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On New Refinements and Applications of Efficient Quadrature Rules Using n-times differentiable mappings

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Abstract. In this paper, new efficient quadrature rules are established using a newly developed special type of kernel for n-times differentiable mappings, having five steps. Some previous inequalities are also recaptured as special cases of our main inequalities. At the end, efficiency of the newly developed quadrature rules are discussed.
Keywords: Ostrowski inequality, Grüss inequality, Quadrature formula,Numerical Integration, peano kerenel..

1 Introduction

In 1938, Ostrowski [13] first announced his inequality result for different differentiable mappings, which is given below:

Theorem 1. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I° *(*I° *is the interior of* I*) and let* $a, b \in I^\circ$ *with* $a < b$*. If* $f' : (a,b) \to \mathbb{R}$ *is bounded on* (a,b) *i.e.* $\|f'\|_\infty = \sup_{t\in[a,b]} |f''(t)| < \infty$*, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty \qquad (1.1)$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

In 1976, Milovanovic et. al [11], proved a generalization of Ostrowski's inequality for n-time differentiable mappings. Up till now, a large number of research papers and books have been written on inequalities and their applications (see for instance [2]-[5], [8] and [14]-[15]). In many practical problems, it is important to bound one quantity by another quantity. The classical inequalities like Ostrowski are very helpful for this purpose. Ostrowski type inequalities have immediate applications in numerical integration, optimization theory, statistics, and integral operator theory.

We indicate another inequality called Grüss inequality [11] which is stated as the integral inequality that establishes a connection between the integral of the product of two functions and the product of the integrals, which is given below.

Theorem 2. *Let $f,\ g : [a,b] \to \mathbb{R}$ be integrable functions such that $\varphi \leq f(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$, for some constants $\varphi,\ \Phi,\ \gamma,\ \Gamma$ and $x \in [a,b]$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx.\frac{1}{b-a}\int_a^b g(x)dx\right| \leq \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma). \tag{1.2}$$

Dragomir et. al [4] combined Ostrowski and Grüss inequality to give a new inequality which they named Ostrowski-Grüss type inequality. Dragomir [3], Liu [6], Alomari [1] and Liu et. al [8] established some companions of ostrowski type integral inequalities.

Recently, Liu [7] proved the following companions of ostrowski type inequalities for 3-step kernels.

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be differentiable in (a,b). If $f' \in L^1[a,b]$, and $\gamma \leq f'(x) \leq \Gamma$, for all $x \in [a,b]$, then for all $x \in \left[a, \frac{a+b}{2}\right]$, we have*

$$\left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{b-a}{4} + \left|x - \frac{3a+b}{4}\right|\right](S-\gamma) \tag{1.3}$$

and

$$\left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{b-a}{4} + \left|x - \frac{3a+b}{4}\right|\right](\Gamma-S). \tag{1.4}$$

More recently, Qayyum et. al [9]-[10] proved companions of Ostrowski inequality for 5-step linear and quadratic kernels but in this paper, we establish our results for 5-step kernel for n-times differentiable mappings. In this paper, new ontrowski inequalities are extended. Using these inequalities, some efficient quadrature rules are established. Some previous inequalities are also recaptured as special cases of our main inequalities. At the end, efficiency of the newly developed quadrature rules are discussed.

2 Derivation of Ostrowski inequalities using 5-step kernel

We will start our work by introducing a new peano kernel defined by $P(x,.) : [a,b] \to \mathbb{R}$

$$P_n(x,t) = \begin{cases} \frac{1}{n!}(t-a)^n, & t \in \left(a, \frac{a+x}{2}\right], \\ \frac{1}{n!}\left(t-\frac{3a+b}{4}\right)^n, & t \in \left(\frac{a+x}{2}, x\right], \\ \frac{1}{n!}\left(t-\frac{a+b}{2}\right)^n, & t \in (x, a+b-x], \\ \frac{1}{n!}\left(t-\frac{a+3b}{4}\right)^n, & t \in \left(a+b-x, \frac{a+2b-x}{2}\right], \\ \frac{1}{n!}(t-b)^n, & t \in \left(\frac{a+2b-x}{2}, b\right], \end{cases} \tag{2.1}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

The following lemma is the main tool to prove the main results.

Lemma 1. *Let $f : [a,b] \to \mathbb{R}$ be an n-times differentiable function such that $f^{(n-1)}(x)$ for $n \in N$ is absolutely continuous on $[a,b]$ then*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b P_n(x,t) f^{(n)}(t)dt \qquad (2.2)\\ =\; &\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1} - \left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}\left(\frac{a+x}{2}\right)\right. \\ &+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1} - \left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}(x) \\ &+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1} - \left(x-\frac{3a+b}{4}\right)^{k+1}\right\} f^{(k)}(a+b-x) \\ &\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1} - (x-a)^{k+1}\right\} f^{(k)}\left(\frac{a+2b-x}{2}\right)\right] \\ &+\frac{(-1)^n}{b-a}\int_a^b f(t)dt \end{aligned}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Proof. The proof of (2.2) is established using mathematical induction.

Take $n = 1$,

$$L.H.S \text{ of } (2.2) = \int_a^b P_1(x,t) f'(t) dt. \tag{2.3}$$

After integrating by parts, we get

$$\frac{1}{b-a} \int_a^b P_1(x,t) f'(t) dt \tag{2.4}$$
$$= \frac{1}{4}\left[f\left(\frac{a+x}{2}\right) + f(x) + f(a+b-x) + f\left(\frac{a+2b-x}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t) dt.$$

We have

$$L.H.S = \int_a^b P_1(x,t) f'(t) dt.$$

Equation (2.3), is identical to the $R.H.S$ of (2.2).

