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REFINEMENTS AND REVERSES OF HÖLDER-MCCARTHY OPERATOR INEQUALITY VIA A CARTWRIGHT-FIELD RESULT

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Abstract. By the use of a classical result of Cartwright and Field we obtain in this paper some new refinements and reverses of Hölder-McCarthy operator inequality in the case $p \in (0,1)$. A comparison for the two upper bounds obtained showing that neither of them is better in general, is also performed.

1. Introduction

Let A be a nonnegative operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$, namely $\langle Ax, x \rangle \geq 0$ for any $x \in H$. We write this as $A \geq 0$.

By the use of the spectral resolution of A and the Hölder inequality, C. A. McCarthy [16] proved that

$$\langle Ax, x \rangle^p \leq \langle A^p x, x \rangle, \ p \in (1, \infty) \tag{1.1}$$

and

$$\langle A^p x, x \rangle \leq \langle Ax, x \rangle^p, \ p \in (0, 1) \tag{1.2}$$

for any $x \in H$ with $\|x\| = 1$.

Let A be a selfadjoint operator on H with

$$mI \leq A \leq MI, \tag{1.3}$$

where I is the *identity operator* and m, M are real numbers with $m < M$.

In [7, Theorem 3] Fujii et al. obtained the following interesting ratio inequality that provides a reverse of the *Hölder-McCarthy inequality* (1.1) for an operator A that satisfy the condition (1.3) with $m > 0$

$$\langle A^p x, x \rangle \leq \left\{ \frac{1}{p^{1/p} q^{1/q}} \frac{M^p - m^p}{(M-m)^{1/p} (mM^p - Mm^p)^{1/q}} \right\}^p \langle Ax, x \rangle^p, \tag{1.4}$$

for any $x \in H$ with $\|x\| = 1$, where $q = p/(p-1)$, $p > 1$.

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If A satisfies the condition (1.3) with $m \geq 0$, then we also have the additive reverse of (1.1) that has been obtained by the author in 2008, see [4]

$$0 \leq \langle A^p x, x\rangle - \langle Ax, x\rangle^p \tag{1.5}$$

$$\leq p \begin{cases} \frac{1}{2}(M-m)\left[\left\|A^{p-1}x\right\|^2 - \left\langle A^{p-1}x, x\right\rangle^2\right]^{1/2} \\ \\ \frac{1}{2}\left(M^{p-1}-m^{p-1}\right)\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2} \end{cases}$$

$$\leq \frac{1}{4}p(M-m)\left(M^{p-1}-m^{p-1}\right),$$

for any $x \in H$ with $\|x\| = 1$, where $p > 1$.

We also have [4]

$$0 \leq \langle A^p x, x\rangle - \langle Ax, x\rangle^p \tag{1.6}$$

$$\leq p \begin{cases} \frac{1}{4}\frac{(M-m)\left(M^{p-1}-m^{p-1}\right)}{M^{p/2}m^{p/2}}\langle Ax, x\rangle\left\langle A^{p-1}x, x\right\rangle, \text{ (for } m > 0), \\ \\ \left(\sqrt{M}-\sqrt{m}\right)\left(M^{(p-1)/2}-m^{(p-1)/2}\right)\left[\langle Ax, x\rangle\left\langle A^{p-1}x, x\right\rangle\right]^{\frac{1}{2}}, \end{cases}$$

$$\leq p \begin{cases} \frac{1}{4}(M-m)\left(M^{p-1}-m^{p-1}\right)\left(\frac{M}{m}\right)^{p/2}, \text{ (for } m > 0), \\ \\ \left(\sqrt{M}-\sqrt{m}\right)\left(M^{(p-1)/2}-m^{(p-1)/2}\right)M^{p/2}, \end{cases}$$

for any $x \in H$ with $\|x\| = 1$, where $p > 1$.

For various related inequalities, see [6]-[10] and [14]-[15].

We have the following inequality that provides a refinement and a reverse for the celebrated *Young's scalar inequality*

$$\begin{aligned} \frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max\{a,b\}} &\leq (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \\ &\leq \frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min\{a,b\}} \end{aligned} \tag{1.7}$$

for any $a, b > 0$ and $\nu \in [0,1]$.

This result was obtained in 1978 by Cartwright and Field [1] who established a more general result for n variables and gave an application for a probability measure supported on a finite interval.

For some new recent reverses and refinements of Young's inequality see [2]-[3], [11]-[12], [13] and [19].

