

**ON JENSEN'S ADDITIVE INEQUALITY FOR POSITIVE
CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN
HILBERT SPACES**

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ABSTRACT. In this paper we obtain some additive refinements and reverses of Jensen's inequality for positive convex/concave functions of selfadjoint operators in Hilbert spaces. Natural applications for power and exponential functions are provided.

1. INTRODUCTION

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0, 1]$, then

$$(1.1) \quad a^{1-\nu}b^\nu \leq (1-\nu)a + \nu b$$

with equality if and only if $a = b$. The inequality (1.2) is also called ν -*weighted arithmetic-geometric mean inequality*.

We have the following inequality that provides a refinement and a reverse for the celebrated Young's inequality

$$(1.2) \quad \frac{1}{2}\nu(1-\nu) \frac{(b-a)^2}{\max\{a, b\}} \leq (1-\nu)a + \nu b - a^{1-\nu}b^\nu \leq \frac{1}{2}\nu(1-\nu) \frac{(b-a)^2}{\min\{a, b\}}$$

for any $a, b > 0$ and $\nu \in [0, 1]$.

This result was obtained in 1978 by Cartwright and Field [2] who established a more general result for n variables and gave an application for a probability measure supported on a finite interval.

We observe that, if $a, b \in [\gamma, \Gamma] \subset (0, \infty)$, then from (1.2) we have

$$(1.3) \quad \frac{1}{2\Gamma}\nu(1-\nu)(b-a)^2 \leq (1-\nu)a + \nu b - a^{1-\nu}b^\nu \leq \frac{1}{2\gamma}\nu(1-\nu)(b-a)^2$$

for $\nu \in [0, 1]$.

Moreover, since

$$\frac{(b-a)^2}{\max\{a, b\}} = \min\{a, b\} \frac{(b-a)^2}{ab} = \min\{a, b\} \left(\frac{b}{a} + \frac{a}{b} - 2 \right)$$

and

$$\frac{(b-a)^2}{\min\{a, b\}} = \max\{a, b\} \frac{(b-a)^2}{ab} = \max\{a, b\} \left(\frac{b}{a} + \frac{a}{b} - 2 \right),$$

1991 *Mathematics Subject Classification*. 47A63, 47A30, 15A60, 26D15, 26D10 .

Key words and phrases. Young's Inequality, Convex functions, Jensen's inequality, Selfadjoint operator, Functions of selfadjoint operators.

then from (1.2) we have the following inequality as well

$$(1.4) \quad \frac{1}{2}\nu(1-\nu)\gamma\left(\frac{b}{a} + \frac{a}{b} - 2\right) \leq (1-\nu)a + \nu b - a^{1-\nu}b^\nu \\ \leq \frac{1}{2}\nu(1-\nu)\Gamma\left(\frac{b}{a} + \frac{a}{b} - 2\right)$$

for any $a, b \in [\gamma, \Gamma] \subset (0, \infty)$ and $\nu \in [0, 1]$.

We recall that *Specht's ratio* is defined by [15]

$$(1.5) \quad S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} & \text{if } h \in (0, 1) \cup (1, \infty), \\ 1 & \text{if } h = 1. \end{cases}$$

It is well known that $\lim_{h \rightarrow 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0, 1)$ and increasing on $(1, \infty)$.

In [16], Tominaga also proved the following additive reverse of Young's inequality

$$(1.6) \quad (1-\nu)a + \nu b - a^{1-\nu}b^\nu \leq S\left(\frac{a}{b}\right)L(a, b)$$

where $a, b > 0$, $\nu \in [0, 1]$ and $L(a, b)$ is the logarithmic mean, namely

$$L(a, b) := \begin{cases} \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \\ b & \text{if } a = b. \end{cases}$$

If for positive numbers a, b we have $a, b \in [\gamma, \Gamma] \subset (0, \infty)$ and $\nu \in [0, 1]$, then by (1.6) we get [16]

$$(1.7) \quad (1-\nu)a + \nu b - a^{1-\nu}b^\nu \leq S\left(\frac{\Gamma}{\gamma}\right)L\left(1, \frac{\Gamma}{\gamma}\right)$$

Kittaneh and Manasrah [10], [11] provided a refinement and an additive reverse for Young inequality as follows:

$$(1.8) \quad r\left(\sqrt{a} - \sqrt{b}\right)^2 \leq (1-\nu)a + \nu b - a^{1-\nu}b^\nu \leq R\left(\sqrt{a} - \sqrt{b}\right)^2$$

where $a, b > 0$, $\nu \in [0, 1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$. The case $\nu = \frac{1}{2}$ reduces (1.8) to an identity.

