Received 06/01/16

FEJÉR TYPE INEQUALITIES FOR HARMONICALLY (s,m)-CONVEX FUNCTIONS

Imran Abbas Baloch, İmdat İşcan and Silvestru Sever Dragomir

Imran Abbas Baloch
Abdus Salam School of Mathematical Sciences
GC University, Lahore, Pakistan.
iabbasbaloch@gmail.com

İmdat İşcan
Department of Mathematics, Faculty of Arts and Sciences
Giresun University, 28200, Giresun, Turkey.
imdat.iscan@giresun.edu.tr

Silvestru Sever Dragomir
Mathematics, College of Engineering and Science
Victoria University
Melbourne City, Australia.
sever.dragomir@vu.edu.au

[2010]Primary 26D15, Secondary 26A51, 26E60, 41A55.

Hermite-Hadamard's inequality, Fejér's inequality, convex function, harmonically (s,m)-convex function, Hölder's inequality, power mean inequality.

ABSTRACT

In this paper, a new weighted identity involving harmonically symmetric functions and differentiable functions is established. By using the notion of harmonic symmetricity, harmonic (s,m)-convexity, analysis and some auxiliary results, some new Fejér type integral inequalities are presented for the class of harmonically (s,m)-convex functions.

1. INTRODUCTION

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called convex function if $f(\lambda x+(1-\lambda)y) \le \lambda f(x)+(1-\lambda)f(y)$ for all $x,y \in I$ and $\lambda \in [0,1]$. There are many results associated with convex functions in the area of inequalities, but one of them is the classical Hermite-Hadamard (see [21]) inequalities:

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}, \tag{1.1}$$

for all $a,b \in I$, with $a<b$. The inequalities in (1.1) hold in reversed direction if f is a concave function. A vast literature have been produced by a number of mathematicians for convex functions but (1.1) is considered to be the most famous inequality for convex mappings due to its usefulness and many applications in various branches of pure and applied mathematics. The definition of classical or usual convex functions has been generalized in a variety of ways and as a consequence many researchers have established a number of Hermite-Hadamard type inequalities by using different generalizations of the classical convexity, see for instance [2]-[23] and the references mentioned in these papers.

One of the generalizations of classical convexity is the harmonic (s,m)-convexity in second sense, which unifies the notion of Harmonically convex and Harmonically s-convex functions in second sense introduced by Imdat Iscan in [12],[13] as stated in the definition below.

Definition 1. [1] The function $f : I \subset (0,\infty) \to \mathbb{R}$ is said to be harmonically (s,m)-convex in second sense, where $s \in (0,1]$ and $m \in (0,1]$ if

$$f\Big(\frac{mxy}{mty+(1-t)x}\Big) = f\Big(\Big(\frac{t}{x}+\frac{1-t}{my}\Big)^{-1}\Big) \le t^s f(x)+m(1-t)^s f(y)$$

$\forall x,y \in I$ and $t \in [0,1]$.

RGMIA Res. Rep. Coll. 19 (2016), Art. 3

Remark 1. *Note that for $s=1$, (s,m)-convexity reduces to harmonically m-convexity and for $m=1$, harmonically (s,m)-convexity reduces to harmonically s-convexity in second sense (see [13]) and for $s,m=1$, harmonically (s,m)-convexity reduces to ordinary harmonically convexity (see [12]).*

Proposition 1. *Let $f:(0,\infty)\to\mathbb{R}$ be a function*
a) if f is (s,m)-convex function in second sense and non-decreasing, then$f$ is harmonically (s,m)-convex function in second sense.
b) if f is harmonically (s,m)-convex function in second sense and non-increasing, then f is (s,m)-convex function in second sense.

Remark 2. *According to proposition 1, every non-decreasing (s,m)-convex function in second sense is also harmonically (s,m)-convex function in second sense.*

Example 1. (see[3]) Let $0<s<1$ and $a,b,c\in\mathbb{R}$, then function $f:(0,\infty)\to\mathbb{R}$ defined by

$$f(x)=\begin{cases} a, & x=0 \\ bx^s+c, & x>0 \end{cases}$$

is non-decreasing s-convex function in second sense for $b\geq 0$ and $0\leq c\leq a$. Hence, by proposition 1, f is harmonically $(s,1)$-convex function.

Proposition 2. *Let $s\in[0,1]$, $m\in(0,1]$, $f:[a,mb]\subset(0,\infty)\to\mathbb{R}$, be an increasing function and $g:[a,mb]\to[a,mb]$, $g(x)=\frac{mab}{a+mb-x}$, $a<mb$. Then f is harmonically (s,m)-convex in second sense on $[a,mb]$ if and only if fog is (s,m)-convex in second sense on $[a,mb]$.*

The following result of the Hermite-Hadamard type holds.