Assume that (2.2) is true for n.

$$\begin{aligned}
&\int_a^b P_{n+1}(x,t) f^{(n+1)}(t) dt \\
= &\sum_{k=0}^{n} \frac{(-1)^{n+k+2}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1} - \left(x - \frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}\left(\frac{a+x}{2}\right)\right. \\
&+ \left\{\left(x - \frac{3a+b}{4}\right)^{k+1} - \left(x - \frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}(x) \\
&+ (-1)^{k+1}\left\{\left(x - \frac{a+b}{2}\right)^{k+1} - \left(x - \frac{3a+b}{4}\right)^{k+1}\right\} f^{(k)}(a+b-x) \\
&\left.+ \left(\frac{-1}{2}\right)^{k+1}\left\{\left(x - \frac{a+b}{2}\right)^{k+1} - (x-a)^{k+1}\right\} f^{(k)}\left(\frac{a+2b-x}{2}\right)\right] \\
&+ (-1)^{n+1}\int_a^b f(t) dt,
\end{aligned}$$

where

$$P_{n+1}(x,t) = \begin{cases} \frac{1}{(n+1)!}(t-a)^{n+1}, & t \in \left(a, \frac{a+x}{2}\right], \\ \frac{1}{(n+1)!}\left(t - \frac{3a+b}{4}\right)^{n+1}, & t \in \left(\frac{a+x}{2}, x\right], \\ \frac{1}{(n+1)!}\left(t - \frac{a+b}{2}\right)^{n+1}, & t \in (x, a+b-x], \\ \frac{1}{(n+1)!}\left(t - \frac{a+3b}{4}\right)^{n+1}, & t \in \left(a+b-x, \frac{a+2b-x}{2}\right], \\ \frac{1}{(n+1)!}(t-b)^{n+1}, & t \in \left(\frac{a+2b-x}{2}, b\right]. \end{cases}$$

After integeration by parts, we get

$$
\begin{aligned}
&\int_a^b P_{n+1}(x,t) f^{(n+1)}(t)dt \\
= {} & \frac{1}{(n+1)!}\Bigg[\frac{1}{2^{n+1}}(x-a)^{n+1} f^{(n)}\left(\frac{a+x}{2}\right) + \left(x-\frac{3a+b}{4}\right)^{n+1} f^{(n)}(x) \\
& -\frac{1}{2^{n+1}}\left(x-\frac{a+b}{2}\right)^{n+1} f^{(n)}\left(\frac{a+x}{2}\right) + (-1)^{n+1}\left(x-\frac{a+b}{2}\right)^{n+1} f^{(n)}(a+b-x) \\
& -\left(x-\frac{a+b}{2}\right)^{n+1} f^{(n)}(x) + \left(\frac{-1}{2}\right)^{n+1}\left(x-\frac{a+b}{2}\right)^{n+1} f^{(n)}\left(\frac{a+2b-x}{2}\right) \\
& -(-1)^{n+1}\left(x-\frac{3a+b}{4}\right)^{n+1} f^{(n)}(a+b-x) \\
& -\left(\frac{-1}{2}\right)^{n+1}(x-a)^{n+1} f^{(n)}\left(\frac{a+2b-x}{2}\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{n!}\Bigg[\int_a^{\frac{a+x}{2}} (t-a)^n f^{(n)}(t)dt + \int_{\frac{a+x}{2}}^{x}\left(t-\frac{3a+b}{4}\right)^n f^{(n)}(t)dt \\
& + \int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^n f^{(n)}(t)dt \\
& + \int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{3a+b}{4}\right)^n f^{(n)}(t)dt + \int_{\frac{a+2b-x}{2}}^{b}(t-b)^n f^{(n)}(t)dt\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= {} & \frac{1}{(n+1)!}\Bigg[\frac{1}{2^{n+1}}(x-a)^{n+1} f^{(n)}\left(\frac{a+x}{2}\right) + \left(x-\frac{3a+b}{4}\right)^{n+1} f^{(n)}(x) \\
& -\frac{1}{2^{n+1}}\left(x-\frac{a+b}{2}\right)^{n+1} f^{(n)}\left(\frac{a+x}{2}\right) + (-1)^{n+1}\left(x-\frac{a+b}{2}\right)^{n+1} f^{(n)}(a+b-x) \\
& -\left(x-\frac{a+b}{2}\right)^{n+1} f^{(n)}(x) + \left(\frac{-1}{2}\right)^{n+1}\left(x-\frac{a+b}{2}\right)^{n+1} f^{(n)}\left(\frac{a+2b-x}{2}\right) \\
& -(-1)^{n+1}\left(x-\frac{3a+b}{4}\right)^{n+1} f^{(n)}(a+b-x) \\
& -\left(\frac{-1}{2}\right)^{n+1}(x-a)^{n+1} f^{(n)}\left(\frac{a+2b-x}{2}\right)\Bigg] - \int_a^b P_n(x,t) f^{(n)}(t)dt
\end{aligned}
$$

$$
\begin{aligned}
= \; & \frac{1}{(n+1)!}\left[\left\{(x-a)^{n+1}-\left(x-\frac{a+b}{2}\right)^{n+1}\right\}\frac{1}{2^{n+1}}f^{(n)}\left(\frac{a+x}{2}\right)\right. \\
& +\left\{\left(x-\frac{3a+b}{4}\right)^{n+1}-\left(x-\frac{a+b}{2}\right)^{n+1}\right\}f^{(n)}(x) \\
& +\left\{\left(x-\frac{a+b}{2}\right)^{n+1}-\left(x-\frac{3a+b}{4}\right)^{n+1}\right\}(-1)^{n+1}f^{(n)}(a+b-x) \\
& \left.+\left\{\left(x-\frac{a+b}{2}\right)^{n+1}-(x-a)^{n+1}\right\}\left(\frac{-1}{2}\right)^{n+1}f^{(n)}\left(\frac{a+2b-x}{2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{k=0}^{n-1}\frac{(-1)^{n+k+2}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}\left(\frac{a+x}{2}\right)\right. \\
& +\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x) \\
& +(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x) \\
& \left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right] \\
& +(-1)^{n+1}\int_a^b f(t)dt
\end{aligned}
$$

$$
\begin{aligned}
= \; & \sum_{k=0}^{n}\frac{(-1)^{n+k+2}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}\left(\frac{a+x}{2}\right)\right. \\
& +\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x) \\
& +(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x) \\
& \left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right] \\
& +(-1)^{n+1}\int_a^b f(t)dt.
\end{aligned}
$$

This completes the proof of lemma 1.

