of the variate-values among themselves and not on the deviations from some central value ([26, p. 48]). Further, its defining integral (1.5) may converge when that of the *variance* $\sigma(w)$,

$$\sigma(w) := \int_{-\infty}^{\infty} (x-E(w))^2\,w(x)\,dx, \tag{1.8}$$

does not. It is, however, more difficult to compute than the standard deviation.

For some recent results concerning integral representations and bounds for $R_G(w)$ see [5], [6], [8] and [9].

For instance, if $w : \mathbb{R} \to [0,\infty)$ is a density function we define by

$$W(x) := \int_{-\infty}^{x} w(t)\,dt, \quad x \in \mathbb{R}$$

its *cumulative function.* Then we have [5], [6]:

$$\begin{aligned} R_G(w) &= 2\operatorname{Cov}(e,W) = \int_{-\infty}^{\infty} (1-W(y))\,W(y)\,dy \\ &= 2\int_{-\infty}^{\infty} xw(x)\,W(x)\,dx - E(w) \\ &= 2\int_{-\infty}^{\infty} (x-E(w))\,(W(x)-\gamma)\,w(x)\,dx \\ &= 2\int_{-\infty}^{\infty} (x-\delta)\left(W(x)-\frac{1}{2}\right) w(x)\,dx \end{aligned} \tag{1.9}$$

for any γ, $\delta \in \mathbb{R}$ and [6]:

$$R_G(w) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y)\,(W(x)-W(y))\,w(x)\,w(y)\,dxdy. \tag{1.10}$$

With the above assumptions, we have the bounds [5]:

$$\frac{1}{2}M_D(w) \le R_G(w) \le 2\sup_{x\in\mathbb{R}} |W(x)-\gamma|\,M_D(w) \le M_D(w), \tag{1.11}$$

for any $\gamma \in [0,1]$, where $W(\cdot)$ is the cumulative distribution of w and $M_D(w)$ is the mean deviation.

Consider the n-tuple of real numbers $a = (a_1, ..., a_n)$ and $p = (p_1, ..., p_n)$ a probability distribution, i.e. $p_i \geq 0$ for each $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$, then by taking $\Omega = \{1, ..., n\}$ and the discrete measure, we can consider from (1.1) and (1.2) that (see [7])

$$R_G(a;p) := \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j \left| \Phi(a_i) - \Phi(a_j) \right|, \tag{1.12}$$

and

$$M_D(a;p) := \frac{1}{2} \sum_{i=1}^{n} p_i \left| \Phi(a_i) - \sum_{j=1}^{n} p_j \Phi(a_j) \right| \tag{1.13}$$

where $a \in I^n := I \times ... \times I$ and $\Phi : I \to \mathbb{R}$.

The quantity $R_G(a;p)$ has been defined in [7] and some results were obtained.

In the case when $\Phi = e$, then we get the special case of Gini mean difference and mean deviation of an empirical distribution that is particularly important for applications,

$$R_G(a;p) := \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j \left| a_i - a_j \right|, \tag{1.14}$$

and

$$M_D(a;p) := \frac{1}{2} \sum_{i=1}^{n} p_i \left| a_i - \sum_{j=1}^{n} p_j a_j \right|. \tag{1.15}$$

The following result incorporates an upper bound for the weighted Gini mean difference [7]:

For any $a \in \mathbb{R}^n$ and any p a probability distribution, we have the inequality:

$$\frac{1}{2} M_D(a;p) \leq R_G(a;p) \leq \inf_{\gamma \in \mathbb{R}} \left[\sum_{i=1}^{n} p_i \left| a_i - \gamma \right| \right] \leq M_D(a;p). \tag{1.16}$$

The constant $\frac{1}{2}$ in the first inequality in (1.16) is sharp.

For some recent results for discrete Gini mean difference and mean deviation, see [7], [11], [14] and [15].

2. General Bounds

We have:

Theorem 1. *Let I be an interval of real numbers and $\Phi : I \to \mathbb{R}$ a Lebesgue measurable function on I. If $w : \Omega \to \mathbb{R}$ is a ν-measurable function with $w(x) \geq 0$ for ν-a.e. (almost every) $x \in \Omega$ and $\int_\Omega w(x) \, d\nu(x) = 1$ and if $f : \Omega \to I$ is a ν-measurable function with $\Phi \circ f \in L_w(\Omega, \nu)$, then*

$$\frac{1}{2} M_D(\Phi, f; w) \leq R_G(\Phi, f; w) \leq I(\Phi, f; w) \leq M_D(\Phi, f; w), \tag{2.1}$$

where

$$I(\Phi, f; w) := \inf_{\gamma \in \mathbb{R}} \int_\Omega w(x) \left| (\Phi \circ f)(x) - \gamma \right| d\nu(x). \tag{2.2}$$

Proof. Using the properties of the integral, we have

$$\begin{aligned}
&R_G(\Phi,f;w)\\
&=\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|(\Phi\circ f)(x)-(\Phi\circ f)(y)|\,d\nu(x)\,d\nu(y)\\
&\geq\frac{1}{2}\int_\Omega w(x)\left|(\Phi\circ f)(x)\int_\Omega w(y)\,d\nu(y)-\int_\Omega w(y)\,(\Phi\circ f)(y)\,d\nu(y)\right|d\nu(x)\\
&=\frac{1}{2}\int_\Omega w(x)\left|(\Phi\circ f)(x)-\int_\Omega w(y)\,(\Phi\circ f)(y)\,d\nu(y)\right|d\nu(x)\\
&=\frac{1}{2}M_D(\Phi,f;w)
\end{aligned}$$

and the first inequality in (2.1) is proved.