If $a, b \in [\gamma, \Gamma] \subset (0, \infty)$, then $\left|\sqrt{a} - \sqrt{b}\right| \leq \sqrt{\Gamma} - \sqrt{\gamma}$ and by (1.8) we get

$$(1.9) \quad (1-\nu)a + \nu b - a^{1-\nu}b^\nu \leq R\left(\sqrt{\Gamma} - \sqrt{\gamma}\right)^2.$$

In the recent paper [5] we obtained the following reverses of Young's inequality as well:

$$(1.10) \quad (1-\nu)a + \nu b - a^{1-\nu}b^\nu \leq \nu(1-\nu)(a-b)(\ln a - \ln b)$$

where $a, b > 0$, $\nu \in [0, 1]$.

Observe that for $a, b \in [\gamma, \Gamma] \subset (0, \infty)$ we have

$$0 \leq (a-b)(\ln a - \ln b) = |a-b||\ln a - \ln b| \leq (\Gamma - \gamma)(\ln \Gamma - \ln \gamma)$$

and by (1.10) we get

$$(1.11) \quad (1-\nu)a + \nu b - a^{1-\nu}b^\nu \leq \nu(1-\nu)(\Gamma - \gamma)(\ln \Gamma - \ln \gamma).$$

For any $a, b > 0$ and $\nu \in [0, 1]$ we have [6]

$$(1.12) \quad \begin{aligned} \frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \min\{a, b\} &\leq (1-\nu)a + \nu b - a^{1-\nu}b^\nu \\ &\leq \frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \max\{a, b\} \end{aligned}$$

This inequality was obtained in the case $a < b$ in [1] as well.

If $a, b \in [\gamma, \Gamma] \subset (0, \infty)$, then by (1.12) we get

$$(1.13) \quad \begin{aligned} \frac{1}{2}\nu(1-\nu)\gamma(\ln a - \ln b)^2 &\leq (1-\nu)a + \nu b - a^{1-\nu}b^\nu \\ &\leq \frac{1}{2}\nu(1-\nu)\Gamma(\ln a - \ln b)^2 \end{aligned}$$

for any $\nu \in [0, 1]$.

If $a, b \in [\gamma, \Gamma] \subset (0, \infty)$ and $\nu \in [0, 1]$, then we have [7]

$$(1.14) \quad (1-\nu)a + \nu b - a^{1-\nu}b^\nu \leq \max\{\iota_{\gamma, \Gamma}(\nu), \iota_{\gamma, \Gamma}(1-\nu)\}$$

where

$$(1.15) \quad \iota_{\gamma, \Gamma}(\nu) := (1-\nu)\gamma + \nu\Gamma - \gamma^{1-\nu}\Gamma^\nu.$$

We consider the function $f_\nu : [0, \infty) \rightarrow [0, \infty)$ defined for $\nu \in (0, 1)$ by

$$f_\nu(x) = 1 - \nu + \nu x - x^\nu.$$

For $[k, K] \subset [0, \infty)$ define

$$(1.16) \quad \Delta_\nu(k, K) := \begin{cases} f_\nu(k) & \text{if } K < 1, \\ \max\{f_\nu(k), f_\nu(K)\} & \text{if } k \leq 1 \leq K, \\ f_\nu(K) & \text{if } 1 < k \end{cases}$$

and

$$(1.17) \quad \delta_\nu(k, K) := \begin{cases} f_\nu(K) & \text{if } K < 1, \\ 0 & \text{if } k \leq 1 \leq K, \\ f_\nu(k) & \text{if } 1 < k. \end{cases}$$

In the recent paper [8] we obtained the following refinement and reverse for the *additive Young's inequality*:

$$(1.18) \quad \delta_\nu(k, K)a \leq (1-\nu)a + \nu b - a^{1-\nu}b^\nu \leq \Delta_\nu(k, K)a,$$

for positive numbers a, b with $\frac{b}{a} \in [k, K] \subset (0, \infty)$ and $\nu \in [0, 1]$ where $\Delta_\nu(k, K)$ and $\delta_\nu(k, K)$ are defined by (1.16) and (1.17) respectively.