Now we will present our results by imposing three different conditions on $f^{(n)}$ and $f^{(n+1)}$.

3 Case A: When $f^{(n)} \in L^1[a,b]$

Theorem 4. *Let $f\colon [a,b] \to \mathbb{R}$ be an n-times differentiable function on (a,b), $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $\gamma \leq f^{(n)}(t) \leq \Gamma$, $\forall\ t \in [a,b]$, then for all $x \in \left[a, \frac{a+b}{2}\right]$, we have*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1} - \left(x - \frac{a+b}{2}\right)^{k+1}\right\}\right.\right. \qquad (3.1)\\
&\times f^{(k)}\left(\frac{a+x}{2}\right) + \left\{\left(x - \frac{3a+b}{4}\right)^{k+1} - \left(x - \frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}(x)\\
&+(-1)^{k+1}\left\{\left(x - \frac{a+b}{2}\right)^{k+1} - \left(x - \frac{3a+b}{4}\right)^{k+1}\right\} f^{(k)}(a+b-x)\\
&\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x - \frac{a+b}{2}\right)^{k+1} - (x-a)^{k+1}\right\} f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]\\
&+\frac{(-1)^n}{b-a}\int_a^b f(t)dt - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(b-a)^2}\frac{1}{(n+1)!}\\
&\left[\times\frac{1}{2^{n+1}}\left(1-(-1)^{n+1}\right)(x-a)^{n+1} + \left(1-(-1)^{n+1}\right)\left(x - \frac{3a+b}{4}\right)^{n+1}\right.\\
&\left.\left.+\left(\frac{-1}{2^{n+1}} + \frac{(-1)^{n+1}}{2^{n+1}} + (-1)^{n+1} - 1\right)\left(x - \frac{a+b}{2}\right)^{n+1}\right]\right|\\
\leq\ &\delta(x)(b-a)(S-\gamma)
\end{aligned}$$

and

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}\right.\right. \tag{3.2}$$
$$\times f^{(k)}\left(\frac{a+x}{2}\right)+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x)$$
$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+\frac{(-1)^{n}}{b-a}\int_a^b f(t)dt-\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2}$$
$$\times\frac{1}{(n+1)!}\left[\frac{1}{2^{n+1}}\left(1-(-1)^{n+1}\right)(x-a)^{n+1}\right.$$
$$+\left(1-(-1)^{n+1}\right)\left(x-\frac{3a+b}{4}\right)^{n+1}$$
$$\left.\left.+\left(\frac{-1}{2^{n+1}}+\frac{(-1)^{n+1}}{2^{n+1}}+(-1)^{n+1}-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right]\right|$$
$$\leq\delta(x)(b-a)(\Gamma-S),$$

where

$$S=\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}, \tag{3.3}$$

$$\begin{aligned}&\delta(x)\\=&\max\left\{\left|\frac{1}{n!}\left(\frac{x-a}{2}\right)^{n}-\frac{\lambda(x)}{b-a}\right|,\left|\frac{1}{n!}\left(x-\frac{3a+b}{4}\right)^{2}-\frac{\lambda(x)}{b-a}\right|,\right.\\&\left.\left|\frac{1}{n!}\left(x-\frac{a+b}{2}\right)^{n}-\frac{\lambda(x)}{b-a}\right|,\left|\frac{1}{4n!}\left(x-\frac{a+b}{2}\right)^{2}-\frac{\lambda(x)}{b-a}\right|,\frac{\lambda(x)}{b-a}\right\}\end{aligned}$$

and

$$\lambda(x)=\frac{1}{(n+1)!}\left[\frac{1}{2^{n+1}}\left(1-(-1)^{n+1}\right)(x-a)^{n+1}\right.$$
$$+\left(1-(-1)^{n+1}\right)\left(x-\frac{3a+b}{4}\right)^{n+1}$$
$$\left.+\left(\frac{-1}{2^{n+1}}+\frac{(-1)^{n+1}}{2^{n+1}}+(-1)^{n+1}-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right].$$

*Proof.*Let

$$\frac{1}{b-a}\int_a^b P_n(x,t)dt \tag{3.4}$$
$$= \frac{1}{(b-a)(n+1)!}\left[\frac{1}{2^{n+1}}\left(1-(-1)^{n+1}\right)(x-a)^{n+1}\right.$$
$$+\left(1-(-1)^{n+1}\right)\left(x-\frac{3a+b}{4}\right)^{n+1}$$
$$\left.+\left(\frac{-1}{2^{n+1}}+\frac{(-1)^{n+1}}{2^{n+1}}+(-1)^{n+1}-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right].$$

Using (3.4), we get

$$\frac{1}{b-a}\int_a^b P_n(x,t)f^{(n)}(t)dt - \frac{1}{(b-a)^2}\int_a^b P_n(x,t)dt\int_a^b f^{(n)}(t)dt \tag{3.5}$$
$$= \sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}\left(\frac{a+x}{2}\right)\right.$$
$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x)$$
$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+\frac{(-1)^n}{b-a}\int_a^b f(t)dt - \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2}$$
$$\times\frac{1}{(n+1)!}\left[\frac{1}{2^{n+1}}\left(1-(-1)^{n+1}\right)(x-a)^{n+1}\right.$$
$$+\left(1-(-1)^{n+1}\right)\left(x-\frac{3a+b}{4}\right)^{n+1}$$
$$\left.+\left(\frac{-1}{2^{n+1}}+\frac{(-1)^{n+1}}{2^{n+1}}+(-1)^{n+1}-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right].$$