Theorem 1. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a harmonically (s,m)-convex function in second sense with $s\in[0,1]$ and $m\in(0,1]$. If $0<a<b<\infty$ and $f\in L[a,b]$, then one has following inequality*

$$\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\leq\min\Big[\frac{f(a)+mf(\frac{b}{m})}{s+1},\frac{f(b)+mf(\frac{a}{m})}{s+1}\Big]$$

Corollary 1. *If we take $m=1$ in Theorem 1, then we get*

$$\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\leq\frac{f(a)+f(b)}{s+1}$$

Corollary 2. *If we take $s=1$ in Theorem 1, then we get*

$$\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\leq\min\Big[\frac{f(a)+mf(\frac{b}{m})}{2},\frac{f(b)+mf(\frac{a}{m})}{2}\Big]$$

Chen and Wu [4], established the following weighted Fejér type inequality for the harmonically convex function as follow

Theorem 2. [4] *Let $f:I\subset\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a harmonically convex function and $a,b\in I$ with $a<b$. If $f\in L([a,b])$, then one has*

$$f\Big(\frac{2ab}{a+b}\Big)\int_a^b\frac{g(x)}{x^2}dx\leq\int_a^b\frac{g(x)f(x)}{x^2}dx\leq\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x^2}dx, \tag{1.2}$$

where $g:[a,b]\to\mathbb{R}$ is non-negative, integrable and satisfies

$$g\Big(\frac{ab}{x}\Big)=g\Big(\frac{ab}{a+b-x}\Big)$$

The main purpose of the present paper is to introduce a new notion of harmonically symmetric functions and to establish an identity involving a harmonically symmetric function and a differentiable function. We will prove some Fejér type inequalities by using this identity related with the second part of the inequality given above by (1.2).We believe that our findings are novel, new and better than those already exist and will open new ways for further research in this field.

2. Main Results

Throughout this section, we take $L(t)=\frac{2ab}{(1-t)a+(1+t)b}$ and $U(t)=\frac{2ab}{(1+t)a+(1-t)b}$. The Beta function, the Gamma function and the integral form of the hypergeometric function are defined as follows to be used in the sequel of paper

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt,\ \alpha,\beta>0$$

$$\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}dt,\ \alpha>0$$

and

$${}_2F_1(\alpha,\beta;\gamma,z)=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt,\ \gamma>\beta>0,\ |z|<1$$

The notion of harmonically symmetric functions is defined as follows:

Definition 2. We say that a function $g:[a,b]\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ is harmonically symmetric with respect to $\frac{2ab}{a+b}$ if

$$g(x)=g\Big(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}\Big)$$

holds for all $x\in[a,b]$.

Now, we give the weighted integral inequality by using which we establish our results in this article.

Lemma 1. *Let $f:I\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a differentiable function on I° and $a,b\in I^\circ$ with $a<b$ and let $g:[a,b]\to[0,\infty)$ be continuous positive mapping and harmonically symmetric to $\frac{2ab}{a+b}$. If $f'\in L([a,b])$, then the following identity holds*

$$\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{g(x)f(x)}{x^2}dx$$
$$=\frac{b-a}{4ab}\int_0^1\left(\int_{L(t)}^{U(t)}\frac{g(x)}{x^2}dx\right)\Big[(U(t))^2f'(U(t))-(L(t))^2f'(L(t))\Big]dt$$

Proof. Since, $g:[a,b]\to[0,\infty)$ is harmonically symmetric to $\frac{2ab}{a+b}$, then $g(U(t))=g(L(t))$. Consider

$$\begin{aligned}
I &= \frac{b-a}{4ab}\int_0^1\left(\int_{L(t)}^{U(t)}\frac{g(x)}{x^2}dx\right)\Big[(U(t))^2f'(U(t))-(L(t))^2f'(L(t))\Big]dt\\
&= \frac{1}{2}\left[\int_0^1\left(\int_{L(t)}^{U(t)}\frac{g(x)}{x^2}dx\right)d[f(U(t))+f(L(t))]\right]\\
&= \frac{1}{2}\Bigg[\left(\int_{L(t)}^{U(t)}\frac{g(x)}{x^2}dx\right)(f(U(t))+f(L(t)))\Big|_0^1\\
&- \frac{b-a}{2ab}\int_0^1(g(U(t))+g(L(t)))(f(U(t))+f(L(t)))dt\Bigg]\\
&= \frac{1}{2}\Bigg[(f(a)+f(b))\left(\int_a^b\frac{g(x)}{x^2}dx\right)-\frac{b-a}{ab}\int_0^1 g(U(t)f(U(t)dt\\
&- \frac{b-a}{ab}\int_0^1 g(L(t)f(L(t)dt\Bigg]\\
&= \frac{1}{2}\left[(f(a)+f(b))\int_a^b\frac{g(x)}{x^2}dx-2\int_a^{\frac{2ab}{a+b}}\frac{g(x)f(x)}{x^2}dx+2\int_{\frac{2ab}{a+b}}^a\frac{g(x)f(x)}{x^2}dx\right]\\
&= \frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{g(x)f(x)}{x^2}dx
\end{aligned}$$

■

Now, we present new Fejér type inequalities for harmonically (s,m)-convex functions, which give the weighted generalization of some of the results established in resent literature.