Now, if $a, b \in [\gamma, \Gamma] \subset (0, \infty)$ and $\nu \in [0, 1]$, then $\frac{b}{a} \in \left[\frac{\gamma}{\Gamma}, \frac{\Gamma}{\gamma}\right]$ and by (1.18) we have

$$(1.19) \quad (1-\nu)a + \nu b - a^{1-\nu}b^\nu \leq \max\left\{f_\nu\left(\frac{\gamma}{\Gamma}\right), f_\nu\left(\frac{\Gamma}{\gamma}\right)\right\}a,$$

and since

$$f_\nu\left(\frac{\gamma}{\Gamma}\right) = \frac{(1-\nu)\Gamma + \nu\gamma - \gamma^\nu\Gamma^{1-\nu}}{\Gamma}$$

and

$$f_\nu \left(\frac{\Gamma}{\gamma} \right) = \frac{(1-\nu)\gamma + \nu\Gamma - \Gamma^\nu \gamma^{1-\nu}}{\gamma}$$

then by (1.19) we get

$$(1.20) \quad (1-\nu)a + \nu b - a^{1-\nu}b^\nu \\ \leq \max \left\{ \frac{(1-\nu)\Gamma + \nu\gamma - \gamma^\nu \Gamma^{1-\nu}}{\Gamma}, \frac{(1-\nu)\gamma + \nu\Gamma - \Gamma^\nu \gamma^{1-\nu}}{\gamma} \right\} a,$$

for any $a, b \in [\gamma, \Gamma] \subset (0, \infty)$ and $\nu \in [0, 1]$.

The following result that provides a vector operator version for the Jensen inequality is well known, see for instance [13] or [14, p. 5]:

Theorem 1. *Let A be a selfadjoint operator on the Hilbert space H and assume that $\text{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$(1.21) \quad f(\langle Ax, x \rangle) \leq \langle f(A)x, x \rangle$$

for each $x \in H$ with $\|x\| = 1$.

As a special case of Theorem 1 we have the *Hölder-McCarthy inequality* [12]: Let A be a selfadjoint positive operator on a Hilbert space H , then

- (i) $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r$ for all $r > 1$ and $x \in H$ with $\|x\| = 1$;
- (ii) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r$ for all $0 < r < 1$ and $x \in H$ with $\|x\| = 1$;
- (iii) If A is invertible, then $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r$ for all $r < 0$ and $x \in H$ with $\|x\| = 1$.

In [3] (see also [4, p. 16]) we obtained the following additive reverse of (1.21):

Theorem 2. *Let I be an interval and $f : I \rightarrow \mathbb{R}$ be a convex and differentiable function on \dot{I} (the interior of I) whose derivative f' is continuous on \dot{I} . If A is a selfadjoint operators on the Hilbert space H with $\text{Sp}(A) \subset \dot{I}$, then*

$$(1.22) \quad (0 \leq) \langle f(A)x, x \rangle - f(\langle Ax, x \rangle) \leq \langle f'(A)Ax, x \rangle - \langle Ax, x \rangle \langle f'(A)x, x \rangle$$

for any $x \in H$ with $\|x\| = 1$.

In the recent paper [9] we established the following multiplicative inequalities:

Theorem 3. *Let $f : [m, M] \rightarrow [0, \infty)$ be a continuous function and assume that*

$$(1.23) \quad 0 < \gamma = \min_{t \in [m, M]} f(t) < \max_{t \in [m, M]} f(t) = \Gamma < \infty.$$

Then for any A a selfadjoint operator with

$$(1.24) \quad m1_H \leq A \leq M1_H,$$

we have the inequality

$$(1.25) \quad \frac{(1-\nu)f(\langle Ax, x \rangle) + \nu \langle f(A)x, x \rangle}{\langle f^\nu(A)x, x \rangle f^{1-\nu}(\langle Ax, x \rangle)} \leq \exp \left[\frac{1}{2} \nu (1-\nu) \left(\frac{\Gamma}{\gamma} - 1 \right)^2 \right]$$

for any $x \in H$ with $\|x\| = 1$, where $\nu \in [0, 1]$.