Denote the L.H.S of (3.5) by $R_n(x)$. If $C \in R$ is an arbitrary constant, then we have

$$R_n(x) = \frac{1}{b-a}\int_a^b \left(f^{(n)}(t)-C\right)\left[P_n(x,t)-\frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right]dt \tag{3.6}$$

Furthermore, we have

$$|R_n(x)| \leq \frac{1}{b-a} \max_{t\in[a,b]} \left| P_n(x,t) - \frac{1}{b-a}\int\limits_a^b P^{(n)}(x,s)ds \right| \int\limits_a^b \left| f^{(n)}(t) - C \right| dt \tag{3.7}$$

Now

$$\left| P_n(x,t) - \frac{1}{b-a}\int\limits_a^b P^{(n)}(x,s)ds \right| \tag{3.8}$$

$$= \max \left\{ \begin{array}{c} \left|\frac{1}{n!}\left(\frac{x-a}{2}\right)^n - \frac{\lambda(x)}{b-a}\right|, \left|\frac{1}{n!}\left(x - \frac{3a+b}{4}\right)^2 - \frac{\lambda(x)}{b-a}\right|, \\ \left|\frac{1}{n!}\left(x - \frac{a+b}{2}\right)^n - \frac{\lambda(x)}{b-a}\right|, \left|\frac{1}{4n!}\left(x - \frac{a+b}{2}\right)^2 - \frac{\lambda(x)}{b-a}\right|, \frac{\lambda(x)}{b-a} \end{array} \right\}$$

$$= \delta(x),$$

where

$$\lambda(x) = \frac{1}{(n+1)!}\left[\frac{1}{2^{n+1}}\left(1-(-1)^{n+1}\right)(x-a)^{n+1} + \left(1-(-1)^{n+1}\right)\left(x-\frac{3a+b}{4}\right)^{n+1} + \left(\frac{-1}{2^{n+1}} + \frac{(-1)^{n+1}}{2^{n+1}} + (-1)^{n+1} - 1\right)\left(x - \frac{a+b}{2}\right)^{n+1}\right].$$

We also have

$$\int\limits_a^b \left| f^{(n)}(t) - \gamma \right| dt = (S-\gamma)(b-a). \tag{3.9}$$

$$\int\limits_a^b \left| f^{(n)}(t) - \Gamma \right| dt = (\Gamma - S)(b-a). \tag{3.10}$$

Using (3.4) to (3.10) and using $C = \gamma$ and $C = \Gamma$ in (3.7), we can obtain (3.1) and (3.2).

Remark 1. If we substitute $n = 2$ in (3.1) and (3.2), we get Qayyum et. al result proved in [9].

Corollary 1. *Substitution of $x = a$ in (3.1) and (3.2) gives*

$$\left| \sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left\{ (-1)^k f^{(k)}(a) + \left(1 + \frac{1}{2^{k+1}} + \frac{(-1)^k}{4^{k+1}}\right) f^{(k)}(b) \right\} + \frac{(-1)^n}{b-a}\int\limits_a^b f(t)dt - \frac{(b-a)^{n-1}}{2^{n+1}(n+1)!}\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right)(1+(-1)^n) \right|$$

$$\leq \delta(a)(b-a)(S-\gamma) \tag{3.11}$$

and

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left\{(-1)^k f^{(k)}(a)+\left(1+\frac{1}{2^{k+1}}+\frac{(-1)^k}{4^{k+1}}\right)f^{(k)}(b)\right\}\right.$$
$$\left.+\frac{(-1)^n}{b-a}\int_a^b f(t)dt-\frac{(b-a)^{n-1}}{2^{n+1}(n+1)!}\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)(1+(-1)^n)\right|$$
$$\leq \delta(a)(b-a)(\Gamma-S). \tag{3.12}$$

Corollary 2. *Substitution of* $x=\frac{a+b}{2}$ *in (3.1) and (3.2) gives*

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{4^{k+1}}\left\{f^{(k)}\left(\frac{3a+b}{4}\right)+\left(1+(-1)^k\right)f^{(k)}\left(\frac{a+b}{2}\right)\right.\right.$$
$$\left.+(-1)^k f^{(k)}\left(\frac{a+3b}{4}\right)\right\}$$
$$\left.+\frac{(-1)^n}{b-a}\int_a^b f(t)dt-\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)\times\frac{2(b-a)^{n-1}}{4^{n+1}(n+1)!}(1+(-1)^n)\right|$$
$$\leq \delta\left(\frac{a+b}{2}\right)(b-a)(S-\gamma) \tag{3.13}$$

and

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{4^{k+1}}\left\{f^{(k)}\left(\frac{3a+b}{4}\right)+\left(1+(-1)^k\right)f^{(k)}\left(\frac{a+b}{2}\right)\right.\right.$$
$$\left.+(-1)^k f^{(k)}\left(\frac{a+3b}{4}\right)\right\}$$
$$\left.+\frac{(-1)^n}{b-a}\int_a^b f(t)dt-\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)\times\frac{2(b-a)^{n-1}}{4^{n+1}(n+1)!}(1+(-1)^n)\right|$$
$$\leq \delta\left(\frac{a+b}{2}\right)(b-a)(\Gamma-S). \tag{3.14}$$