Theorem 3. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° and $a, \frac{b}{m} \in I^\circ$, $m \in (0,1]$ with $a < b$ and let $g : [a,b] \to [0,\infty)$ be continuous positive mapping and harmonically symmetric to $\frac{2ab}{a+b}$ such that $f' \in L([a,b])$. If $|f'|^q$ is harmonically (s,m)-convex on $[a, \frac{b}{m}]$ for $q \geq 1$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{g(x)f(x)}{x^2}dx\right|$$

$$\begin{aligned}
\leq\;& \frac{b-a}{8ab}a^{\frac{2}{q}}\|g\|_\infty \Big\{\lambda_1^{1-\frac{1}{q}}(a,b)\Big(\big\{2^2B(s+1,2).{}_2F_1(2,s+1,s+3;\frac{b-a}{b})\\
-\;& 2^{1-s}B(s+1,1).{}_2F_1(2,s+1,s+2;\frac{b-a}{2b}) + \frac{1}{2^s}B(s+2,1).{}_2F_1(2,s+2,s+3;\frac{b-a}{2b})\big\}|f'(b)|^q\\
+\;& \frac{m2^{2-s}b^2}{(b+a)^2}B(1,s+2).{}_2F_1(2,1,s+3;\frac{b-a}{b+a})|f'(\frac{a}{m})|^q\Big)^{\frac{1}{q}} + \lambda_2^{1-\frac{1}{q}}(a,b)\\
&\times\Big(\big\{2^2B(2,s+1).{}_2F_1(2,2,s+3;\frac{b-a}{b}) - \frac{2^{2-s}b^2}{(b+a)^2}B(1,s+1).{}_2F_1(2,1,s+2;\frac{b-a}{b+a})\\
-\;& \frac{2^{2-s}b^2}{(b+a)^2}B(2,s+1).{}_2F_1(2,2,s+3;\frac{b-a}{b+a})\big\}|f'(a)|^q
\end{aligned}$$

$$+\frac{m}{2^s}B(s+2,1).{}_2F_1(2,s+2,s+3;\frac{b-a}{2b})|f'(\frac{b}{m})|^q\Big)^{\frac{1}{q}}\Big\} \tag{2.1}$$

Proof. From Lemma 1 and hölder's inequality, we get

$$\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{g(x)f(x)}{x^2}dx\right| \leq \frac{b-a}{8ab}\|g\|_\infty$$

$$\times\Big\{\Big(\int_0^1 (1-t)(U(t))^2dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 (1-t)(U(t))^2|f'(U(t))|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 (1-t)(L(t))^2dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 (1-t)(L(t))^2|f'(L(t))|^q dt\Big)^{\frac{1}{q}}\Big\} \tag{2.2}$$

By the harmonic (s,m)-convexity of $|f'|^q$ on $[a,b]$ for $q \geq 1$, we have

$$\begin{aligned}
\int_0^1 (1-t)(U(t))^2|f'(U(t))|^q dt &= \int_0^1 (1-t)\Big(\frac{2ab}{(1+t)a+(1-t)b}\Big)^2\\
\times \left|f'\Big(\frac{2ab}{(1+t)a+(1-t)b}\Big)\right|^q dt &\leq \frac{1}{2^s}|f'(b)|^q\int_0^1 (1-t)(1+t)^s\Big(\frac{2ab}{(1+t)a+(1-t)b}\Big)^2 dt\\
&+ m\frac{1}{2^s}|f'(\frac{a}{m})|^q\int_0^1 (1-t)^{s+1}\Big(\frac{2ab}{(1+t)a+(1-t)b}\Big)^2 dt
\end{aligned}$$

$$\begin{aligned}
&= \big\{2^2a^2B(s+1,2).{}_2F_1(2,s+1,s+3;\frac{b-a}{b}) - \frac{a^2}{2^{s-1}}B(s+1,1).{}_2F_1(2,s+1,s+2;\frac{b-a}{2b})\\
&+\frac{a^2}{2^s}B(s+2,1).{}_2F_1(2,s+2,s+3;\frac{b-a}{2b})\big\}|f'(b)|^q + \frac{ma^2b^2}{2^{s-2}(b+a)^2}B(1,s+2).{}_2F_1(2,1,s+3;\frac{b-a}{b+a})|f'(\frac{a}{m})|^q
\end{aligned} \tag{2.3}$$

and

$$\int_0^1 (1-t)(L(t))^2|f'(L(t))|^q dt = \int_0^1 (1-t)\Big(\frac{2ab}{(1-t)a+(1+t)b}\Big)^2$$