By the triangle inequality for modulus we have

$$\begin{aligned}
|(\Phi\circ f)(x)-(\Phi\circ f)(y)| &= |(\Phi\circ f)(x)-\gamma+\gamma-(\Phi\circ f)(y)| \\
&\leq |(\Phi\circ f)(x)-\gamma|+|(\Phi\circ f)(y)-\gamma|
\end{aligned}\tag{2.3}$$

for any $x,y\in\Omega$ and $\gamma\in\mathbb{R}$.

Now, if we multiply (2.3) by $\frac{1}{2}w(x)\,w(y)$ and integrate, we get

$$\begin{aligned}
&R_G(\Phi,f;w)\\
&=\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|(\Phi\circ f)(x)-(\Phi\circ f)(y)|\,d\nu(x)\,d\nu(y)\\
&\leq\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,[|(\Phi\circ f)(x)-\gamma|+|(\Phi\circ f)(y)-\gamma|]\,d\nu(x)\,d\nu(y)\\
&=\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|(\Phi\circ f)(x)-\gamma|\,d\nu(x)\,d\nu(y)\\
&+\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|(\Phi\circ f)(y)-\gamma|\,d\nu(x)\,d\nu(y)\\
&=\frac{1}{2}\int_\Omega w(x)\,|(\Phi\circ f)(x)-\gamma|\,d\nu(x)+\frac{1}{2}\int_\Omega w(y)\,|(\Phi\circ f)(y)-\gamma|\,d\nu(y)\\
&=\int_\Omega w(x)\,|(\Phi\circ f)(x)-\gamma|\,d\nu(x)
\end{aligned}\tag{2.4}$$

for any $\gamma\in\mathbb{R}$.

Taking the infimum over $\gamma\in\mathbb{R}$ in (2.4) we get the second part of (2.1).

Since, obviously

$$\begin{aligned}
I(\Phi,f;w)&=\inf_{\gamma\in\mathbb{R}}\int_\Omega w(x)\,|(\Phi\circ f)(x)-\gamma|\,d\nu(x)\\
&\leq\int_\Omega w(x)\left|(\Phi\circ f)(x)-\int_\Omega w(y)\,(\Phi\circ f)(y)\,d\nu(y)\right|d\nu(x)\\
&=M_D(\Phi,f;w),
\end{aligned}$$

the last part of (2.1) is thus proved. □

By the Cauchy-Bunyakowsky-Schwarz (CBS) inequality, if $(\Phi \circ f)^2 \in L_w(\Omega, \nu)$, then we have

$$\begin{aligned}
&\left[\int_\Omega w(x)\left|(\Phi \circ f)(x) - \int_\Omega w(y)(\Phi \circ f)(y)\,d\nu(y)\right| d\nu(x)\right]^2 \\
&\leq \int_\Omega w(x)\left[(\Phi \circ f)(x) - \int_\Omega w(y)(\Phi \circ f)(y)\,d\nu(y)\right]^2 d\nu(x) \\
&= \int_\Omega w(x)(\Phi \circ f)^2(x)\,d\nu(x) \\
&- 2\int_\Omega w(y)(\Phi \circ f)(y)\,d\nu(y)\int_\Omega w(x)(\Phi \circ f)(x)\,d\nu(x) \\
&+ \left[\int_\Omega w(y)(\Phi \circ f)(y)\,d\nu(y)\right]^2 \int_\Omega w(x)\,d\nu(x) \\
&= \int_\Omega w(x)(\Phi \circ f)^2(x)\,d\nu(x) - \left[\int_\Omega w(x)(\Phi \circ f)(x)\,d\nu(x)\right]^2.
\end{aligned}$$

By considering the *generalized* (Φ, f)*-dispersion*

$$\sigma(\Phi, f; w) := \left(\int_\Omega w(x)(\Phi \circ f)^2(x)\,d\nu(x) - \left[\int_\Omega w(x)(\Phi \circ f)(x)\,d\nu(x)\right]^2\right)^{1/2},$$

then we have

$$M_D(\Phi, f; w) \leq \sigma(\Phi, f; w) \tag{2.5}$$

provided $(\Phi \circ f)^2 \in L_w(\Omega, \nu)$.

If there exists the constants m, M so that

$$-\infty < m \leq \Phi(t) \leq M < \infty \text{ for almost any } t \in I \tag{2.6}$$

then by the reverse CBS inequality

$$\sigma(\Phi, f; w) \leq \frac{1}{2}(M - m), \tag{2.7}$$

by (2.1) and by (2.5) we can state the following result:

Corollary 1. *Let I be an interval of real numbers and $\Phi : I \to \mathbb{R}$ a Lebesgue measurable function on I satisfying the condition (2.6) for some constants m, M. If $w : \Omega \to \mathbb{R}$ is a ν-measurable function with $w(x) \geq 0$ for ν -a.e. $x \in \Omega$ and $\int_\Omega w(x)\,d\nu(x) = 1$ and if $f : \Omega \to I$ is a ν-measurable function with $(\Phi \circ f)^2 \in L_w(\Omega, \nu)$, then we have the chain of inequalities*

$$\begin{aligned}
\frac{1}{2}M_D(\Phi, f; w) &\leq R_G(\Phi, f; w) \leq I(\Phi, f; w) \leq M_D(\Phi, f; w) \\
&\leq \sigma(\Phi, f; w) \leq \frac{1}{2}(M - m).
\end{aligned} \tag{2.8}$$

