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SOME ADDITIVE INEQUALITIES FOR HEINZ OPERATOR MEAN

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Abstract. In this paper we obtain some new additive inequalities for Heinz operator mean.

1. Introduction

Throughout this paper A, B are positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notations for operators and $\nu \in [0,1]$

$$A\nabla_\nu B := (1-\nu) A + \nu B,$$

the *weighted operator arithmetic mean,* and

$$A\sharp_\nu B := A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^\nu A^{1/2},$$

the *weighted operator geometric mean* [14]. When $\nu = \frac{1}{2}$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

Define the *Heinz operator mean* by

$$H_\nu (A, B) := \frac{1}{2} (A\sharp_\nu B + A\sharp_{1-\nu} B).$$

The following interpolatory inequality is obvious

$$A\sharp B \le H_\nu (A, B) \le A\nabla B \tag{1.1}$$

for any $\nu \in [0,1]$.

We recall that *Specht's ratio* is defined by [16]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1, \infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \ne 1$. The function is decreasing on $(0,1)$ and increasing on $(1, \infty)$.

The following result provides an upper and lower bound for the Heinz mean in terms of the operator geometric mean $A\sharp B$:

Theorem 1 (Dragomir, 2015 [6])**.** *Assume that A, B are positive invertible operators and the constants $M > m > 0$ are such that*

$$mA \le B \le MA. \tag{1.3}$$

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Then we have

$$\omega_{\nu}(m,M)\, A\sharp B \leq H_{\nu}(A,B) \leq \Omega_{\nu}(m,M)\, A\sharp B, \tag{1.4}$$

where

$$\Omega_{\nu}(m,M) := \begin{cases} S\left(m^{|2\nu-1|}\right) \text{ if } M<1, \\ \max\left\{S\left(m^{|2\nu-1|}\right), S\left(M^{|2\nu-1|}\right)\right\} \text{ if } m \leq 1 \leq M, \\ S\left(M^{|2\nu-1|}\right) \text{ if } 1<m \end{cases} \tag{1.5}$$

and

$$\omega_{\nu}(m,M) := \begin{cases} S\left(M^{\left|\nu-\frac{1}{2}\right|}\right) \text{ if } M<1, \\ 1 \text{ if } m \leq 1 \leq M, \\ S\left(m^{\left|\nu-\frac{1}{2}\right|}\right) \text{ if } 1<m, \end{cases} \tag{1.6}$$

where $\nu \in [0,1]$.

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h>0. \tag{1.7}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h>0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h>0$.

We have:

Theorem 2 (Dragomir, 2015 [7])**.** *Assume that* A, B *are positive invertible operators and the constants* $M>m>0$ *are such that the condition (1.3) is valid. Then for any* $\nu \in [0,1]$ *we have*

$$(A\sharp B \leq)\, H_{\nu}(A,B) \leq \exp\left[\Theta_{\nu}(m,M)-1\right] A\sharp B \tag{1.8}$$

where

$$\Theta_{\nu}(m,M) := \begin{cases} K\left(m^{|2\nu-1|}\right) \text{ if } M<1, \\ \max\left\{K\left(m^{|2\nu-1|}\right), K\left(M^{|2\nu-1|}\right)\right\} \text{ if } m \leq 1 \leq M, \\ K\left(M^{|2\nu-1|}\right) \text{ if } 1<m \end{cases} \tag{1.9}$$

and

$$(0 \leq)\, H_{\nu}(A,B) - A\sharp B \leq \frac{1}{4m^{1-\nu}} \max_{x\in[m,M]} D\left(x^{2\nu-1}\right) A, \tag{1.10}$$

where the function $D:(0,\infty)\to[0,\infty)$ *is defined by* $D(x) = (x-1)\ln x$.