$$\times\ \left|f'\big(\frac{2ab}{(1-t)a+(1+t)b}\big)\right|^q dt \le \frac{1}{2^s}|f'(a)|^q\int_0^1(1-t)(1+t)^s\big(\frac{2ab}{(1-t)a+(1+t)b}\big)^2dt$$
$$+\ m\frac{1}{2^s}|f'(\frac{b}{m})|^q\int_0^1(1-t)^{s+1}\big(\frac{2ab}{(1-t)a+(1+t)b}\big)^2dt$$

$$(2.4)\quad =\big\{2^2a^2B(2,s+1).{}_2F_1(2,2,s+3;\frac{b-a}{b})-\frac{2^{2-s}a^2b^2}{(b+a)^2}B(1,s+1).{}_2F_1(2,1,s+2;\frac{b-a}{b+a})$$
$$-\frac{2^{2-s}a^2b^2}{(b+a)^2}B(2,s+1).{}_2F_1(2,2,s+3;\frac{b-a}{b+a})\big\}|f'(a)|^q+\frac{ma^2}{2^s}B(s+2,1).{}_2F_1(2,s+2,s+3;\frac{b-a}{2b})|f'(\frac{b}{m})|^q$$

Moreover,

$$(2.5)\quad \int_0^1(1-t)(U(t))^2dt=\int_0^1(1-t)\big(\frac{2ab}{(1+t)a+(1-t)b}\big)^2dt$$
$$=(\frac{2ab}{b-a})^2\ln(\frac{a+b}{2a})-\frac{(2ab)^2}{b^2-a^2}:=\lambda_1(a,b)$$

and

$$(2.6)\quad \int_0^1(1-t)(L(t))^2dt=\int_0^1(1-t)\big(\frac{2ab}{(1-t)a+(1+t)b}\big)^2dt$$
$$=\frac{(2ab)^2}{b^2-a^2}+(\frac{2ab}{b-a})^2\ln(\frac{a+b}{2b}):=\lambda_2(a,b)$$

A combination of (2.2), (2.3), (2.4), (2.5) and (2.6) gives required result. This completes the proof. ∎

Corollary 3. *Suppose the assumptions of the Theorem 3 are satisfied. If $g(x)=\frac{ab}{b-a}$ for all $x\in[a,b]$, then one has the following inequality*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{g(x)f(x)}{x^2}dx\right|$$
$$\le\ \frac{a^{\frac{2}{q}}}{8}\Bigg\{\lambda_1^{1-\frac{1}{q}}(a,b)\Bigg(\big\{2^2B(s+1,2).{}_2F_1(2,s+1,s+3;\frac{b-a}{b})$$
$$-\ 2^{1-s}B(s+1,1).{}_2F_1(2,s+1,s+2;\frac{b-a}{2b})+\frac{1}{2^s}B(s+2,1).{}_2F_1(2,s+2,s+3;\frac{b-a}{2b})\big\}|f'(b)|^q$$
$$+\ \frac{m2^{2-s}b^2}{(b+a)^2}B(1,s+2).{}_2F_1(2,1,s+3;\frac{b-a}{b+a})|f'(\frac{a}{m})|^q\Bigg)^{\frac{1}{q}}+\ \lambda_2^{1-\frac{1}{q}}(a,b)$$
$$\times\Bigg(\big\{2^2B(2,s+1).{}_2F_1(2,2,s+3;\frac{b-a}{b})-\frac{2^{2-s}b^2}{(b+a)^2}B(1,s+1).{}_2F_1(2,1,s+2;\frac{b-a}{b+a})$$
$$-\ \frac{2^{2-s}b^2}{(b+a)^2}B(2,s+1).{}_2F_1(2,2,s+3;\frac{b-a}{b+a})\big\}|f'(a)|^q$$
$$(2.7)\qquad +\frac{m}{2^s}B(s+2,1).{}_2F_1(2,s+2,s+3;\frac{b-a}{2b})|f'(\frac{b}{m})|^q\Bigg)^{\frac{1}{q}}\Bigg\}$$

Theorem 4. *Let $f:I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I° and $a,\frac{b}{m}\in I^\circ$, $m\in(0,1]$ with $a<b$ and let $g:[a,b]\to[0,\infty)$ be continuous positive mapping and harmonically symmetric to $\frac{2ab}{a+b}$ such that $f'\in L([a,b])$. If $|f'|^q$ is harmonically (s,m)-convex on $[a,\frac{b}{m}]$ for $q>1$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{g(x)f(x)}{x^2}dx\right|\le\frac{a(b-a)}{8b}.\|g\|_\infty$$
$$\times\Bigg\{\Bigg(\big\{2B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{b})-2^{1-s}B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})\big\}|f'(b)|^q$$

$$+m2^{2q-s}(\frac{b}{b+a})^{2q}B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})|f'(\frac{a}{m})|^q\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\{2B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b})-2^{2q-s}(\frac{b}{b+a})^{2q}B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})\}|f'(a)|^q$$

$$+\frac{m}{2^s}B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})|f'(\frac{b}{m})|^q\Bigg)^{\frac{1}{q}} \tag{2.8}$$