We observe that, in the discrete case we obtain from (2.1) the inequality (1.16) while for the univariate case with $\int_{-\infty}^{\infty} w(t)\,dt = 1$ we have

$$\frac{1}{2}M_D(w) \leq R_G(w) \leq I(w) \leq M_D(w) \leq \sigma(\Phi, f; w) \tag{2.9}$$

where

$$I(w) := \inf_{\gamma \in \mathbb{R}} \int_{-\infty}^{\infty} w(x)|x-\gamma|\,dx. \tag{2.10}$$

If w is supported on the finite interval $[a,b]$, namely $\int_a^b w(x)\,dx = 1$, then we have the chain of inequalities

$$\frac{1}{2}M_D(w) \le R_G(w) \le I(w) \le M_D(w) \le \sigma(\Phi, f; w) \le \frac{1}{2}(b-a). \tag{2.11}$$

3. Bounds for Various Classes of Functions

In the case of functions of bounded variation we have:

Theorem 2. *Let $\Phi : [a,b] \to \mathbb{R}$ be a function of bounded variation on the closed interval $[a,b]$. If $w : \Omega \to \mathbb{R}$ is a ν-measurable function with $w(x) \ge 0$ for ν -a.e. $x \in \Omega$ and $\int_\Omega w(x)\,d\nu(x) = 1$ and if $f : \Omega \to [a,b]$ is a ν-measurable function with $\Phi \circ f \in L_w(\Omega, \nu)$, then*

$$R_G(\Phi, f; w) \le \frac{1}{2}\bigvee_a^b(\Phi), \tag{3.1}$$

where $\bigvee_a^b(\Phi)$ is the total variation of Φ on $[a,b]$.

Proof. Using the inequality (2.4) we have

$$R_G(\Phi, f; w) \le \int_\Omega w(x)|(\Phi \circ f)(x) - \gamma|\,d\nu(x) \tag{3.2}$$

for any $\gamma \in \mathbb{R}$.

By the triangle inequality, we have

$$\begin{aligned}&\left|(\Phi \circ f)(x) - \frac{1}{2}[\Phi(a) + \Phi(b)]\right| \\ &\le \frac{1}{2}|\Phi(a) - \Phi(f(x))| + \frac{1}{2}|\Phi(b) - \Phi(f(x))|\end{aligned} \tag{3.3}$$

for any $x \in \Omega$.

Since $\Phi : [a,b] \to \mathbb{R}$ is of bounded variation and d is a division of $[a,b]$, namely

$$d \in \mathcal{D}([a,b]) := \{d := \{a = t_0 < t_1 < ... < t_n = b\}\},$$

then

$$\bigvee_a^b(\Phi) = \sup_{d \in \mathcal{D}([a,b])} \sum_{i=0}^{n-1} |\Phi(t_{i+1}) - \Phi(t_i)| < \infty.$$

Taking the division $d_0 := \{a = t_0 < t < t_2 = b\}$ we then have

$$|\Phi(t) - \Phi(a)| + |\Phi(b) - \Phi(t)| \le \bigvee_a^b(\Phi)$$

for any $t \in [a,b]$ and then

$$|\Phi(f(x)) - \Phi(a)| + |\Phi(b) - \Phi(f(x))| \le \bigvee_a^b(\Phi) \tag{3.4}$$

for any $x \in \Omega$.

On making use of (3.3) and (3.4) we get

$$\left|(\Phi \circ f)(x) - \frac{1}{2}[\Phi(a) + \Phi(b)]\right| \leq \frac{1}{2}\bigvee_a^b(\Phi) \tag{3.5}$$

for any $x \in \Omega$.

If we multiply (3.5) by $w(x)$ and integrate, then we obtain

$$\int_\Omega w(x)\left|(\Phi \circ f)(x) - \frac{1}{2}[\Phi(a) + \Phi(b)]\right| \leq \frac{1}{2}\bigvee_a^b(\Phi). \tag{3.6}$$

Finally, by choosing $\gamma = \frac{1}{2}[\Phi(a) + \Phi(b)]$ in (3.2) and making use of (3.6) we deduce the desired result (3.1). □

In the case of absolutely continuous functions we have:

Theorem 3. *Let $\Phi : [a, b] \to \mathbb{R}$ be an absolutely continuous function on the closed interval $[a, b]$. If $w : \Omega \to \mathbb{R}$ is a ν-measurable function with $w(x) \geq 0$ for ν -a.e. $x \in \Omega$ and $\int_\Omega w(x)\, d\nu(x) = 1$ and if $f : \Omega \to [a, b]$ is a ν-measurable function with $\Phi \circ f \in L_w(\Omega, \nu)$, then*

$$R_G(\Phi, f; w) \leq \begin{cases} \|\Phi'\|_{[a,b],\infty} R_G(f; w) \ \text{ if } \Phi' \in L_\infty([\alpha, \beta]), \\ \\ \frac{1}{2^{1/p}}\|\Phi'\|_{[a,b],p} R_G^{1/q}(f; w) \ \text{ if } \Phi' \in L_p([\alpha, \beta]), \\ \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases}, \tag{3.7}$$

where the Lebesgue norms are defined by

$$\|g\|_{[\alpha,\beta],p} := \begin{cases} \operatorname{essup}_{t\in[\alpha,\beta]} |g(t)| \ \text{ if } p = \infty, \\ \\ \left(\int_\alpha^\beta |g(t)|^p\, dt\right)^{1/p} \ \text{ if } p \geq 1 \end{cases}$$

and $L_p([\alpha, \beta]) := \left\{g \middle| \ g \text{ measurable and } \|g\|_{[\alpha,\beta],p} < \infty\right\}$, $p \in [1, \infty]$.