Corollary 3. *Substitution of* $x = \frac{3a+b}{4}$ *in (3.1) and (3.2) gives*

$$\begin{aligned}
&\left| \sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \frac{(b-a)^{k+1}}{4^{k+1}} \left[\frac{\left\{1+(-1)^k\right\}}{2^{k+1}} f^{(k)}\left(\frac{7a+b}{8}\right) + (-1)^k f^{(k)}\left(\frac{3a+b}{4}\right) \right.\right. \\
&\left. +f^{(k)}\left(\frac{a+3b}{4}\right) + \frac{1}{2^{k+1}}\left(1+(-1)^k\right) f^{(k)}\left(\frac{a+7b}{8}\right)\right] + \frac{(-1)^n}{b-a}\int_a^b f(t)dt \\
&\left. -\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right)\frac{1}{(n+1)!}\frac{(b-a)^{n-1}}{4^{n+1}}\left\{1+(-1)^n + \frac{1}{2^n} + \frac{(-1)^n}{2^n}\right\} \right| \\
\leq\ & \delta\left(\frac{3a+b}{4}\right)(b-a)(S-\gamma) \qquad (3.15)
\end{aligned}$$

and

$$\begin{aligned}
&\left| \sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \frac{(b-a)^{k+1}}{4^{k+1}} \left[\frac{1}{2^{k+1}}\left\{1+(-1)^k\right\} f^{(k)}\left(\frac{7a+b}{8}\right) \right.\right. \\
&+(-1)^k f^{(k)}\left(\frac{3a+b}{4}\right) + f^{(k)}\left(\frac{a+3b}{4}\right) \\
&\left. +\frac{1}{2^{k+1}}\left\{1+(-1)^k\right\} f^{(k)}\left(\frac{a+7b}{8}\right)\right] + \frac{(-1)^n}{b-a}\int_a^b f(t)dt - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(b-a)^2} \\
&\left. \times\frac{1}{(n+1)!}\frac{(b-a)^{n+1}}{4^{n+1}}\left\{1+(-1)^n + \frac{1}{2^n} + \frac{(-1)^n}{2^n}\right\} \right| \\
\leq\ & \delta\left(\frac{3a+b}{4}\right)(b-a)(\Gamma - S). \qquad (3.16)
\end{aligned}$$

4 Case B: When $f^{(n+1)} \in L^2[a,b]$

Theorem 5. *Let* $f : [a,b] \to \mathbb{R}$ *be an n-times differentiable function on* (a,b), $f^{(n+1)} \in L^2[a,b]$, *then for all* $x \in \left[a, \frac{a+b}{2}\right]$, *we have*

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{f^{(k)}\left(\frac{a+x}{2}\right)}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}\right.\right. \tag{4.1}$$
$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x)$$
$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+\frac{(-1)^n}{b-a}\int_a^b f(t)dt-\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2}$$
$$\times\frac{1}{(n+1)!}\left[\frac{1}{2^{n+1}}\left(1-(-1)^{n+1}\right)(x-a)^{n+1}\right.$$
$$+\left(1-(-1)^{n+1}\right)\left(x-\frac{3a+b}{4}\right)^{n+1}$$
$$\left.\left.+\left(\frac{-1}{2^{n+1}}+\frac{(-1)^{n+1}}{2^{n+1}}+(-1)^{n+1}-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right]\right|$$

$$\leq \frac{b-a}{\pi}\left\|f^{(n+1)}\right\|_2\left[\frac{1}{(n!)^2(2n+1)}\left\{\frac{(x-a)^{2n+1}}{2^{2n}}+2\left(x-\frac{3a+b}{4}\right)^{2n+1}\right.\right.$$
$$\left.-\left(\frac{1}{2^{2n}}+2\right)\left(x-\frac{a+b}{2}\right)^{2n+1}\right\}-\frac{1}{b-a}\frac{1}{(n+1)!}\left\{\frac{(x-a)^{n+1}}{2^{n+1}}(1+(-1)^n)\right.$$
$$+(1+(-1)^n)\left(x-\frac{3a+b}{4}\right)^{n+1}+\left(\frac{-1}{2^{n+1}}-\frac{(-1)^n}{2^{n+1}}-(-1)^n-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}$$

*Proof.*Substitute $C = f^{(n)}\left(\frac{a+b}{2}\right)$, in $R_n(x)$ given in (3.5) and use the Cauchy

Inequality, then we get

$$\begin{aligned}
&|R_n(x)| \qquad (4.2)\\
\le\ & \frac{1}{b-a}\int_a^b \left|f^{(n)}(t)-f^{(n)}\left(\frac{a+b}{2}\right)\right|\left|P^{(n)}(x,t)-\frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right| dt\\
\le\ & \frac{1}{b-a}\left[\int_a^b \left(f^{(n)}(t)-f^{(n)}\left(\frac{a+b}{2}\right)\right)^2 dt\right]^{\frac{1}{2}}\\
&\times\left[\int_a^b\left(P^{(n)}(x,t)-\frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right)^2 dt\right]^{\frac{1}{2}}.
\end{aligned}$$

Use the Diaz-Metcalf inequality [12] or [16], to get

$$\int_a^b \left(f^{(n)}(t)-f^{(n)}\left(\frac{a+b}{2}\right)\right)^2 dt \le \frac{(b-a)^2}{\pi^2}\left\|f^{(n+1)}\right\|_2^2.$$

Therefore, using the above relations, we obtain (4.1).

Corollary 4. *Substitution of $x=a$ in (4.1) gives*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left\{(-1)^k f^{(k)}(a)+\left(1+\frac{1}{2^{k+1}}+\frac{(-1)^k}{4^{k+1}}\right)f^{(k)}(b)\right\}\right.\\
&\left.+\frac{(-1)^n}{b-a}\int_a^b f(t)dt-\frac{(b-a)^{n-1}}{2^{n+1}(n+1)!}\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)(1+(-1)^n)\right|
\end{aligned}$$

$$\begin{aligned}
\le\ & \frac{b-a}{\pi}\left\|f^{(n+1)}\right\|_2\left[\frac{(b-a)^{2n+1}}{(n!)^2(2n+1)}\frac{1}{2^{2n}}\right.\\
&\left.-\frac{1}{b-a}\frac{1}{(n+1)!}\left\{\frac{(1+(-1)^n)}{2^{n+1}}(b-a)^{n+1}\right\}^2\right]^{\frac{1}{2}}. \qquad (4.3)
\end{aligned}$$