By the use of (1.7) we obtain in this paper some new refinements and reverses of Hölder-McCarthy operator inequality in the case $p \in (0,1)$. A comparison for the two upper bounds obtained showing that neither of them is better in general, is also performed.

2. Some Refinements and Reverse Results

We have:

Theorem 1. *Let m, M be real numbers so that $M > m > 0$. If A is a selfadjoint operator satisfying the condition (1.3) above, then for any $p \in (0,1)$ we have*

$$\begin{aligned}(2.1)\qquad \frac{p(1-p)}{2}\frac{m}{M}\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right) &\le \frac{p(1-p)}{2M}\langle Ax,x\rangle\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right)\\ &\le 1-\frac{\langle A^px,x\rangle}{\langle Ax,x\rangle^p}\\ &\le \frac{p(1-p)}{2m}\langle Ax,x\rangle\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right)\\ &\le \frac{p(1-p)}{2}\frac{M}{m}\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right)\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

In particular,

$$\begin{aligned}(2.2)\qquad \frac{1}{8}\frac{m}{M}\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right) &\le \frac{\langle Ax,x\rangle}{8M}\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right)\\ &\le 1-\frac{\langle A^{1/2}x,x\rangle}{\langle Ax,x\rangle^{1/2}} \le \frac{\langle Ax,x\rangle}{8m}\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right)\\ &\le \frac{1}{8}\frac{M}{m}\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right),\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. If a, $b \in [m, M]$, then by Cartwright-Field inequality (1.7) we have

$$\frac{1}{2M}p(1-p)(b-a)^2 \le (1-p)a + pb - a^{1-p}b^p \le \frac{1}{2m}p(1-p)(b-a)^2$$

or, equivalently

$$\begin{aligned}(2.3)\qquad \frac{1}{2M}p(1-p)(b^2-2ab+a^2) &\le (1-p)a+pb-a^{1-p}b^p\\ &\le \frac{1}{2m}p(1-p)(b^2-2ab+a^2),\end{aligned}$$

for any $p \in (0,1)$.

Fix $a \in [m, M]$ and by using the operator functional calculus for A with $mI \le A \le MI$ we have

$$\begin{aligned}(2.4)\qquad \frac{1}{2M}p(1-p)(A^2-2aA+a^2I) &\le (1-p)aI+pA-a^{1-p}A^p\\ &\le \frac{1}{2m}p(1-p)(A^2-2aA+a^2I).\end{aligned}$$

Then for any $x \in H$ with $\|x\| = 1$ we have from (2.4) that

$$\begin{aligned} &\frac{1}{2M} p(1-p)\left(\left\langle A^2 x, x\right\rangle - 2a\langle Ax, x\rangle + a^2\right) \\ &\le (1-p)\, a + p\langle Ax, x\rangle - a^{1-p}\left\langle A^p x, x\right\rangle \\ &\le \frac{1}{2m} p(1-p)\left(\left\langle A^2 x, x\right\rangle - 2a\langle Ax, x\rangle + a^2\right), \end{aligned} \tag{2.5}$$

for any $a \in [m, M]$.

If we choose in (2.5) $a = \langle Ax, x\rangle \in [m, M]$, then we get for any $x \in H$ with $\|x\| = 1$ that

$$\begin{aligned} \frac{1}{2M} p(1-p)\left(\left\langle A^2 x, x\right\rangle - \langle Ax, x\rangle^2\right) &\le \langle Ax, x\rangle - \langle Ax, x\rangle^{1-p}\left\langle A^p x, x\right\rangle \\ &\le \frac{1}{2m} p(1-p)\left(\left\langle A^2 x, x\right\rangle - \langle Ax, x\rangle^2\right), \end{aligned}$$

and by division with $\langle Ax, x\rangle > 0$ we obtain the second and third inequalities in (2.1).

The rest is obvious. □

Remark 1. *It is well known that, if $mI \le A \le MI$ with $M > 0$, then, see for instance* [17, p. 27], *we have*

$$(1 \le)\, \frac{\left\langle A^2 x, x\right\rangle}{\langle Ax, x\rangle^2} \le \frac{(m+M)^2}{4mM}$$

for any $x \in H$ with $\|x\| = 1$, which implies that

$$(0 \le)\, \frac{\left\langle A^2 x, x\right\rangle}{\langle Ax, x\rangle^2} - 1 \le \frac{(M-m)^2}{4mM}.$$

Using (2.1) and by denoting $h = \frac{M}{m}$ we get

$$(0 \le)\, 1 - \frac{\langle A^p x, x\rangle}{\langle Ax, x\rangle^p} \le \frac{p(1-p)}{8}(h-1)^2 \tag{2.6}$$

and, in particular,

$$(0 \le)\, 1 - \frac{\left\langle A^{1/2} x, x\right\rangle}{\langle Ax, x\rangle^{1/2}} \le \frac{1}{32}(h-1)^2, \tag{2.7}$$

for any $x \in H$ with $\|x\| = 1$.