Proof. From Lemma 1 and hölder's inequality, we get

$$\left|\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{g(x)f(x)}{x^2}dx\right|\leq\frac{b-a}{8ab}\|g\|_\infty\Bigg(\int_0^1(1-t)^{\frac{q}{q-1}}dt\Bigg)^{1-\frac{1}{q}}$$

$$\times\Bigg\{\Bigg(\int_0^1(U(t))^{2q}|f'(U(t))|^qdt\Bigg)^{\frac{1}{q}}+\Bigg(\int_0^1(L(t))^{2q}|f'(L(t))|^qdt\Bigg)^{\frac{1}{q}}\Bigg\} \tag{2.9}$$

By the harmonic (s,m)-convexity of $|f'|^q$ on $[a,b]$ for $q>1$, we have

$$\int_0^1(U(t))^{2q}|f'(U(t))|^qdt=\int_0^1(\frac{2ab}{(1+t)a+(1-t)b})^{2q}$$

$$\times\left|f'(\frac{2ab}{(1+t)a+(1-t)b})\right|^qdt\leq\frac{1}{2^s}|f'(b)|^q\int_0^1(1+t)^s(\frac{2ab}{(1+t)a+(1-t)b})^{2q}dt$$

$$+\ m\frac{1}{2^s}|f'(\frac{a}{m})|^q\int_0^1(1-t)^s(\frac{2ab}{(1+t)a+(1-t)b})^{2q}dt$$

$$=a^{2q}\{2B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{b})-2^{1-s}B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})\}|f'(b)|^q$$
$$+m2^{2q-s}(\frac{ab}{b+a})^{2q}B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})|f'(\frac{a}{m})|^q \tag{2.10}$$

and

$$\int_0^1(L(t))^{2q}|f'(L(t))|^qdt=\int_0^1(\frac{2ab}{(1-t)a+(1+t)b})^{2q}$$

$$\times\left|f'(\frac{2ab}{(1-t)a+(1+t)b})\right|^qdt\leq\frac{1}{2^s}|f'(a)|^q\int_0^1(1+t)^s(\frac{2ab}{(1-t)a+(1+t)b})^{2q}dt$$

$$+\ m\frac{1}{2^s}|f'(\frac{b}{m})|^q\int_0^1(1-t)^s(\frac{2ab}{(1-t)a+(1+t)b})^{2q}dt$$

$$=a^{2q}\{2B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b})-2^{2q-s}(\frac{b}{b+a})^{2q}B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})\}|f'(a)|^q$$
$$+\frac{ma^{2q}}{2^s}B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})|f'(\frac{b}{m})|^q \tag{2.11}$$

By putting (2.10) and (2.11) in (2.9), we get desired result. ∎

Corollary 4. *Suppose the assumptions of the Theorem* 3 *are satisfied. If* $g(x)=\frac{ab}{b-a}$ *for all* $x\in[a,b]$, *then one has the following inequality*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{g(x)f(x)}{x^2}dx\right|\leq\frac{a^2}{8}$$

$$\times\Bigg\{\Bigg(\{2B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{b})-2^{1-s}B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})\}|f'(b)|^q$$

$$+m2^{2q-s}(\frac{b}{b+a})^{2q}B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})|f'(\frac{a}{m})|^q\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\Big\{2B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b})-2^{2q-s}(\frac{b}{b+a})^{2q}B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})\Big\}|f'(a)|^q$$

$$+\frac{m}{2^s}B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})|f'(\frac{b}{m})|^q\Bigg)^{\frac{1}{q}} \tag{2.12}$$

Theorem 5. *Let $f: I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I° and $a,\frac{b}{m}\in I^\circ$, $m\in(0,1]$ with $a<b$ and let $g:[a,b]\to[0,\infty)$ be continuous positive mapping and harmonically symmetric to $\frac{2ab}{a+b}$ such that $f'\in L([a,b])$. If $|f'|^q$ is harmonically (s,m)-convex on $[a,\frac{b}{m}]$ for $q>1$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{g(x)f(x)}{x^2}dx\right|\leq 2^{1-\frac{1}{q}}\frac{a(b-a)}{8b}\|g\|_\infty$$

$$\Bigg(2B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{b})|f'(b)|^q+2B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b})|f'(a)|^q$$

$$+\frac{m|f'(\frac{b}{m})|^q-|f'(b)|^q}{2^s}.B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})$$

$$+2^{2q-s}(\frac{b}{b+a})^{2q}(m|f'(\frac{a}{m})|^q-|f'(a)|^q)B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})\Bigg)^{\frac{1}{q}} \tag{2.13}$$

Proof. From Lemma 1 and hölder's inequality, we get

$$\left|\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{g(x)f(x)}{x^2}dx\right|\leq\frac{b-a}{8ab}\|g\|_\infty\left(\int_0^1(1-t)^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(\int_0^1(U(t))^{2q}|f'(U(t))|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1(L(t))^{2q}|f'(L(t))|^q dt\right)^{\frac{1}{q}}\right\} \tag{2.14}$$