The following bounds for the Heinz mean $H_{\nu}(A,B)$ in terms of $A\nabla B$ are also valid:

Theorem 3 (Dragomir, 2015 [7])**.** *With the assumptions of Theorem 4 we have*

$$(0 \leq)\, A\nabla B - H_{\nu}(A,B) \leq \nu(1-\nu)\,\Upsilon(m,M)\, A, \tag{1.11}$$

where

$$\Upsilon(m,M) := \begin{cases} (m-1)\ln m \text{ if } M<1, \\ \max\{(m-1)\ln m,(M-1)\ln M\} \text{ if } m\le 1\le M, \\ (M-1)\ln M \text{ if } 1<m \end{cases} \tag{1.12}$$

and

$$A\nabla B\exp\left[-4\nu(1-\nu)(\digamma(m,M)-1)\right] \le H_\nu(A,B)\ (\le A\nabla B) \tag{1.13}$$

where

$$\digamma(m,M) := \begin{cases} K(m) \text{ if } M<1, \\ \max\{K(m),K(M)\} \text{ if } m\le 1\le M, \\ K(M) \text{ if } 1<m. \end{cases} \tag{1.14}$$

For other recent results on geometric operator mean inequalities, see [2]-[13], [15] and [17]-[18].

Motivated by the above results, we establish in this paper some inequalities for the quantities

$$H_\nu(A,B)-A\sharp B \text{ and } A\nabla B-H_\nu(A,B)$$

under various assumptions for positive invertible operators A, B.

2. Bounds for $H_\nu(A,B)-A\sharp B$

First we notice the following simple result:

Theorem 4. *Assume that A, B are positive invertible operators and the constants $M>m>0$ are such that the condition (1.3) holds. If we consider the function $f_\nu:[0,\infty)\to\mathbb{R}$ for $\nu\in[0,1]$ defined by*

$$f_\nu(x)=\frac{1}{2}\left(x^\nu+x^{1-\nu}\right),$$

then we have

$$f_\nu(m)A\le H_\nu(A,B)\le f_\nu(M)A. \tag{2.1}$$

Proof. We observe that

$$f_\nu'(x)=\frac{1}{2}\left(\nu x^{\nu-1}+(1-\nu)x^{-\nu}\right),$$

which is positive for $x\in(0,\infty)$.

Therefore f_ν is increasing on $(0,\infty)$ and

$$f_\nu(m)=\min_{x\in[m,M]}f_\nu(x)\le f_\nu(x)\le\max_{x\in[m,M]}f_\nu(x)=f_\nu(M)$$

for any $x\in[m,M]$.

Using the continuous functional calculus, we have for any operator X with $mI\le X\le MI$ that

$$f_\nu(m)I\le\frac{1}{2}\left(X^\nu+X^{1-\nu}\right)\le f_\nu(M)I. \tag{2.2}$$

From (1.3) we have, by multiplying both sides with $A^{-1/2}$ that

$$mI \leq A^{-1/2}BA^{-1/2} \leq MI.$$

Now, writing the inequality (2.2) for $X = A^{-1/2}BA^{-1/2}$, we get

$$f_\nu(m) I \leq \frac{1}{2}\left[\left(A^{-1/2}BA^{-1/2}\right)^{\nu} + \left(A^{-1/2}BA^{-1/2}\right)^{1-\nu}\right] \leq f_\nu(M) I. \tag{2.3}$$

Finally, if we multiply both sides of (2.3) by $A^{1/2}$ we get the desired result (2.1). □

Corollary 1. *Let A, B be two positive operators. For positive real numbers m, m', M, M', put $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ and let $\nu \in [0,1]$.*

(i) If $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$, then

$$f_\nu(h') A \leq H_\nu(A,B) \leq f_\nu(h) A. \tag{2.4}$$

(ii) If $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$, then

$$\frac{f_\nu(h)}{h} A \leq H_\nu(A,B) \leq \frac{f_\nu(h')}{h'} A. \tag{2.5}$$

Proof. If the condition (i) is valid, then we have for $X = A^{-1/2}BA^{-1/2}$

$$I < \frac{M'}{m'} I = h'I \leq X \leq hI = \frac{M}{m} I,$$

which, by (2.2) gives the desired result (2.4).

If the condition (ii) is valid, then we have

$$0 < \frac{1}{h} I \leq X \leq \frac{1}{h'} I < I,$$

which, by (2.2) gives

$$f_\nu\left(\frac{1}{h}\right) A \leq H_\nu(A,B) \leq f_\nu\left(\frac{1}{h'}\right) A$$

that is equivalent to (2.5), since

$$f_\nu\left(\frac{1}{h}\right) = \frac{f_\nu(h)}{h}.$$

□

We need the following lemma:

Lemma 1. *Consider the function $g_\nu : [0,\infty) \to \mathbb{R}$ for $\nu \in (0,1)$ defined by*

$$g_\nu(x) = \frac{1}{2}\left(x^\nu + x^{1-\nu}\right) - \sqrt{x} \geq 0. \tag{2.6}$$

Then $g_\nu(0) = g_\nu(1) = 0$, g_ν is increasing on $(0, x_\nu)$ with a local maximum in

$$x_\nu := \left(\frac{\nu}{1-\nu}\right)^{\frac{2}{1-2\nu}} \in (0,1), \tag{2.7}$$

is decreasing on $(x_\nu, 1)$ with a local minimum in $x = 1$ and increasing on $(1,\infty)$ with $\lim_{x\to\infty} g_\nu(x) = \infty$.