Proof. Since f is absolutely continuous, then we have

$$\Phi(t) - \Phi(s) = \int_s^t \Phi'(u)\, du$$

for any $t, s \in [a, b]$.

Using the Hölder integral inequality we have

$$\begin{aligned} |\Phi(t) - \Phi(s)| &= \left|\int_s^t \Phi'(u)\, du\right| \\ &\leq \begin{cases} \|\Phi'\|_{[a,b],\infty} |t - s| \ \text{ if } p = \infty, \\ \\ \|\Phi'\|_{[a,b],p} |t - s|^{1/q} \ \text{ if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \end{aligned} \tag{3.8}$$

for any $t, s \in [a, b]$.

Using (3.8) we then have

$$|(\Phi\circ f)(x)-(\Phi\circ f)(y)| \leq \begin{cases} \|\Phi'\|_{[a,b],\infty}|f(x)-f(y)| \text{ if } p=\infty, \\ \\ \|\Phi'\|_{[a,b],p}|f(x)-f(y)|^{1/q} \text{ if } p>1, \frac{1}{p}+\frac{1}{q}=1 \end{cases} \tag{3.9}$$

for any x, $y\in\Omega$.

If we multiply (3.9) by $\frac{1}{2}w(x)w(y)$ and integrate, then we get

$$\frac{1}{2}\int_\Omega\int_\Omega w(x)w(y)|(\Phi\circ f)(x)-(\Phi\circ f)(y)|\,d\nu(x)\,d\nu(y) \leq \begin{cases} \frac{1}{2}\|\Phi'\|_{[a,b],\infty}\int_\Omega\int_\Omega w(x)w(y)|f(x)-f(y)|\,d\nu(x)\,d\nu(y) \text{ if } p=\infty, \\ \\ \frac{1}{2}\|\Phi'\|_{[a,b],p}\int_\Omega\int_\Omega w(x)w(y)|f(x)-f(y)|^{1/q}\,d\nu(x)\,d\nu(y) \\ \\ \text{if } p>1, \frac{1}{p}+\frac{1}{q}=1. \end{cases} \tag{3.10}$$

This proves the first branch of (3.7).

Using Jensen's integral inequality for concave function $\Psi(t)=t^s$, $s\in(0,1)$ we have for $s=\frac{1}{q}<1$ that

$$\begin{aligned} &\int_\Omega\int_\Omega w(x)w(y)|f(x)-f(y)|^{1/q}\,d\nu(x)\,d\nu(y) \\ &\leq\left(\int_\Omega\int_\Omega w(x)w(y)|f(x)-f(y)|\,d\nu(x)\,d\nu(y)\right)^{1/q}, \end{aligned}$$

which implies that

$$\begin{aligned} &\frac{1}{2}\|\Phi'\|_{[a,b],p}\int_\Omega\int_\Omega w(x)w(y)|f(x)-f(y)|^{1/q}\,d\nu(x)\,d\nu(y) \\ &\leq\frac{1}{2}\|\Phi'\|_{[a,b],p}\left(\int_\Omega\int_\Omega w(x)w(y)|f(x)-f(y)|\,d\nu(x)\,d\nu(y)\right)^{1/q} \\ &=\|\Phi'\|_{[a,b],p}\left(\frac{1}{2^q}\int_\Omega\int_\Omega w(x)w(y)|f(x)-f(y)|\,d\nu(x)\,d\nu(y)\right)^{1/q} \\ &=\|\Phi'\|_{[a,b],p}\left(\frac{1}{2^{q-1}}\frac{1}{2}\int_\Omega\int_\Omega w(x)w(y)|f(x)-f(y)|\,d\nu(x)\,d\nu(y)\right)^{1/q} \\ &=\frac{1}{2^{\frac{q-1}{q}}}\|\Phi'\|_{[a,b],p}(R_G(f;w))^{1/q}=\frac{1}{2^{1/p}}\|\Phi'\|_{[a,b],p}R_G^{1/q}(f;w) \end{aligned}$$

and the second part of (3.7) is proved. □

The function $\Phi:[a,b]\to\mathbb{R}$ is called of *r-H-Hölder type* with the given constants $r\in(0,1]$ and $H>0$ if

$$|\Phi(t)-\Phi(s)|\leq H|t-s|^r$$

for any t, $s\in[a,b]$.

In the case when $r=1$, namely, there is the constant $L>0$ such that

$$|\Phi(t)-\Phi(s)|\leq L|t-s|$$

for any $t,s\in[a,b]$, the function Φ is called L-Lipschitzian on $[a,b]$.

We have:

Theorem 4. *Let $\Phi : [a,b] \to \mathbb{R}$ be a function of r-H-Hölder type on the closed interval $[a,b]$. If $w : \Omega \to \mathbb{R}$ is a ν-measurable function with $w(x) \geq 0$ for ν-a.e. $x \in \Omega$ and $\int_\Omega w(x)\, d\nu(x) = 1$ and if $f : \Omega \to [a,b]$ is a ν-measurable function with $\Phi \circ f \in L_w(\Omega, \nu)$, then*

$$R_G(\Phi, f; w) \leq \frac{1}{2^{1-r}} H R_G^r(f; w). \tag{3.11}$$

In particular, if Φ is L-Lipschitzian on $[a,b]$, then

$$R_G(\Phi, f; w) \leq L R_G(f; w). \tag{3.12}$$

Proof. We have

$$|(\Phi \circ f)(x) - (\Phi \circ f)(y)| \leq H |f(x) - f(y)|^r \tag{3.13}$$

for any $x, y \in \Omega$.