Moreover, if f is convex on $[m, M]$, then for any $\nu \in [0, 1]$,

$$(1.26) \quad \frac{f^\nu(\langle Ax, x \rangle)}{\langle f^\nu(A)x, x \rangle} \leq \exp \left[\frac{1}{2} \nu (1-\nu) \left(\frac{\Gamma}{\gamma} - 1 \right)^2 \right]$$

while, if f is concave on $[m, M]$, then

$$(1.27) \quad \frac{\langle f(A)x, x \rangle}{f(\langle Ax, x \rangle)} \leq \exp \left[\frac{1}{2} \nu (1 - \nu) \left(\frac{\Gamma}{\gamma} - 1 \right)^2 \right] \frac{\langle f^\nu(A)x, x \rangle}{f^\nu(\langle Ax, x \rangle)}.$$

For some meaningful examples of functions satisfying the above inequalities (1.26)-(1.27), see [9].

Motivated by the above results, in this paper we obtain several additive refinements and reverses of Jensen's inequality for positive convex/concave functions of selfadjoint operators in Hilbert spaces. Natural applications for power functions are provided.

2. UPPER BOUNDS

By using the definition of $\iota_{\gamma, \Gamma}(\nu)$ from (1.15), we define for $\gamma, \Gamma \in (0, \infty)$ and $\nu \in [0, 1]$, the function

$$(2.1) \quad \begin{aligned} \varphi(\gamma, \Gamma, \nu) &:= \max \{ \iota_{\gamma, \Gamma}(\nu), \iota_{\gamma, \Gamma}(1 - \nu) \} \\ &= \max \{ (1 - \nu)\gamma + \nu\Gamma - \gamma^{1-\nu}\Gamma^\nu, \nu\gamma + (1 - \nu)\Gamma - \gamma^\nu\Gamma^{1-\nu} \}. \end{aligned}$$

We observe that

$$\varphi(\gamma, \Gamma, \nu) = \varphi(\Gamma, \gamma, \nu) = \varphi(\gamma, \Gamma, 1 - \nu)$$

for $\gamma, \Gamma \in (0, \infty)$ and $\nu \in [0, 1]$.

Using the inequality (1.3) for $(a, b) = (\gamma, \Gamma)$ and $(a, b) = (\Gamma, \gamma)$ we get

$$(2.2) \quad \varphi(\gamma, \Gamma, \nu) \leq \frac{1}{2\gamma} \nu (1 - \nu) (\Gamma - \gamma)^2$$

for any $0 < \gamma < \Gamma$ and $\nu \in [0, 1]$.

From Tominaga's inequality (1.6) we have

$$\varphi(\gamma, \Gamma, \nu) \leq S \left(\frac{\Gamma}{\gamma} \right) L(\gamma, \Gamma)$$

while from (1.7) we have

$$\varphi(\gamma, \Gamma, \nu) \leq S \left(\frac{\Gamma}{\gamma} \right) L \left(1, \frac{\Gamma}{\gamma} \right) = \frac{1}{\gamma} S \left(\frac{\Gamma}{\gamma} \right) L(\gamma, \Gamma)$$

giving that

$$(2.3) \quad \varphi(\gamma, \Gamma, \nu) \leq S \left(\frac{\Gamma}{\gamma} \right) L(\gamma, \Gamma) \begin{cases} 1 & \text{if } \gamma \leq 1, \\ \frac{1}{\gamma} & \text{if } \gamma > 1, \end{cases}$$

for any $0 < \gamma < \Gamma$ and $\nu \in [0, 1]$.

From (1.9) we have

$$(2.4) \quad \varphi(\gamma, \Gamma, \nu) \leq \max \{ \nu, 1 - \nu \} \left(\sqrt{\Gamma} - \sqrt{\gamma} \right)^2$$

for any $0 < \gamma < \Gamma$ and $\nu \in [0, 1]$, while from (1.10) we have

$$(2.5) \quad \varphi(\gamma, \Gamma, \nu) \leq \nu (1 - \nu) (\Gamma - \gamma) (\ln \Gamma - \ln \gamma)$$

for any $0 < \gamma < \Gamma$ and $\nu \in [0, 1]$.

From (1.13) we also have

$$(2.6) \quad \varphi(\gamma, \Gamma, \nu) \leq \frac{1}{2} \nu (1 - \nu) \Gamma (\ln \Gamma - \ln \gamma)^2$$

for any $0 < \gamma < \Gamma$ and $\nu \in [0, 1]$.