Proof. (i). If $\nu \in \left(0, \frac{1}{2}\right)$, then

$$g_{\nu}'(x) = \frac{1}{2}\left(\frac{\nu}{x^{1-\nu}} + \frac{1-\nu}{x^{\nu}} - \frac{1}{x^{1/2}}\right) = \frac{1}{2}\frac{\nu + (1-\nu)\, x^{1-2\nu} - x^{\frac{1-2\nu}{2}}}{x^{1-\nu}}.$$

If we denote $u = x^{\frac{1-2\nu}{2}}$, then we have

$$\nu + (1-\nu)\, x^{1-2\nu} - x^{\frac{1-2\nu}{2}} = (1-\nu)\, u^2 - u + \nu. = (1-\nu)\left(u - \frac{\nu}{1-\nu}\right)(u-1) = (1-\nu)\left(x^{\frac{1-2\nu}{2}} - \frac{\nu}{1-\nu}\right)\left(x^{\frac{1-2\nu}{2}} - 1\right).$$

We observe that $g_{\nu}'(x) = 0$ only for $x = 1$ and $x_{\nu} = \left(\frac{\nu}{1-\nu}\right)^{\frac{2}{1-2\nu}} \in (0,1)$. Also $g_{\nu}'(x) > 0$ for $x \in (0, x_{\nu}) \cup (1, \infty)$ and $g_{\nu}'(x) < 0$ for $x \in (x_{\nu}, 1)$. These imply the desired conclusion.

(ii) If $\nu \in \left(\frac{1}{2}, 1\right)$, then

$$g_{\nu}'(x) = \frac{1}{2}\frac{1 - \nu + \nu x^{2\nu-1} - x^{\frac{2\nu-1}{2}}}{x^{\nu}}.$$

If we denote $z = x^{\frac{2\nu-1}{2}}$, then we have

$$1 - \nu + \nu x^{2\nu-1} - x^{\frac{2\nu-1}{2}} = \nu z^2 - z + 1 - \nu = \nu\left(z - \frac{1-\nu}{\nu}\right)(z-1) = \nu\left(x^{\frac{2\nu-1}{2}} - \frac{1-\nu}{\nu}\right)\left(x^{\frac{2\nu-1}{2}} - 1\right).$$

We observe that $g_{\nu}'(x) = 0$ only for $x = 1$ and $x_{\nu} = \left(\frac{1-\nu}{\nu}\right)^{\frac{2}{2\nu-1}} = \left(\frac{\nu}{1-\nu}\right)^{\frac{2}{1-2\nu}} \in (0,1)$. Also $g_{\nu}'(x) > 0$ for $x \in (0, x_{\nu}) \cup (1, \infty)$ and $g_{\nu}'(x) < 0$ for $x \in (x_{\nu}, 1)$. These imply the desired conclusion. □

The above lemma allows us to obtain various bounds for the nonnegative quantity

$$H_{\nu}(A, B) - A \sharp B$$

when some conditions for the involved operators A and B are known.

Theorem 5. *Assume that A, B are positive invertible operators with $B \leq A$. Then for $\nu \in (0,1)$ we have*

$$(0 \leq)\, H_{\nu}(A, B) - A \sharp B \leq g_{\nu}(x_{\nu})\, A, \tag{2.8}$$

where g_{ν} is defined by (2.6) and x_{ν} by (2.7).

Proof. From Lemma 1 we have for $\nu \in (0,1)$ that

$$0 \leq \frac{1}{2}\left(x^{\nu} + x^{1-\nu}\right) - \sqrt{x} \leq g_{\nu}(x_{\nu})$$

for any $x \in [0,1]$.