Corollary 5. *Substitution of* $x = \frac{a+b}{2}$ *in (4.1) gives*

$$\begin{aligned}
&\Bigg|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \frac{(b-a)^{k+1}}{4^{k+1}} \left\{ f^{(k)}\left(\frac{3a+b}{4}\right) + \left(1+(-1)^k\right) f^{(k)}\left(\frac{a+b}{2}\right) \right. \\
&\left. + (-1)^k f^{(k)}\left(\frac{a+3b}{4}\right) \right\} \qquad (4.4) \\
&+ \frac{(-1)^n}{b-a} \int_a^b f(t)dt - \left(f^{(n-1)}(b) - f^{(n-1)}(a)\right) \times \frac{2(b-a)^{n-1}}{4^{n+1}(n+1)!} \left(1+(-1)^n\right) \Bigg| \\
\leq\ & \frac{b-a}{\pi} \left\| f^{(n+1)} \right\|_2 \left[\frac{1}{(n!)^2 (2n+1)} \frac{4}{4^{2n+1}} (b-a)^{2n+1} \right. \\
&\left. - \frac{1}{b-a} \frac{1}{(n+1)!} \left\{ \frac{2(b-a)^{n+1}}{4^{n+1}} \left(1+(-1)^n\right) \right\}^2 \right]^{\frac{1}{2}}.
\end{aligned}$$

Corollary 6. *Substitution of* $x = \frac{3a+b}{4}$ *in (4.1) gives*

$$\begin{aligned}
&\Bigg|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \frac{(b-a)^{k+1}}{4^{k+1}} \left[\frac{\left\{1+(-1)^k\right\}}{2^{k+1}} f^{(k)}\left(\frac{7a+b}{8}\right) \right. \\
&+ (-1)^k f^{(k)}\left(\frac{3a+b}{4}\right) + f^{(k)}\left(\frac{a+3b}{4}\right) \\
&\left. + \frac{1}{2^{k+1}} \left(1+(-1)^k\right) f^{(k)}\left(\frac{a+7b}{8}\right) \right] + \frac{(-1)^n}{b-a} \int_a^b f(t)dt - \left(f^{(n-1)}(b) - f^{(n-1)}(a)\right) \\
&\times \frac{1}{(n+1)!} \frac{(b-a)^{n-1}}{4^{n+1}} \left\{ 1 + (-1)^n + \frac{1}{2^n} + \frac{(-1)^n}{2^n} \right\} \Bigg| \qquad (4.5) \\
\leq\ & \frac{b-a}{\pi} \left\| f^{(n+1)} \right\|_2 \left[\frac{1}{(n!)^2 (2n+1)} \frac{(b-a)^{2n+1}}{4^{2n+1}} \left\{ \frac{4}{2^{2n+1}} - 2 \right\} \right. \\
&\left. - \frac{1}{b-a} \frac{1}{(n+1)!} \frac{(b-a)^{n+1}}{4^{n+1}} \left\{ \left(1+(-1)^n\right) \left(2 + \frac{1}{2^{n+1}}\right) \right\}^2 \right]^{\frac{1}{2}}.
\end{aligned}$$

5 Case C: When $f^{(n)} \in L^2[a,b]$.

Theorem 6. *Let $f:[a;b]\to R$ be an n-times differentiable function on (a,b), with $f^{(n)}\in L^2[a,b]$. Then, we have*

$$\begin{aligned}
&\Bigg|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}\right. \qquad (5.1)\\
&\times f^{(k)}\left(\frac{a+x}{2}\right)+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x)\\
&+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x)\\
&\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]\\
&+\frac{(-1)^n}{b-a}\int_a^b f(t)dt-\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2}\frac{1}{(n+1)!}\\
&\times\left[\frac{1}{2^{n+1}}\left(1-(-1)^{n+1}\right)(x-a)^{n+1}+\left(1-(-1)^{n+1}\right)\left(x-\frac{3a+b}{4}\right)^{n+1}\right.\\
&\left.+\left(\frac{-1}{2^{n+1}}+\frac{(-1)^{n+1}}{2^{n+1}}+(-1)^{n+1}-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right]\Bigg|
\end{aligned}$$

$$\begin{aligned}
\leq\ &\frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a}\left[\frac{1}{(n!)^2(2n+1)}\left\{\frac{(x-a)^{2n+1}}{2^{2n}}+2\left(x-\frac{3a+b}{4}\right)^{2n+1}\right.\right.\\
&\left.-\left(\frac{1}{2^{2n}}+2\right)\left(x-\frac{a+b}{2}\right)^{2n+1}\right\}\\
&-\frac{1}{b-a}\frac{1}{(n+1)!}\left\{\frac{(x-a)^{n+1}}{2^{n+1}}(1+(-1)^n)+(1+(-1)^n)\left(x-\frac{3a+b}{4}\right)^{n+1}\right.\\
&\left.\left.+\left(\frac{-1}{2^{n+1}}-\frac{(-1)^n}{2^{n+1}}-(-1)^n-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right\}^2\right]^{\frac{1}{2}}
\end{aligned}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$, where

$$\begin{aligned}
&\sigma\left(f^{(n)}\right) \qquad (5.2)\\
=\ &\left\|f^{(n)}\right\|_2^2-\frac{\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)^2}{b-a}=\left\|f^{(n)}\right\|_2^2-k^2(b-a),
\end{aligned}$$

where S is as defined in Theorem 4.