If we multiply (3.13) by $\frac{1}{2} w(x) w(y)$ and integrate, then we get

$$\begin{aligned} &\frac{1}{2} \int_\Omega \int_\Omega w(x) w(y) |(\Phi \circ f)(x) - (\Phi \circ f)(y)| \, d\nu(x)\, d\nu(y) \\ &\leq \frac{1}{2} H \int_\Omega \int_\Omega w(x) w(y) |f(x) - f(y)|^r \, d\nu(x)\, d\nu(y). \end{aligned} \tag{3.14}$$

By Jensen's integral inequality for concave functions we also have

$$\begin{aligned} &\int_\Omega \int_\Omega w(x) w(y) |f(x) - f(y)|^r \, d\nu(x)\, d\nu(y) \\ &\leq \left(\int_\Omega \int_\Omega w(x) w(y) |f(x) - f(y)| \, d\nu(x)\, d\nu(y) \right)^r. \end{aligned} \tag{3.15}$$

Therefore, by (3.14) and (3.15) we get

$$\begin{aligned} R_G(\Phi, f; w) &\leq \frac{1}{2} H \left(\int_\Omega \int_\Omega w(x) w(y) |f(x) - f(y)| \, d\nu(x)\, d\nu(y) \right)^r \\ &= \frac{1}{2^{1-r}} H \left(\frac{1}{2} \int_\Omega \int_\Omega w(x) w(y) |f(x) - f(y)| \, d\nu(x)\, d\nu(y) \right)^r \\ &= \frac{1}{2^{1-r}} H R_G^r(f; w) \end{aligned}$$

and the inequality (3.11) is proved. □

We have:

Theorem 5. *Let $\Phi, \Psi : [a,b] \to \mathbb{R}$ be continuos functions on $[a,b]$ and differentiable on (a,b) with $\Psi'(t) \neq 0$ for $t \in (a,b)$. If $w : \Omega \to \mathbb{R}$ is a ν-measurable function with $w(x) \geq 0$ for ν -a.e. $x \in \Omega$ and $\int_\Omega w(x)\, d\nu(x) = 1$ and if $f : \Omega \to [a,b]$ is a ν-measurable function with $\Phi \circ f \in L_w(\Omega, \nu)$, then*

$$\inf_{t \in (a,b)} \left| \frac{\Phi'(t)}{\Psi'(t)} \right| R_G(\Psi, f; w) \leq R_G(\Phi, f; w) \leq \sup_{t \in (a,b)} \left| \frac{\Phi'(t)}{\Psi'(t)} \right| R_G(\Psi, f; w). \tag{3.16}$$

Proof. By the Cauchy's mean value theorem, for any $t, s \in [a,b]$ with $t \neq s$ there exists a ξ between t and s such that

$$\frac{\Phi(t) - \Phi(s)}{\Psi(t) - \Psi(s)} = \frac{\Phi'(\xi)}{\Psi'(\xi)}.$$

This implies that

$$\inf_{t\in(a,b)}\left|\frac{\Phi'(t)}{\Psi'(t)}\right||\Psi(t)-\Psi(s)|\leq|\Phi(t)-\Phi(s)| \leq\sup_{t\in(a,b)}\left|\frac{\Phi'(t)}{\Psi'(t)}\right||\Psi(t)-\Psi(s)| \tag{3.17}$$

for any $t,\ s\in[a,b]$.

Therefore, we have

$$\inf_{t\in(a,b)}\left|\frac{\Phi'(t)}{\Psi'(t)}\right||\Psi(f(x))-\Psi(f(y))|\leq|\Phi(f(x))-\Phi(f(y))| \leq\sup_{t\in(a,b)}\left|\frac{\Phi'(t)}{\Psi'(t)}\right||\Psi(f(x))-\Psi(f(y))| \tag{3.18}$$

for any $x,y\in\Omega$.

If we multiply (3.18) by $\frac{1}{2}w(x)w(y)$ and integrate, we get the desired result (3.16). □

Corollary 2. *Let $\Phi:[a,b]\to\mathbb{R}$ be a continuos function on $[a,b]$ and differentiable on (a,b). If w is as in Theorem 5, then we have*

$$\inf_{t\in(a,b)}|\Phi'(t)|R_G(f;w)\leq R_G(\Phi,f;w)\leq\sup_{t\in(a,b)}|\Phi'(t)|R_G(f;w). \tag{3.19}$$

We also have:

Theorem 6. *Let $\Phi:[a,b]\to\mathbb{R}$ be an absolutely continuous function on the closed interval $[a,b]$. If $w:\Omega\to\mathbb{R}$ is a ν-measurable function with $w(x)\geq 0$ for ν -a.e. $x\in\Omega$ and $\int_\Omega w(x)d\nu(x)=1$ and if $f:\Omega\to[a,b]$ is a ν-measurable function with $\Phi\circ f\in L_w(\Omega,\nu)$, then*

$$\begin{aligned} &R_G(\Phi,f;w)\\ &\leq\begin{cases}\|\Phi'\|_{[a,b],\infty}M(f;w) & \text{if } p=\infty,\\ \\ \|\Phi'\|_{[a,b],p}M^{1/q}(f;w) & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1\end{cases}\\ &\leq\begin{cases}\frac{1}{2}(b-a)\|\Phi'\|_{[a,b],\infty} & \text{if } p=\infty,\\ \\ \frac{1}{2^{1/q}}(b-a)^{1/q}\|\Phi'\|_{[a,b],p} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1,\end{cases}\end{aligned} \tag{3.20}$$

where $M(f;w)$ is defined by

$$M(f;w):=\int_\Omega w(x)\left|f(x)-\frac{a+b}{2}\right|d\nu(x). \tag{3.21}$$

Proof. From the inequality (3.8) we have

$$\begin{aligned} &\left|(\Phi\circ f)(x)-\Phi\left(\frac{a+b}{2}\right)\right|\\ &\leq\begin{cases}\|\Phi'\|_{[a,b],\infty}\left|f(x)-\frac{a+b}{2}\right| & \text{if } p=\infty,\\ \\ \|\Phi'\|_{[a,b],p}\left|f(x)-\frac{a+b}{2}\right|^{1/q} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1\end{cases}\end{aligned} \tag{3.22}$$

for any $x\in\Omega$.