Using the continuous functional calculus, we have for any operator X with $0 \leq X \leq I$ that

$$0 \leq \frac{1}{2}\left(X^{\nu}+X^{1-\nu}\right)-X^{1/2} \leq g_{\nu}\left(x_{\nu}\right) \tag{2.9}$$

for $\nu \in (0,1)$.

By multiplying both sides of the inequality $0 \leq B \leq A$ with $A^{-1/2}$ we get

$$0 \leq A^{-1/2}BA^{-1/2} \leq I.$$

If we use the inequality (2.9) for $X = A^{-1/2}BA^{-1/2}$, then we get

$$\begin{aligned} 0 &\leq \frac{1}{2}\left[\left(A^{-1/2}BA^{-1/2}\right)^{\nu}+\left(A^{-1/2}BA^{-1/2}\right)^{1-\nu}\right]-\left(A^{-1/2}BA^{-1/2}\right)^{1/2} \\ &\leq g_{\nu}\left(x_{\nu}\right) I \end{aligned} \tag{2.10}$$

for $\nu \in (0,1)$.

Finally, if we multiply both sides of (2.10) with $A^{1/2}$, then we get the desired result (2.8). □

Theorem 6. *Assume that A, B are positive invertible operators and the constants $M > m \geq 0$ are such that the condition (1.3) holds. Let $\nu \in (0,1)$.*

(i) If $0 \leq m < M \leq 1$, then

$$\gamma_{\nu}(m,M) A \leq H_{\nu}(A,B) - A\sharp B \leq \Gamma_{\nu}(m,M) A, \tag{2.11}$$

where

$$\gamma_{\nu}(m,M) := \begin{cases} g_{\nu}(m) & \text{if } 0 \leq m < M \leq x_{\nu}, \\ \min\{g_{\nu}(m), g_{\nu}(M)\} & \text{if } 0 \leq m \leq x_{\nu} \leq M \leq 1, \\ g_{\nu}(M) & \text{if } x_{\nu} \leq m < M \end{cases} \tag{2.12}$$

and

$$\Gamma_{\nu}(m,M) := \begin{cases} g_{\nu}(M) & \text{if } 0 \leq m < M \leq x_{\nu}, \\ g_{\nu}(x_{\nu}) & \text{if } 0 \leq m \leq x_{\nu} \leq M \leq 1, \\ g_{\nu}(m) & \text{if } x_{\nu} \leq m \leq M \leq 1, \end{cases} \tag{2.13}$$

where g_{ν} is defined by (2.6) and x_{ν} by (2.7).

(ii) If $1 \leq m < M < \infty$, then

$$g_{\nu}(m) A \leq H_{\nu}(A,B) - A\sharp B \leq g_{\nu}(M) A. \tag{2.14}$$

Proof. (i) If $0 \leq m < M \leq 1$ then by Lemma 1 we have for $\nu \in (0,1)$ that

$$\begin{aligned} &\begin{cases} g_\nu(m) \text{ if } 0 \leq m < M \leq x_\nu \\ \\ \min\{g_\nu(m), g_\nu(M)\} \text{ if } 0 \leq m \leq x_\nu \leq M \leq 1 \\ \\ g_\nu(M) \text{ if } x_\nu \leq m < M \end{cases} \\ &\leq g_\nu(x) \\ &\leq \begin{cases} g_\nu(M) \text{ if } 0 \leq m < M \leq x_\nu \\ \\ g_\nu(x_\nu) \text{ if } 0 \leq m \leq x_\nu \leq M \leq 1 \\ \\ g_\nu(m) \text{ if } x_\nu \leq m < M \leq 1 \end{cases} \end{aligned}$$

for any $x \in [m, M]$.

Now, on making use of a similar argument to the one in the proof of Theorem 5, we obtain the desired result (2.13).

(ii) Obvious by the properties of function g_ν. □

The interested reader may obtain similar bounds for other locations of $0 \leq m < M < \infty$. The details are omitted.