Therefore, by the inequality (1.14), we can state that for any $a, b \in [\gamma, \Gamma] \subset (0, \infty)$ (with $\gamma < \Gamma$) we have the following reverse of Young's inequality

$$(2.7) \quad (1 - \nu) a + \nu b - a^{1-\nu} b^\nu \leq \varphi(\gamma, \Gamma, \nu) \leq \Phi(\gamma, \Gamma, \nu)$$

for any $\nu \in [0, 1]$, where, as pointed out above, the upper bound $\Phi(\gamma, \Gamma, \nu)$ can be one of the right hand side of the inequalities (2.2)-(2.6), namely

$$(2.8) \quad \Phi(\gamma, \Gamma, \nu) := \frac{1}{2} \nu (1 - \nu) (\Gamma - \gamma)^2, S\left(\frac{\Gamma}{\gamma}\right) L(\gamma, \Gamma) \begin{cases} 1 & \text{if } \gamma \leq 1, \\ \frac{1}{\gamma} & \text{if } \gamma > 1, \end{cases} \\ \max\{\nu, 1 - \nu\} \left(\sqrt{\Gamma} - \sqrt{\gamma}\right)^2, \nu (1 - \nu) (\Gamma - \gamma) (\ln \Gamma - \ln \gamma), \\ \frac{1}{2} \nu (1 - \nu) \Gamma (\ln \Gamma - \ln \gamma)^2,$$

for any $0 < \gamma < \Gamma$ and $\nu \in [0, 1]$.

Theorem 4. *Let $f : [m, M] \rightarrow [0, \infty)$ be a continuous function and assume that it satisfies the condition (1.23). Then for any A a selfadjoint operator with the property (1.24), we have the inequality*

$$(2.9) \quad 0 \leq (1 - \nu) f(\langle Ax, x \rangle) + \nu \langle f(A)x, x \rangle - f^{1-\nu}(\langle Ax, x \rangle) \langle f^\nu(A)x, x \rangle \\ \leq \varphi(\gamma, \Gamma, \nu) \leq \Phi(\gamma, \Gamma, \nu)$$

for any $x \in H$ with $\|x\| = 1$.

Moreover, if f is convex on $[m, M]$, then

$$(2.10) \quad f^{1-\nu}(\langle Ax, x \rangle) [f^\nu(\langle Ax, x \rangle) - \langle f^\nu(A)x, x \rangle] \leq \varphi(\gamma, \Gamma, \nu) \leq \Phi(\gamma, \Gamma, \nu)$$

for any $\nu \in [0, 1]$ and for any $x \in H$ with $\|x\| = 1$.

Proof. Let $t \in [m, M]$ and $x \in H$ with $\|x\| = 1$. Then $f(t), f(\langle Ax, x \rangle) \in [\gamma, \Gamma]$ and by (2.7) we have

$$(2.11) \quad (1 - \nu) f(\langle Ax, x \rangle) + \nu f(t) - f^{1-\nu}(\langle Ax, x \rangle) f^\nu(t) \leq \varphi(\gamma, \Gamma, \nu) \leq \Phi(\gamma, \Gamma, \nu)$$

for any $t \in [m, M]$, $\nu \in [0, 1]$ and $x \in H$ with $\|x\| = 1$.

If we use the functional calculus for the operator A with $m1_H \leq A \leq M1_H$, then by (2.11) we get

$$(2.12) \quad (1 - \nu) f(\langle Ax, x \rangle) 1_H + \nu f(A) - f^{1-\nu}(\langle Ax, x \rangle) f^\nu(A) \\ \leq \varphi(\gamma, \Gamma, \nu) 1_H \leq \Phi(\gamma, \Gamma, \nu) 1_H$$

for any $\nu \in [0, 1]$ and $x \in H$ with $\|x\| = 1$.

If we take in (2.12) the inner product over $y \in H$ with $\|y\| = 1$, then we get

$$(2.13) \quad (1 - \nu) f(\langle Ax, x \rangle) + \nu \langle f(A)y, y \rangle - f^{1-\nu}(\langle Ax, x \rangle) \langle f^\nu(A)y, y \rangle \\ \leq \varphi(\gamma, \Gamma, \nu) \leq \Phi(\gamma, \Gamma, \nu)$$

for any $\nu \in [0, 1]$, which by taking $y = x$, we get the desired inequality (2.9).