Now, if we multiply (3.22) by $w(x)$ and integrate, then we get

$$\int_{\Omega} w(x)\left|(\Phi\circ f)(x)-\Phi\left(\frac{a+b}{2}\right)\right| d\nu(x) \tag{3.23}$$
$$\leq \begin{cases} \|\Phi'\|_{[a,b],\infty}\int_{\Omega} w(x)\left|f(x)-\frac{a+b}{2}\right| d\nu(x) \text{ if } p=\infty, \\ \\ \|\Phi'\|_{[a,b],p}\int_{\Omega} w(x)\left|f(x)-\frac{a+b}{2}\right|^{1/q} d\nu(x) \text{ if } p>1, \frac{1}{p}+\frac{1}{q}=1. \end{cases}$$

By Jensen's integral inequality for concave functions we have

$$\int_{\Omega} w(x)\left|f(x)-\frac{a+b}{2}\right|^{1/q} d\nu(x) \leq \left(\int_{\Omega} w(x)\left|f(x)-\frac{a+b}{2}\right| d\nu(x)\right)^{1/q}. \tag{3.24}$$

On making use of (3.2), (3.23) and (3.24) we get the first inequality in (3.20).

The last part of (3.20) follows by the fact that

$$\left|f(x)-\frac{a+b}{2}\right| \leq \frac{1}{2}(b-a)$$

for any $x\in\Omega$. $\square$

4. Bounds for Special Convexity

When some convexity properties for the function Φ are assumed, then other bounds can be derived as follows.

Theorem 7. *Let $w:\Omega\to\mathbb{R}$ be a ν-measurable function with $w(x)\geq 0$ for ν -a.e. $x\in\Omega$ and $\int_{\Omega} w(x)\,d\nu(x)=1$ and $f:\Omega\to[a,b]$ be a ν-measurable function with $\Phi\circ f\in L_w(\Omega,\nu)$. Assume also that $\Phi:[a,b]\to\mathbb{R}$ is a continuous function on $[a,b]$.*

(i) If $|\Phi|$ is concave on $[a,b]$, then

$$R_G(\Phi,f;w) \leq |\Phi(E(f;w))|, \tag{4.1}$$

(ii) If $|\Phi|$ is convex on $[a,b]$, then

$$R_G(\Phi,f;w) \leq \frac{1}{b-a}\left[(b-E(f;w))|\Phi(a)|+(E(f;w)-a)\Phi|(b)|\right]. \tag{4.2}$$

Proof. (i) If $|\Phi|$ is concave on $[a,b]$, then by Jensen's inequality we have

$$\int_{\Omega} w(x)|(\Phi\circ f)(x)|\,d\nu(x) \leq \left|\Phi\left(\int_{\Omega} w(x)f(x)\,d\nu(x)\right)\right|. \tag{4.3}$$

From (3.2) for $\gamma=0$ we also have

$$R_G(\Phi,f;w) \leq \int_{\Omega} w(x)|(\Phi\circ f)(x)|\,d\nu(x). \tag{4.4}$$

This is an inequality of interest in itself.

On utilizing (4.3) and (4.4) we get (4.1).

(ii) Since $|\Phi|$ is convex on $[a,b]$, then for any $t\in[a,b]$ we have

$$|\Phi(t)| = \left|\Phi\left(\frac{(b-t)a+b(t-a)}{b-a}\right)\right| \leq \frac{(b-t)|\Phi(a)|+(t-a)\Phi|(b)|}{b-a}.$$

This implies that

$$|(\Phi\circ f)(x)| \leq \frac{(b-f(x))|\Phi(a)|+(f(x)-a)\Phi|(b)|}{b-a} \tag{4.5}$$

for any $x \in \Omega$.

If we multiply (4.5) by $w(x)$ and integrate, then we get

$$\begin{aligned}&\int_{\Omega} w(x) \left|(\Phi \circ f)(x)\right| d\nu(x) \\ &\leq \frac{1}{b-a}\left[\left(b\int_{\Omega} w(x)\, d\nu(x) - \int_{\Omega} w(x) f(x)\, d\nu(x)\right)|\Phi(a)|\right. \\ &\left.+\left(\int_{\Omega} w(x) f(x)\, d\nu(x) - a\int_{\Omega} w(x)\, d\nu(x)\right)\Phi|(b)|\right],\end{aligned}$$

which, together with (4.4), produces the desired result (4.2). □

In order to state other results we need the following definitions:

Definition 1 ([19])**.** *We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0,1]$ we have*

$$f(tx + (1-t)y) \leq f(x) + f(y).$$

It is important to note that $P(I)$ contains all nonnegative monotone, convex and *quasi convex functions,* i.e. functions satisfying

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$$

for all x, $y \in I$ and $t \in [0,1]$.