The following particular case holds:

Corollary 2. *Let A, B be two positive operators. For positive real numbers m, m', M, M', put $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ and let $\nu \in (0,1)$.*

(i) If $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$, then

$$g_\nu(h') A \leq H_\nu(A, B) - A\sharp B \leq g_\nu(h) A. \tag{2.15}$$

(ii) If $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$, then

$$\tilde{\gamma}_\nu(h, h') A \leq H_\nu(A, B) - A\sharp B \leq \tilde{\Gamma}_\nu(h, h') A, \tag{2.16}$$

where

$$\tilde{\gamma}_\nu(h, h') := \begin{cases} \frac{g_\nu(h)}{h} \textit{ if } 0 \leq \frac{1}{h} < \frac{1}{h'} \leq x_\nu, \\ \\ \min\left\{\frac{g_\nu(h)}{h}, \frac{g_\nu(h')}{h'}\right\} \textit{ if } 0 \leq \frac{1}{h} \leq x_\nu \leq \frac{1}{h'} \leq 1, \\ \\ \frac{g_\nu(h')}{h'} \textit{ if } x_\nu \leq \frac{1}{h} < \frac{1}{h'} \end{cases} \tag{2.17}$$

and

$$\tilde{\Gamma}_\nu(h, h') := \begin{cases} \frac{g_\nu(h')}{h'} \textit{ if } 0 \leq \frac{1}{h} < \frac{1}{h'} \leq x_\nu, \\ \\ g_\nu(x_\nu) \textit{ if } 0 \leq \frac{1}{h} \leq x_\nu \leq \frac{1}{h'} \leq 1, \\ \\ \frac{g_\nu(h)}{h} \textit{ if } x_\nu \leq \frac{1}{h} < \frac{1}{h'} \leq 1. \end{cases} \tag{2.18}$$

3. Bounds for $A\nabla B - H_{\nu}(A,B)$

In order to provide some upper and lower bounds for the quantity

$$A\nabla B - H_{\nu}(A,B)$$

where A, B are positive invertible operators, we need the following lemma.

Lemma 2. *Consider the function $h_{\nu}:[0,\infty)\to\mathbb{R}$ for $\nu\in(0,1)$ defined by*

$$h_{\nu}(x)=\frac{x+1}{2}-\frac{1}{2}\left(x^{\nu}+x^{1-\nu}\right)\geq 0. \tag{3.1}$$

Then h_{ν} is decreasing on $[0,1)$ and increasing on $(1,\infty)$ with $x=1$ its global minimum. We have $h_{\nu}(0)=\frac{1}{2}$, $\lim_{x\to\infty}h_{\nu}(x)=\infty$ and h_{ν} is convex on $(0,\infty)$.

Proof. We have

$$h_{\nu}'(x)=\frac{1}{2}\left(1-\frac{\nu}{x^{1-\nu}}-\frac{1-\nu}{x^{\nu}}\right)$$

and

$$h_{\nu}''(x)=\frac{1}{2}\nu(1-\nu)\left(x^{\nu-2}+x^{-\nu-1}\right)$$

for any $x\in(0,\infty)$ and $\nu\in(0,1)$.

We observe that $h_{\nu}'(1)=0$ and $h_{\nu}''(x)>0$ for any $x\in(0,\infty)$ and $\nu\in(0,1)$. These imply that the equation $h_{\nu}'(x)=0$ has only one solution on $(0,\infty)$, namely $x=1$. Since $h_{\nu}'(x)<0$ for $x\in(0,1)$ and $h_{\nu}'(x)>0$ for $x\in(1,\infty)$, then we deduce the desired conclusion. □

Theorem 7. *Assume that A, B are positive invertible operators, the constants $M>m\geq 0$ are such that the condition (1.3) holds and $\nu\in(0,1)$. Then we have*

$$\delta_{\nu}(m,M)A\leq A\nabla B-H_{\nu}(A,B)\leq\Delta_{\nu}(m,M)A, \tag{3.2}$$

where

$$\delta_{\nu}(m,M):=\begin{cases} h_{\nu}(M) & \text{if } M<1,\\ 0 & \text{if } m\leq 1\leq M,\\ h_{\nu}(m) & \text{if } 1<m\end{cases} \tag{3.3}$$

and

$$\Delta_{\nu}(m,M):=\begin{cases} h_{\nu}(m) & \text{if } M<1,\\ \max\{h_{\nu}(m),h_{\nu}(M)\} & \text{if } m\leq 1\leq M,\\ h_{\nu}(M) & \text{if } 1<m,\end{cases} \tag{3.4}$$

where h_{ν} is defined by (3.1).