By the power-mean inequality ($a^r+b^r\leq 2^{1-r}(a+b)^r$ for $a>0$, $b>0$ and $r<1$), we have

$$\left(\int_0^1(U(t))^{2q}|f'(U(t))|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1(L(t))^{2q}|f'(L(t))|^q dt\right)^{\frac{1}{q}}$$

$$\leq 2^{1-\frac{1}{q}}\left(\int_0^1(U(t))^{2q}|f'(U(t))|^q dt+\int_0^1(L(t))^{2q}|f'(L(t))|^q dt\right)^{\frac{1}{q}} \tag{2.15}$$

Since, $|f'|^q$ is harmonically (s,m)-convex on $[a,b]$ for $q>1$, we obtain

$$\int_0^1(U(t))^{2q}|f'(U(t))|^q dt+\int_0^1(L(t))^{2q}|f'(L(t))|^q dt$$

$$\leq\frac{1}{2^s}|f'(b)|^q\int_0^1(1+t)^s(\frac{2ab}{(1+t)a+(1-t)b})^{2q}dt$$

$$+m\frac{1}{2^s}|f'(\frac{a}{m})|^q\int_0^1(1-t)^s(\frac{2ab}{(1+t)a+(1-t)b})^{2q}dt$$

$$+\frac{1}{2^s}|f'(a)|^q\int_0^1(1+t)^s(\frac{2ab}{(1-t)a+(1+t)b})^{2q}dt$$

$$+m\frac{1}{2^s}|f'(\frac{b}{m})|^q\int_0^1(1-t)^s(\frac{2ab}{(1-t)a+(1+t)b})^{2q}dt$$

$$= a^{2q}\{2B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{b}) - 2^{1-s}B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})\}|f'(b)|^q$$

$$+m2^{2q-s}(\frac{ab}{b+a})^{2q}B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})|f'(\frac{a}{m})|^q$$

$$+a^{2q}\{2B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b}) - 2^{2q-s}(\frac{b}{b+a})^{2q}B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})\}|f'(a)|^q$$

$$+\frac{ma^{2q}}{2^s}B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})|f'(\frac{b}{m})|^q$$

using (2.15) in (2.14), we get

$$\left(\int_0^1 (U(t))^{2q}|f'(U(t))|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1 (L(t))^{2q}|f'(L(t))|^q dt\right)^{\frac{1}{q}}$$

$$\le 2^{1-\frac{1}{q}}a^2\Bigg(2B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{b})|f'(b)|^q + 2B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b})|f'(a)|^q$$

$$+\frac{m|f'(\frac{b}{m})|^q - |f'(b)|^q}{2^s}.B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})$$

$$+2^{2q-s}(\frac{b}{b+a})^{2q}(m|f'(\frac{a}{m})|^q - |f'(a)|^q)B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})\Bigg)^{\frac{1}{q}} \quad (2.16)$$

Applying (2.17) in (2.14), we obtain the required inequality. ∎

Corollary 5. *Suppose the assumptions of the Theorem* 3 *are satisfied. If* $g(x) = \frac{ab}{b-a}$ *for all* $x \in [a,b]$, *then one has the following inequality*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{g(x)f(x)}{x^2}dx\right| \le 2^{1-\frac{1}{q}}\frac{a^2}{8}$$

$$\Bigg(2B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{b})|f'(b)|^q + 2B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b})|f'(a)|^q$$

$$+\frac{m|f'(\frac{b}{m})|^q - |f'(b)|^q}{2^s}.B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})$$

$$+2^{2q-s}(\frac{b}{b+a})^{2q}(m|f'(\frac{a}{m})|^q - |f'(a)|^q)B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})\Bigg)^{\frac{1}{q}} \quad (2.17)$$

Theorem 6. *Let* $f : I \subseteq (0,\infty) \to \mathbb{R}$ *be a differentiable function on* I° *and* $a, \frac{b}{m} \in I^\circ$, $m \in (0,1]$ *with* $a < b$ *and let* $g : [a,b] \to [0,\infty)$ *be continuous positive mapping and harmonically symmetric to* $\frac{2ab}{a+b}$ *such that* $f' \in L([a,b])$. *If* $|f'|$ *is harmonically* (s,m)*-convex on* $[a,\frac{b}{m}]$, *then the following inequality holds for* $q > 1$

$$\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{g(x)f(x)}{x^2}dx\right| \le \frac{(b-a)}{8ab}\|g\|_\infty(\frac{1}{sq+1})^{\frac{1}{q}}$$

$$\times\Bigg\{2^{2-s}(\frac{ab}{b+a})^2\big((2^{sq+1}-1)|f'(b)| + m|f'(\frac{a}{m})|\big)\left(B(1,\frac{2q-1}{q-1}).{}_2F_1(\frac{2q}{q-1},1,\frac{3q-2}{q-1};\frac{b-a}{b+a})\right)^{\frac{q-1}{q}}$$

$$+\frac{a^2}{2^s}\big((2^{sq+1}-1)|f'(a)| + m|f'(\frac{b}{m})|\big)\left(B(\frac{2q-1}{q-1},1).{}_2F_1(\frac{2q}{q-1},\frac{2q-1}{q-1},\frac{3q-2}{q-1};\frac{b-a}{b})\right)^{\frac{q-1}{q}}\Bigg\} \quad (2.18)$$