*Proof.*Let $R_n(x)$ is defined as in (3.5). If we choose $C = \frac{1}{b-a}\int_a^b f^{(n)}(s)ds$ in (3.6) and use the Cauchy inequality and (3.5), then we get

$$
\begin{aligned}
&|R_n(x)| \\
\leq\ & \frac{1}{b-a}\int_a^b \left| f^{(n)}(t) - \frac{1}{b-a}\int_a^b f^{(n)}(s)ds \right| \left| P_n(x,t) - \frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds \right| dt \\
\leq\ & \frac{1}{b-a}\left[\int_a^b \left(f^{(n)}(t) - \frac{1}{b-a}\int_a^b f^{(n)}(s)ds \right)^2 dt \right]^{\frac{1}{2}} \\
& \times \left[\int_a^b \left(P_n(x,t) - \frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds \right)^2 dt \right]^{\frac{1}{2}} \\
=\ & \frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a}\left[\int_a^b (P_n(x,t))^2 - \frac{1}{b-a}\left(\int_a^b P^{(n)}(x,t)dt \right)^2 \right]^{\frac{1}{2}} \\
\leq\ & \frac{\sqrt{\sigma\left(f^{\ (n)}\right)}}{b-a}\left[\frac{1}{(n!)^2(2n+1)} \left\{ \frac{1}{2^{2n}}(x-a)^{2n+1} + 2\left(x - \frac{3a+b}{4}\right)^{2n+1} \right.\right. \\
& \left. - \left(\frac{1}{2^{2n}} + 2\right)\left(x - \frac{a+b}{2}\right)^{2n+1} \right\} \\
& - \frac{1}{b-a}\frac{1}{(n+1)!}\left\{ \frac{1}{2^{n+1}}\left(1+(-1)^n\right)(x-a)^{n+1} \right. \\
& + \left(1+(-1)^n\right)\left(x - \frac{3a+b}{4}\right)^{n+1} + \left(\frac{-1}{2^{n+1}} + \frac{(-1)^{n+1}}{2^{n+1}} + (-1)^{n+1} - 1 \right) \\
& \left.\left. \times \left(x - \frac{a+b}{2}\right)^{n+1} \right\}^2 \right]^{\frac{1}{2}}.
\end{aligned}
$$

Hence theorem is completed.

Corollary 7. *Substitution of $x = a$ in (5.1) gives*

$$
\begin{aligned}
&\left| \sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}} \left\{ (-1)^k f^{(k)}(a) + \left(1 + \frac{1}{2^{k+1}} + \frac{(-1)^k}{4^{k+1}}\right) f^{(k)}(b) \right\} \right. \\
& \left. + \frac{(-1)^n}{b-a}\int_a^b f(t)dt - \frac{(b-a)^{n-1}}{2^{n+1}(n+1)!}\left(f^{(n-1)}(b) - f^{(n-1)}(a) \right)\left(1+(-1)^n\right) \right|
\end{aligned}
$$

$$\leq \frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a}\left[\frac{-1}{(n!)^2(2n+1)}\frac{(b-a)^{2n+1}}{4^{3n+1}}\right.$$
$$\left.-\frac{1}{b-a}\frac{1}{(n+1)!}\left\{\frac{(b-a)^{n+1}}{2^{n+1}}\left(1+(-1)^n\right)\right\}^2\right]^{\frac{1}{2}}. \quad (5.3)$$

Corollary 8. *Substitution of* $x=\frac{a+b}{2}$ *in (5.1) gives*

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{4^{k+1}}\left\{f^{(k)}\left(\frac{3a+b}{4}\right)+\left(1+(-1)^k\right)f^{(k)}\left(\frac{a+b}{2}\right)\right.\right.$$
$$\left.+(-1)^k f^{(k)}\left(\frac{a+3b}{4}\right)\right\}+\frac{(-1)^n}{b-a}\int_a^b f(t)dt$$
$$\left.-\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)\frac{2(b-a)^{n-1}}{4^{n+1}(n+1)!}\left(1+(-1)^n\right)\right| \quad (5.4)$$

$$\leq \frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a}\left[\frac{1}{(n!)^2(2n+1)}\frac{(b-a)^{2n+1}}{2^{2n}}\left(1+\frac{1}{2^{2n+1}}\right)\right.$$
$$\left.-\frac{1}{b-a}\frac{1}{(n+1)!}\left\{\frac{2(b-a)^{n+1}}{4^{n+1}}\left(1+(-1)^n\right)\right\}^2\right]^{\frac{1}{2}}.$$

Corollary 9. *Substitution of* $x=\frac{3a+b}{4}$ *in (5.1) gives*

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{4^{k+1}}\left[\frac{\left(1+(-1)^k\right)}{2^{k+1}}f^{(k)}\left(\frac{7a+b}{8}\right)\right.\right.$$
$$+(-1)^k f^{(k)}\left(\frac{3a+b}{4}\right)+f^{(k)}\left(\frac{a+3b}{4}\right)$$
$$\left.+\frac{1}{2^{k+1}}\left(1+(-1)^k\right)f^{(k)}\left(\frac{a+7b}{8}\right)\right]+\frac{(-1)^n}{b-a}\int_a^b f(t)dt \quad (5.5)$$
$$-\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)\cdot\frac{1}{(n+1)!}\frac{(b-a)^{n-1}}{4^{n+1}}$$
$$\left.\times\left(1+(-1)^n+\frac{1}{2^n}+\frac{(-1)^n}{2^n}\right)\right|$$

$$\leq \frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a}\left[\frac{1}{(n!)^2(2n+1)}\frac{2(b-a)^{2n+1}}{4^{2n+1}}\left(\frac{1}{2^{2n}}+1\right)\right.$$
$$\left.-\frac{1}{b-a}\frac{1}{(n+1)!}\left\{\frac{(b-a)^{n+1}}{4^{n+1}}\left(1+(-1)^n\right)\left(1+\frac{1}{2^n}\right)\right\}^2\right]^{\frac{1}{2}}.$$

Remark 2. By choosing $n = 1$ in case A, B and C, we get all results obtained in [10].

Remark 3. By choosing $n = 2$ in case A, B and C, we get all results obtained in [9].