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ \ h > 0. \tag{2.8}$$

The function K is decreasing on $(0, 1)$ and increasing on $[1, \infty)$, $K(h) \ge 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

Observe that for any $h > 0$

$$K(h) - 1 = \frac{(h-1)^2}{4h} = K\left(\frac{1}{h}\right) - 1.$$

From (2.6) we then have

$$(0 \le)\, 1 - \frac{\langle A^p x, x\rangle}{\langle Ax, x\rangle^p} \le \frac{p(1-p)}{2} h\,[K(h) - 1] \tag{2.9}$$

and, in particular,

$$(0 \leq) \, 1 - \frac{\langle A^{1/2}x, x\rangle}{\langle Ax, x\rangle^{1/2}} \leq \frac{1}{8} h \left[K(h) - 1 \right], \tag{2.10}$$

for any $x \in H$ with $\|x\| = 1$.

Also, if $a, b > 0$ then

$$K\left(\frac{b}{a}\right) - 1 = \frac{(b-a)^2}{4ab}.$$

Since $\min\{a,b\} \max\{a,b\} = ab$ if $a, b > 0$, then

$$\frac{(b-a)^2}{\max\{a,b\}} = \frac{\min\{a,b\}(b-a)^2}{ab} = 4\min\{a,b\}\left[K\left(\frac{b}{a}\right) - 1\right]$$

and

$$\frac{(b-a)^2}{\min\{a,b\}} = \frac{\max\{a,b\}(b-a)^2}{ab} = 4\max\{a,b\}\left[K\left(\frac{b}{a}\right) - 1\right]$$

and the inequality (1.7) can be written as

$$\begin{aligned} 2\nu(1-\nu)\min\{a,b\}\left[K\left(\frac{b}{a}\right) - 1\right] &\leq (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \\ &\leq 2\nu(1-\nu)\max\{a,b\}\left[K\left(\frac{b}{a}\right) - 1\right] \end{aligned} \tag{2.11}$$

for any $a, b > 0$ and $\nu \in [0,1]$.

Theorem 2. *Let m, M be real numbers so that $M > m > 0$. If A is a selfadjoint operator satisfying the condition (1.3) above, then for any $p \in (0,1)$ we have*

$$\begin{aligned} (0 \leq) \, 1 - \frac{\langle A^p x, x\rangle}{\langle Ax, x\rangle^p} &\leq p(1-p)\left[K(h) - 1\right]\left(2 + \frac{\langle |A - \langle Ax, x\rangle I| x, x\rangle}{\langle Ax, x\rangle}\right) \\ &\leq p(1-p)\left[K(h) - 1\right]\left[2 + \left(\frac{\langle A^2x, x\rangle}{\langle Ax, x\rangle^2} - 1\right)^{1/2}\right] \\ &\leq p(1-p)\left[K(h) - 1\right]\left[2 + (K(h) - 1)^{1/2}\right] \end{aligned} \tag{2.12}$$

for any $x \in H$ with $\|x\| = 1$.

In particular, we have

$$\begin{aligned} (0 \leq) \, 1 - \frac{\langle A^{1/2}x, x\rangle}{\langle Ax, x\rangle^{1/2}} &\leq \frac{1}{4}\left[K(h) - 1\right]\left(2 + \frac{\langle |A - \langle Ax, x\rangle I| x, x\rangle}{\langle Ax, x\rangle}\right) \\ &\leq \frac{1}{4}\left[K(h) - 1\right]\left[2 + \left(\frac{\langle A^2x, x\rangle}{\langle Ax, x\rangle^2} - 1\right)^{1/2}\right] \\ &\leq \frac{1}{4}\left[K(h) - 1\right]\left[2 + (K(h) - 1)^{1/2}\right] \end{aligned} \tag{2.13}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. From (2.11) we have for any a, $b > 0$ and $p \in [0,1]$ that

$$(1-p)a + pb - a^{1-p}b^p \leq p(1-p)(a+b+|b-a|)\left[K\left(\frac{b}{a}\right)-1\right] \tag{2.14}$$

since

$$\max\{a,b\} = \frac{1}{2}(a+b+|b-a|).$$

If a, $b \in [m,M]$, then $\frac{b}{a} \in \left[\frac{m}{M}, \frac{M}{m}\right]$ and by the properties of Kantorovich's constant K, we have

$$1 \leq K\left(\frac{b}{a}\right) \leq K\left(\frac{M}{m}\right) = K(h) \text{ for any } a, b \in [m,M].$$

Therefore, by (2.14) we have

$$(1-p)a + pb - a^{1-p}b^p \leq p(1-p)(a+b+|b-a|)[K(h)-1]$$

for any a, $b \in [m,M]$ and $p \in [0,1]$.