Proof. From Lemma 1 and by using the harmonic (s,m)-convexity of $|f'|$ on $[a,b]$, we get

$$\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx-\int_a^b \frac{g(x)f(x)}{x^2}dx\right|\le \frac{b-a}{8ab}\|g\|_\infty$$

$$\times\left[\int_0^1(1-t)(U(t))^2|f'(U(t))|dt+\int_0^1(1-t)(L(t))^2|f'(L(t))|dt\right]$$

$$\le \frac{b-a}{8ab}\|g\|_\infty\left\{\int_0^1(1-t)(U(t))^2\left[(\frac{1+t}{2})^s|f'(b)|+m(\frac{1-t}{2})^s|f'(\frac{a}{m})|\right]\right.$$

$$\left.+\int_0^1(1-t)(L(t))^2\left[(\frac{1+t}{2})^s|f'(a)|+m(\frac{1-t}{2})^s|f'(\frac{b}{m})|\right]\right\} \tag{2.19}$$

Now, by using h$\ddot{o}$lder's inequality, we get

$$\int_0^1(1-t)(U(t))^2\left[(\frac{1+t}{2})^s|f'(b)|+m(\frac{1-t}{2})^s|f'(\frac{a}{m})|\right]dt$$

$$\le\left(\int_0^1(1-t)^{\frac{q}{q-1}}(U(t))^{\frac{2q}{q-1}}dt\right)^{\frac{q-1}{q}}$$

$$\times\left\{\left(\int_0^1(\frac{1+t}{2})^{sq}dt\right)^{\frac{1}{q}}|f'(b)|+m\left(\int_0^1(\frac{1-t}{2})^{sq}dt\right)^{\frac{1}{q}}|f'(\frac{a}{m})|\right\}$$

$$=2^{2-s}(\frac{ab}{b+a})^2(\frac{1}{sq+1})^{\frac{1}{q}}\left((2^{sq+1}-1)|f'(b)|+m|f'(\frac{a}{m})|\right)\left(B(1,\frac{2q-1}{q-1}).{}_2F_1(\frac{2q}{q-1},1,\frac{3q-2}{q-1};\frac{b-a}{b+a})\right)^{\frac{q-1}{q}}. \tag{2.20}$$

Similarly, one has

$$\int_0^1(1-t)(L(t))^2\left[(\frac{1+t}{2})^s|f'(a)|+m(\frac{1-t}{2})^s|f'(\frac{b}{m})|\right]$$

$$=\frac{a^2}{2^s}(\frac{1}{sq+1})^{\frac{1}{q}}\left((2^{sq+1}-1)|f'(a)|+m|f'(\frac{b}{m})|\right)\left(B(\frac{2q-1}{q-1},1).{}_2F_1(\frac{2q}{q-1},\frac{2q-1}{q-1},\frac{3q-2}{q-1};\frac{b-a}{b})\right)^{\frac{q-1}{q}}. \tag{2.21}$$

∎

Corollary 6. *Suppose the assumptions of the Theorem 3 are satisfied. If $g(x)=\frac{ab}{b-a}$ for all $x\in[a,b]$, then one has the following inequality*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx-\int_a^b \frac{g(x)f(x)}{x^2}dx\right|\le \frac{1}{8}(\frac{1}{sq+1})^{\frac{1}{q}}$$

$$\times\left\{2^{2-s}(\frac{ab}{b+a})^2\left((2^{sq+1}-1)|f'(b)|+m|f'(\frac{a}{m})|\right)\left(B(1,\frac{2q-1}{q-1}).{}_2F_1(\frac{2q}{q-1},1,\frac{3q-2}{q-1};\frac{b-a}{b+a})\right)^{\frac{q-1}{q}}\right.$$

$$\left.+\frac{a^2}{2^s}\left((2^{sq+1}-1)|f'(a)|+m|f'(\frac{b}{m})|\right)\left(B(\frac{2q-1}{q-1},1).{}_2F_1(\frac{2q}{q-1},\frac{2q-1}{q-1},\frac{3q-2}{q-1};\frac{b-a}{b})\right)^{\frac{q-1}{q}}\right\} \tag{2.22}$$

REFERENCES

[1] I. A. Baloch, I.Işcan, Some Ostrowski Type Inequalities For Harmonically (s,m)-convex functoins in Second Sense, International Journal of Analysis,Volume 2015, Article ID 672675, 9 pages.
http://dx.doi.org/10.1155/2015/672675

[2] P. S. Bullen, Handbook of Means and Their Inequalities, Mathematics and its Applications, Volume 560, Kluwer Academic Publishers,Dordrecht/Boston/London, 2003.