Proof. Using Lemma 2 we have

$$\begin{cases} h_\nu(M) \text{ if } M<1, \\ 0 \text{ if } m\le 1\le M, \\ h_\nu(m) \text{ if } 1<m, \end{cases} \le h_\nu(x)$$

$$\le \begin{cases} h_\nu(m) \text{ if } M<1, \\ \max\{h_\nu(m), h_\nu(M)\} \text{ if } m\le 1\le M, \\ h_\nu(M) \text{ if } 1<m \end{cases}$$

for any $x\in[m,M]$ and $\nu\in(0,1)$.

Using the continuous functional calculus, we have for any operator X with $mI\le X\le MI$ that

$$\delta_\nu(m,M)I \le \frac{X+I}{2}-\frac{1}{2}\left(X^\nu+X^{1-\nu}\right)\le \Delta_\nu(m,M)I. \tag{3.5}$$

From (1.3) we have, by multiplying both sides with $A^{-1/2}$ that

$$mI\le A^{-1/2}BA^{-1/2}\le MI.$$

Now, writing the inequality (3.5) for $X=A^{-1/2}BA^{-1/2}$, we get

$$\begin{aligned} &\delta_\nu(m,M)I \\ &\le \frac{A^{-1/2}BA^{-1/2}+I}{2}-\frac{1}{2}\left(\left(A^{-1/2}BA^{-1/2}\right)^\nu+\left(A^{-1/2}BA^{-1/2}\right)^{1-\nu}\right) \\ &\le \Delta_\nu(m,M)I. \end{aligned} \tag{3.6}$$

Finally, if we multiply both sides of (3.6) by $A^{1/2}$ we get the desired result (3.2). □

Corollary 3. *Let A, B be two positive operators. For positive real numbers m, m', M, M', put $h:=\frac{M}{m}$, $h':=\frac{M'}{m'}$ and let $\nu\in(0,1)$.*

(i) If $0<mI\le A\le m'I<M'I\le B\le MI$, then

$$h_\nu(h')A\le A\nabla B-H_\nu(A,B)\le h_\nu(h)A. \tag{3.7}$$

(ii) If $0<mI\le B\le m'I<M'I\le A\le MI$, then

$$\frac{h_\nu(h')}{h'}A\le A\nabla B-H_\nu(A,B)\le\frac{h_\nu(h)}{h}A. \tag{3.8}$$

References

[1] S. S. Dragomir, Some new reverses of Young's operator inequality, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 130. [http://rgmia.org/papers/v18/v18a130.pdf].

[2] S. S. Dragomir, On new refinements and reverses of Young's operator inequality, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 135. [http://rgmia.org/papers/v18/v18a135.pdf].

[3] S. S. Dragomir, Some inequalities for operator weighted geometric mean, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 139. [http://rgmia.org/papers/v18/v18a139.pdf].

[4] S. S. Dragomir, Refinements and reverses of Hölder-McCarthy operator inequality, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 143. [http://rgmia.org/papers/v18/v18a143.pdf].

[5] S. S. Dragomir, Some reverses and a refinement of Hölder operator inequality, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 147. [http://rgmia.org/papers/v18/v18a147.pdf].

[6] S. S. Dragomir, Some inequalities for Heinz operator mean, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 163. [Online http://rgmia.org/papers/v18/v18a163.pdf].

[7] S. S. Dragomir, Further inequalities for Heinz operator mean, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 167. [Online http://rgmia.org/papers/v18/v18a167.pdf].
[8] S. Furuichi, On refined Young inequalities and reverse inequalities, *J. Math. Inequal.* **5** (2011), 21-31.
[9] S. Furuichi, Refined Young inequalities with Specht's ratio, *J. Egyptian Math. Soc.* **20** (2012), 46–49.
[10] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.* **361** (2010), 262-269.
[11] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Lin. Multilin. Alg.,* **59** (2011), 1031-1037.
[12] F. Kittaneh, M. Krnić, N. Lovričević and J. Pečarić, Improved arithmetic-geometric and Heinz means inequalities for Hilbert space operators, *Publ. Math. Debrecen*, 2012, **80**(3-4), 465–478.
[13] M. Krnić and J. Pečarić, Improved Heinz inequalities via the Jensen functional, *Cent. Eur. J. Math.* **11** (9) 2013,1698-1710.
[14] F. Kubo and T. Ando, Means of positive operators, *Math. Ann.* **264** (1980), 205–224.
[15] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.
[16] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.* **74** (1960), pp. 91-98.
[17] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.H.
[18] G. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.,* **5** (2011), 551-556.

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