Fix $a \in [m,M]$ and by using the operator functional calculus for A with $mI \leq A \leq MI$, we have

$$(1-p)aI + pA - a^{1-p}A^p \leq p(1-p)[K(h)-1](aI + A + |A - aI|). \tag{2.15}$$

Then for any $x \in H$ with $\|x\| = 1$ we get from (2.15) that

$$\begin{aligned}(1-p)a + p\langle Ax,x\rangle - a^{1-p}\langle A^p x,x\rangle \\ \leq p(1-p)[K(h)-1](a + \langle Ax,x\rangle + \langle |A-aI|x,x\rangle),\end{aligned} \tag{2.16}$$

for any $a \in [m,M]$ and $p \in [0,1]$.

Now, if we take $a = \langle Ax,x\rangle \in [m,M]$, where $x \in H$ with $\|x\| = 1$ in (2.16), then we obtain

$$\begin{aligned}&\langle Ax,x\rangle - \langle Ax,x\rangle^{1-p}\langle A^p x,x\rangle \\ &\leq p(1-p)[K(h)-1](2\langle Ax,x\rangle + \langle |A - \langle Ax,x\rangle I|x,x\rangle),\end{aligned}$$

which, by division with $\langle Ax,x\rangle > 0$ provides the first inequality in (2.12).

By Schwarz inequality, we have for $x \in H$ with $\|x\| = 1$ that

$$\begin{aligned}\langle |A - \langle Ax,x\rangle I|x,x\rangle &\leq \left\langle (A - \langle Ax,x\rangle I)^2 x,x\right\rangle^{1/2} \\ &= \left\langle \left(A^2 - 2\langle Ax,x\rangle A + \langle Ax,x\rangle^2 I\right)x,x\right\rangle^{1/2} \\ &= \left(\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right)^{1/2},\end{aligned}$$

which proves the second part of (2.12).

Since

$$\frac{\left\langle A^2x,x\right\rangle}{\langle Ax,x\rangle^2} - 1 \leq \frac{(M-m)^2}{4mM} = K(h) - 1$$

for $x \in H$ with $\|x\| = 1$, then the last part of (2.12) is thus proved. □

3. A Comparison for Upper Bounds

We observe that the inequality (2.9) provides for the quantity

$$(0 \leq) \, 1 - \frac{\langle A^p x, x\rangle}{\langle Ax, x\rangle^p}, \; x \in H \text{ with } \|x\| = 1,$$

the following upper bound

$$B_1 (p, h) := \frac{p (1 - p)}{2} h [K (h) - 1], \tag{3.1}$$

while the inequality (2.12) gives the upper bound

$$B_2 (p, h) := p (1 - p) [K (h) - 1] \left[2 + (K (h) - 1)^{1/2}\right], \tag{3.2}$$

where $p \in (0, 1)$ and $h > 1$.

Now, if we depict the 3D plot for the difference of the bounds B_1 and B_2, namely

$$D (x, y) := B_1 (y, x) - B_2 (y, x)$$

on the box $[1, 8] \times [0, 1]$, see Figure 1

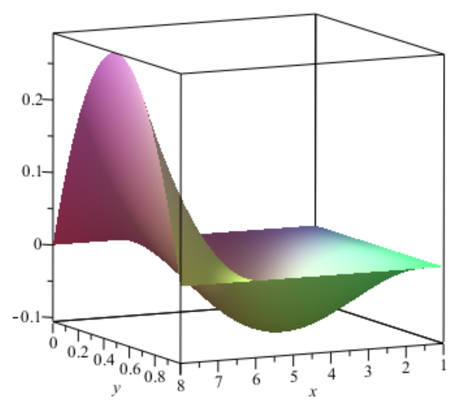

Figure 1. Plot of the difference $D (x, y)$

then we observe that it takes both positive and negative values, showing that the bounds $B_1 (p, h)$ and $B_2 (p, h)$ can not be compared in general, namely neither of them is better for any $p \in (0, 1)$ and $h > 1$.

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