[3] W.W. Breckner,"*Stetigkeitsaussagen für eine klasse verallgemeinerter konvexer funktonen in topologischen linearen Räumen*,Publ.Inst.Math.(Beograd),23,(1978),13-20.

[4] F. Chen and S. Wu, Fej*é*r and Hermite-Hadamard type inequalities for harmonically convex functions, Journal of Applied Mathematics 2014, Article ID 386806, 6 pages.

[5] F. Chen and S.Wu, Hermite-Hadamard type inequalities for harmonically s-convex functions, Sci. World J. 2014 (2014) 7, Article ID 279158.

[6] S. S. Dragomir, R.P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett. 11 (5) (1998) 91-95.

[7] S. S. Dragomir, C.E.M. Pearce, Selected topics on Hermite-Hadamard type inequalities and applications, RGMIA Monographs, Victoria University, 2000.

[8] V. N. Huy and N. T. Chung, Some generalizations of the Fej*é*r and Hermite-Hadamard inequalities in H*ö*lder spaces, J. Appl. Math. Inform. 29 (2011), no. 3-4, 859-868.

[9] J. Hua, B.-Y. Xi, and F. Qi, Hemite-Hadamard type inequalities for geometrically- arithmetically s-convex functions, Commun. Korean Math. Soc. 29 (2014), No. 1, pp. 51-63.

[10] J. Hua, B. -Y. Xi and F. Qi, Inequalities of HermiteHadamard type involving an s-convex function with applications, Applied Mathematics and Computation, 246 (2014), 752-760.

[11] I.Işcan, Hemite-Hadamard type inequalities for GA-s-convex functions, Le Matematiche, LXIX (2014)-Fasc. II, pp. 129-146.

[12] I.Işcan, Hermite-Hadamard type inequalities for harmonically convex functions, Hacettepe Journal of Mathematics and Statistics 43 (6) (2014), 935-942.

[13] I.Işcan, Ostrowski type inequalities for harmonically s-convex functions, Konuralp Journal of Mathematics, vol. 3, no. 1, pp. 63-74, 2015.

[14] I.Işcan, Hermite-Hadamard and Simpson-like type inequalities for differentiable harmonically convex functions, Journal of Mathematics, 2014, Article ID 346305, 10 pages.

[15] I.Işcan and S. Wu, Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals, Applied Mathematica and Computation, 238 (2014), 237-244.

[16] A. P. Ji, T. Y. Zhang, F. Qi, Integral Inequalities of Hermite-Hadamard Type for (α,m)- GA-Convex Functions, Journal of Function Spaces and Applications, 2013 (2013), Article ID 823856, 8 pages.

[17] M. A. Latif, New Hermite-Hadamard type integral inequalities for GA-convex functions with applications, Volume 34, Issue 4 (Nov 2014).

[18] M. V. Mihai, M. A. Noor, K. I. Noor and M. U. Awan, Some integral inequalities for harmonic h-convex functions involving hypergeometric functions, Applied Mathematics and Computa- tion 252 (2015) 257-262.

[19] M. A. Noor, K. I. Noor and M. U. Awana, Integral inequalities for coordinated harmonically convex functions, , Complex Var. Elliptic Eqn. (2014).

[20] M. A. Noor, K. I. Noor, M. U. Awana and S. Costache, Some integral inequalities for har- monically h-convex functions, U.P.B Sci. Bull. Serai A. (2015).

[21] J. E. Pečarič, F. Proschan, Y. L. Tong, Convex Functions, Partial Orderings and Statistical Applications, Mathematics in Science and Engineering, vol. 187, 1992.

[22] M. Z. Sarikaya, On new Hermite Hadamard Fej*é*r type integral inequalities, Stud. Univ. Babeş-Bolyai Math. 57 (2012), no. 3, 377-386.

[23] Y. Shuang, H. P. Yin, F. Qi, Hermite-Hadamard type integral inequalities for geometric- arithmetically s-convex functions, Analysis 33, (2013), 1001-1010.

[24] B. -Y. Xi and F. Qi, Hemite-Hadamard type inequalities for geometrically r-convex functions, Studia Scientiarum Mathematicarum Hungarica 51 (4), 530-546 (2014).

[25] T. Y. Zhang, A. P. Ji, F. Qi, Some inequalities of Hermite-Hadamard type for GA-Convex functions with applications to means. Le Matematiche, 48 (2013), no. 1, 229-239.

[26] T. -Y. Zhang, A. -P. Ji and F. Qi, Integral inequalities of Hermite-Hadamard type for har- monically quasi-convex functions, Proceedings of the Jangjeon Mathematical Society, 16(3) (2013), 399-407.