For some results on P-functions see [19] and [28] while for quasi convex functions, the reader can consult [18].

Definition 2 ([3])**.** *Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

For some properties of this class of functions see [1], [2], [3], [4], [16], [17], [25], [27] and [29].

Theorem 8. *Let $w : \Omega \to \mathbb{R}$ be a ν-measurable function with $w(x) \geq 0$ for ν -a.e. $x \in \Omega$ and $\int_{\Omega} w(x)\, d\nu(x) = 1$ and $f : \Omega \to [a,b]$ be a ν-measurable function with $\Phi \circ f \in L_w(\Omega, \nu)$. Assume also that $\Phi : [a,b] \to \mathbb{R}$ is a continuous function on $[a,b]$.*

(i) If $|\Phi|$ belongs to the class P on $[a,b]$, then

$$R_G(\Phi, f; w) \leq |\Phi(a)| + \Phi|(b)|; \tag{4.6}$$

(ii) If $|\Phi|$ is quasi convex on $[a,b]$, then

$$R_G(\Phi, f; w) \leq \max\{|\Phi(a)|, \Phi|(b)|\}; \tag{4.7}$$

(iii) If $|\Phi|$ is Breckner s-convex on $[a,b]$, then

$$R_G(\Phi, f; w) \le \frac{1}{(b-a)^s}\left[|\Phi(a)| \int_\Omega w(x)(b-f(x))^s d\nu(x) + \Phi|(b)| \int_\Omega w(x)(f(x)-a)^s d\nu(x)\right] \le \frac{1}{(b-a)^s}\left[|\Phi(a)|(b-E(f;w))^s d\nu(x) + \Phi|(b)|(E(f;w)-a)^s d\nu(x)\right]. \tag{4.8}$$

Proof. (i) Since $|\Phi|$ belongs to the class P on $[a,b]$, then for any $t \in [a,b]$ we have

$$|\Phi(t)| = \left|\Phi\left(\frac{(b-t)a + b(t-a)}{b-a}\right)\right| \le |\Phi(a)| + \Phi|(b)|.$$

This implies that

$$|(\Phi \circ f)(x)| \le |\Phi(a)| + \Phi|(b)| \tag{4.9}$$

for any $x \in \Omega$.

If we multiply (4.9) by $w(x)$ and integrate, then we get

$$\int_\Omega w(x)|(\Phi \circ f)(x)| d\nu(x) \le |\Phi(a)| + \Phi|(b)|, \tag{4.10}$$

which, together with (4.4), produces the desired result (4.6).

(ii) Goes in a similar way.

(iii) By Breckner s-convexity we have

$$|\Phi(t)| = \left|\Phi\left(\frac{(b-t)a + b(t-a)}{b-a}\right)\right| \le \left(\frac{b-t}{b-a}\right)^s |\Phi(a)| + \left(\frac{t-a}{b-a}\right)^s \Phi|(b)|$$

for any $t \in [a,b]$.

This implies that

$$|(\Phi \circ f)(x)| \le \frac{1}{(b-a)^s}\left[(b-f(x))^s |\Phi(a)| + (f(x)-a)^s \Phi|(b)|\right] \tag{4.11}$$

for any $x \in \Omega$.

If we multiply (4.11) by $w(x)$ and integrate, then we get

$$\int_\Omega w(x)|(\Phi \circ f)(x)| d\nu(x) \le \frac{1}{(b-a)^s}\left[|\Phi(a)| \int_\Omega w(x)(b-f(x))^s d\nu(x) + \Phi|(b)| \int_\Omega w(x)(f(x)-a)^s d\nu(x)\right], \tag{4.12}$$

which, together with (4.4), produces the first part of (4.8).

The last part follows by Jensen's integral inequality for concave functions, namely

$$\int_\Omega w(x)(b-f(x))^s d\nu(x) \le \left(b - \int_\Omega w(x) f(x) d\nu(x)\right)^s$$

and

$$\int_\Omega w(x)(f(x)-a)^s d\nu(x) \le \left(\int_\Omega w(x) f(x) d\nu(x) - a\right)^s,$$

where $s \in (0,1)$. $\square$

5. Some Examples

Let $f:\Omega\rightarrow[0,\infty)$ be a ν-measurable function and $w:\Omega\rightarrow\mathbb{R}$ a ν-measurable function with $w(x)\geq 0$ for ν -a.e. $x\in\Omega$ and $\int_{\Omega}w(x)\,d\nu(x)=1$. We define, for the function $\Phi(t)=t^{p}$, $p>0$, the *generalized* (p,f)*-mean difference* $R_{G}(p,f;w)$ by

$$R_{G}(p,f;w):=\frac{1}{2}\int_{\Omega}\int_{\Omega}w(x)\,w(y)\,|f^{p}(x)-f^{p}(y)|\,d\nu(x)\,d\nu(y) \tag{5.1}$$

and the *generalized* (p,f)*-mean deviation* $M_{D}(p,f;w)$ by

$$M_{D}(p,f;w):=\int_{\Omega}w(x)\,|f^{p}(x)-E(p,f;w)|\,d\nu(x), \tag{5.2}$$

where

$$E(p,f;w):=\int_{\Omega}f^{p}(y)\,w(y)\,d\nu(y) \tag{5.3}$$

is the *generalized* (p,f)-expectation.