6 Derivation of Numerical Quadrature Rules

We propose some new quadrature rules involving higher order derivatives of the function f. In fact, the following new quadrature rules can be obtained while investigating error bounds using theorem 5.

$$Q_{n,1}(f) := \int_a^b f(t)dt$$
$$\approx \sum_{k=0}^{n-1}\frac{(b-a)^{k+2}}{2^{k+1}(k+1)!}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]$$
$$+\left[f^{(n-1)}(b)-f^{(n-1)}(a)\right]\frac{(b-a)^n}{2^{n+1}(n+1)!}\left(1+(-1)^n\right)$$

$$Q_{n,2}(f) := \int_a^b f(t)dt$$
$$\approx \sum_{k=0}^{n-1}\frac{(b-a)^{k+2}(-1)^k}{4^{k+1}(k+1)!}\left[f^{(k)}\left(\frac{3a+b}{4}\right)\right.$$
$$\left.+\left\{1+(-1)^k\right\}f^{(k)}\left(\frac{a+b}{2}\right)+(-1)^k f^{(k)}\left(\frac{a+3b}{4}\right)\right]$$
$$+\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)\times\frac{2}{4^{n+1}}\frac{(b-a)^n}{(n+1)!}\left((-1)^n+1\right)$$

$$Q_{n,3}(f) := \int_a^b f(t)dt$$

$$\approx \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!} \frac{(b-a)^{k+2}}{4^{k+1}} \left[\frac{1}{2^{k+1}} \left(1+(-1)^k\right) \left(f^{(k)}\left(\frac{7a+b}{8}\right) + f^{(k)}\left(\frac{a+7b}{8}\right)\right) + \left\{ (-1)^k f^{(k)}\left(\frac{3a+b}{4}\right) + f^{(k)}\left(\frac{a+3b}{4}\right)\right\}\right] + \left[f^{(n-1)}(b) - f^{(n-1)}(a)\right] \times \frac{(b-a)^n}{4^{n+1}(n+1)!}\left((-1)^n+1\right)\left(\frac{1}{2^n}+1\right)$$

Performance of the efficient quadrature rules

	Method	$n:\ Q_{n,1}(f)$	$n:\ Q_{n,2}(f)$	$n:\ Q_{n,3}(f)$	Exact Value
1.	$\int_0^1 f_1(x)dx$	**2:** 2.83333	**2:** 2.83333	**2:** 2.83333	2.83333
	Error:	0	0	0	
2.	$\int_0^1 f_2(x)dx$	**6:** 0.30117	**4:** 0.301172	**4:** 0.30117	0.301169
	Error:	1.1381×10^{-6}	3.08726×10^{-6}	1.38925×10^{-6}	
3.	$\int_0^1 f_3(x)dx$	**6:** 0.909328	**4:** 0.909324	**4:** 0.909327	0.909331
	Error:	2.33999×10^{-6}	7.13925×10^{-6}	3.21638×10^{-6}	
4.	$\int_0^1 f_4(x)dx$	**5:** 0.793022	**4:** 0.793031	**4:** 0.793031	0.793031
	Error:	8.63182×10^{-6}	2.9641×10^{-7}	1.33626×10^{-7}	
5.	$\int_0^1 f_5(x)dx$	**11:** 1.46266	**7:** 1.46265	**6:** 1.46266	1.46265
	Error:	5.8789×10^{-6}	2.29707×10^{-6}	5.20247×10^{-6}	
6.	$\int_0^1 f_6(x)dx$	**11:** 1.31384	**6:** 1.31383	**6:** 1.31383	1.31383
	Error	7.37624×10^{-6}	2.13363×10^{-6}	1.73918×10^{-6}	
7.	$\int_0^1 f_7(x)dx$	**6:** 1.34146	**4:** 1.34137	**4:** 1.34147	1.34147
	Error:	1.4808×10^{-7}	5.42574×10^{-7}	2.44601×10^{-7}	
8.	$\int_0^1 f_8(x)dx$	**9:** 0.62977	**5:** 0.629762	**4:** 0.629774	0.629769
	Error:	1.18074×10^{-6}	6.3567×10^{-6}	5.647×10^{-6}	

Table: 1

$$\begin{aligned} f_1(x) &= x^2+x+2, & f_2(x) &= x\sin x, \\ f_3(x) &= e^x \sin x\ , & f_4(x) &= x^2+\sin x, \\ f_5(x) &= e^{x^2}, & f_6(x) &= e^x\cos\left(e^x-2x\right), \\ f_7(x) &= x+\cos x, & f_8(x) &= \log\left(x^2+2\right)\sin\left[\log\left(x^2+2\right)\right]. \end{aligned} \tag{6.1}$$

From the above table, we observe that all three quadrature rules show exact value of the integral of f_1 for $n = 2$. For any polynomial of degree k, $n = k+1$ will give exact value of the integral f_1. Acceptable error estimates can be obtained for smaller values of n to save computational time.

The integral of f_5 , $Q_{n,3}(f)$ report an error of the order of 10^{-6} for $n = 6$ while the other two quadrature rules give a similar error for $n = 7$ and $n = 11$. Similarly for all other functions $Q_{n,3}(f)$ report errors of the order of 10^{-6} or 10^{-7} for relatively smaller values of n as compared to the other two quadrature rules. Specifically, $Q_{n,3}(f)$ give an excellent estimate for the integrals of f_5 and f_8 at $n = 6$ and $n = 4$ respectively. In general $Q_{n,3}(f)$ gave better results as compared to the rest of the quadrature rules for much smaller values of n. Therefore we can conclude that overall $Q_{n,3}(f)$ is computationally more efficient both in terms of error approximation, simplicity, and time. As a rough estimate we integrated $\log\left(x^2+2\right)\sin\left[\log\left(x^2+2\right)\right]$ using the built in algorithms of Mathematica 10.0 which took 26.30 seconds to give its approximate answer. To obtain similar approximation for the integral of f_8, $Q_{n,3}(f)$ took less than a second.

Based on this analysis, we can conjecture that $Q_{n,3}(f)$ is the most efficient quadrature rule, while $Q_{n,2}(f)$ comes second in terms of performance. It should be noted that if desired the value of n can be adjusted to improve the error bounds or decrease computational time.

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