If $f:\Omega\rightarrow[a,b]\subset[0,\infty)$ is a ν-measurable function, then by (3.1) we have

$$R_{G}(p,f;w)\leq\frac{1}{2}(b^{p}-a^{p}). \tag{5.4}$$

By (3.7) we have

$$R_{G}(p,f;w)\leq p\delta_{p}(a,b)\,R_{G}(f;w), \tag{5.5}$$

where

$$\delta_{p}(a,b):=\begin{cases} b \text{ if } p\geq 1,\\ \\ a \text{ if } p\in(0,1)\end{cases}$$

and

$$R_{G}(p,f;w)\leq\frac{p}{2^{1/\alpha}}\left[\frac{b^{\alpha(p-1)+1}-a^{\alpha(p-1)+1}}{\alpha(p-1)+1}\right]^{1/\alpha}R_{G}^{1/\beta}(f;w), \tag{5.6}$$

where $\alpha>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

From (3.20) we also have

$$\begin{aligned} R_{G}(p,f;w) &\leq\begin{cases}\delta_{p}(a,b)\,M(f;w),\\ \\ p\left(\frac{b^{\alpha(p-1)+1}-a^{\alpha(p-1)+1}}{\alpha(p-1)+1}\right)^{1/\alpha}M^{1/\beta}(f;w) \text{ if } \alpha>1,\frac{1}{\alpha}+\frac{1}{\beta}=1\end{cases}\\ &\leq\begin{cases}\frac{1}{2}(b-a)\,\delta_{p}(a,b),\\ \\ \frac{1}{2^{1/\beta}}(b-a)^{1/\beta}p\left(\frac{b^{\alpha(p-1)+1}-a^{\alpha(p-1)+1}}{\alpha(p-1)+1}\right)^{1/\alpha} \text{ if } \alpha>1,\frac{1}{\alpha}+\frac{1}{\beta}=1,\end{cases}\end{aligned} \tag{5.7}$$

where $M(f;w)$ is defined by (3.21).

If $p\in(0,1)$, then the function $|\Phi(t)|=t^{p}$ is concave on $[a,b]\subset[0,\infty)$ and by (4.1) we have

$$R_{G}(p,f;w)\leq E^{p}(f;w). \tag{5.8}$$

For $p \geq 1$ the function $|\Phi(t)| = t^p$ is convex on $[a,b] \subset [0,\infty)$ and by (4.2) we have

$$R_G(p,f;w) \leq \frac{1}{b-a}\left[(b-E(f;w))\,a^p + (E(f;w)-a)\,b^p\right]. \tag{5.9}$$

Let $f : \Omega \to [0,\infty)$ be a ν-measurable function and $w : \Omega \to \mathbb{R}$ a ν-measurable function with $w(x) \geq 0$ for ν -a.e. $x \in \Omega$ and $\int_\Omega w(x)\,d\nu(x) = 1$. We define, for the function $\Phi(t) = \ln t$, the *generalized* $(\ln, f)$*-mean difference* $R_G(\ln, f; w)$ by

$$R_G(\ln,f;w) := \frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|\ln f(x) - \ln f(y)|\,d\nu(x)\,d\nu(y) \tag{5.10}$$

and the *generalized* (p,f)*-mean deviation* $M_D(\ln,f;w)$ by

$$M_D(\ln,f;w) := \int_\Omega w(x)\,|\ln f(x) - E(\ln,f;w)|\,d\nu(x), \tag{5.11}$$

where

$$E(\ln,f;w) := \int_\Omega w(y)\ln f(y)\,d\nu(y) \tag{5.12}$$

is the *generalized* $(\ln, f)$*-expectation.*

If $f : \Omega \to [a,b] \subset [0,\infty)$ is a ν-measurable function, then by (3.1) we have

$$R_G(\ln,f;w) \leq \frac{1}{2}(\ln b - \ln a). \tag{5.13}$$

By (3.7) we have

$$R_G(\ln,f;w) \leq \begin{cases} \frac{1}{a}R_G(f;w), \\ \\ \frac{1}{2^{1/p}}\left(\frac{b^{p-1}-a^{p-1}}{(p-1)b^{p-1}a^{p-1}}\right)^{1/p} R_G^{1/q}(f;w) & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1. \end{cases} \tag{5.14}$$

By (3.20) we have

$$\begin{aligned} R_G(\ln,f;w) &\leq \begin{cases} \frac{1}{a}M(f;w), \\ \\ \left(\frac{b^{p-1}-a^{p-1}}{(p-1)b^{p-1}a^{p-1}}\right)^{1/p} M^{1/q}(f;w) & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \end{cases} \\ &\leq \begin{cases} \frac{1}{2}\left(\frac{b}{a}-1\right), \\ \\ \frac{1}{2^{1/q}}(b-a)^{1/q}\left(\frac{b^{p-1}-a^{p-1}}{(p-1)b^{p-1}a^{p-1}}\right)^{1/p} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1. \end{cases} \end{aligned} \tag{5.15}$$

Now, observe that the function $|\Phi(t)| = |\ln t|$ is convex on $(0,1)$ and concave on $[1,\infty)$. If $f : \Omega \to [a,b] \subset (0,1)$ is a ν-measurable function, then by (4.2) we have

$$R_G(\ln,f;w) \leq \frac{1}{b-a}\left[(b-E(f;w))\,|\ln a| + (E(f;w)-a)\,|\ln b|\right] \tag{5.16}$$

and if $f : \Omega \to [a,b] \subset [1,\infty)$, then by (4.1) we have

$$R_G(\ln,f;w) \leq \ln(E(f;w)). \tag{5.17}$$

The interested reader may state similar bounds for functions Φ such as $\Phi(t) = \exp t$, $t \in \mathbb{R}$ or $\Phi(t) = t\ln t$, $t > 0$. We omit the details.

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