If f is convex on $[m, M]$, then by Jensen's inequality we have $f(\langle Ax, x \rangle) \leq \langle f(A)x, x \rangle$, for $x \in H$ with $\|x\| = 1$, then

$$\begin{aligned} & f(\langle Ax, x \rangle) - f^{1-\nu}(\langle Ax, x \rangle) \langle f^\nu(A)x, x \rangle \\ & \leq (1-\nu) f(\langle Ax, x \rangle) + \nu \langle f(A)x, x \rangle - f^{1-\nu}(\langle Ax, x \rangle) \langle f^\nu(A)x, x \rangle \\ & \leq \varphi(\gamma, \Gamma, \nu) \leq \Phi(\gamma, \Gamma, \nu), \end{aligned}$$

which implies the desired result (2.10). \square

Remark 1. *If for some $\nu \in (0, 1)$ the function f^ν is concave, then $\langle f^\nu(A)x, x \rangle \leq f^\nu(\langle Ax, x \rangle)$ for any $x \in H$ with $\|x\| = 1$, then by (2.10) we have the meaningful inequality*

$$(2.14) \quad 0 \leq f^{1-\nu}(\langle Ax, x \rangle) [f^\nu(\langle Ax, x \rangle) - \langle f^\nu(A)x, x \rangle] \leq \varphi(\gamma, \Gamma, \nu) \leq \Phi(\gamma, \Gamma, \nu)$$

for any $x \in H$ with $\|x\| = 1$.

If we consider the convex function $f(t) = t^r$, $r \geq 1$ and take $\nu \in (0, 1)$ with $r\nu < 1$, then the function f^ν is concave and by (2.14) we get for any $x \in H$ with $\|x\| = 1$ that

$$(2.15) \quad \begin{aligned} 0 & \leq \langle Ax, x \rangle^{(1-\nu)r} [\langle Ax, x \rangle^{\nu r} - \langle A^{\nu r}x, x \rangle] \leq \varphi(m^r, M^r, \nu) \\ & \leq \Phi(m^r, M^r, \nu), \end{aligned}$$

where $m1_H \leq A \leq M1_H$ with $0 < m < M$. Since $\langle Ax, x \rangle^{(1-\nu)r} \geq m^{(1-\nu)r}$ for any $x \in H$ with $\|x\| = 1$, then by (2.15) we get the following additive reverse of the Hölder-McCarthy inequality

$$(2.16) \quad \begin{aligned} (0 \leq) \langle Ax, x \rangle^{\nu r} - \langle A^{\nu r}x, x \rangle & \leq \frac{1}{m^{(1-\nu)r}} \varphi(m^r, M^r, \nu) \\ & \leq \frac{1}{m^{(1-\nu)r}} \Phi(m^r, M^r, \nu) \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

3. SOME INEQUALITIES VIA CARTWRIGHT-FIELD RESULT

We have:

Theorem 5. *Let $f : [m, M] \rightarrow [0, \infty)$ be a continuous function and assume that it satisfies the condition (1.23). Then for any A a selfadjoint operator with the property (1.24), we have the inequality*

$$(3.1) \quad \begin{aligned} (0 \leq) & \frac{1}{2\Gamma} \nu (1-\nu) (\langle f(A)x, x \rangle - f(\langle Ax, x \rangle))^2 \\ & \leq \frac{1}{2\Gamma} \nu (1-\nu) (\langle f^2(A)x, x \rangle - 2f(\langle Ax, x \rangle) \langle f(A)x, x \rangle + f^2(\langle Ax, x \rangle)) \\ & \leq (1-\nu) f(\langle Ax, x \rangle) + \nu \langle f(A)x, x \rangle - f^{1-\nu}(\langle Ax, x \rangle) \langle f^\nu(A)x, x \rangle \\ & \leq \frac{1}{2\gamma} \nu (1-\nu) (\langle f^2(A)x, x \rangle - 2f(\langle Ax, x \rangle) \langle f(A)x, x \rangle + f^2(\langle Ax, x \rangle)) \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Moreover, if f is convex on $[m, M]$, then

$$(3.2) \quad \begin{aligned} (0 \leq) & \frac{1}{2\Gamma} \nu (1-\nu) (\langle f(A)x, x \rangle - f(\langle Ax, x \rangle))^2 \\ & \leq \langle f(A)x, x \rangle - f^{1-\nu}(\langle Ax, x \rangle) \langle f^\nu(A)x, x \rangle \end{aligned}$$

and

$$(3.3) \quad f^{1-\nu}(\langle Ax, x \rangle) [f^\nu(\langle Ax, x \rangle) - \langle f^\nu(A)x, x \rangle] \\ \leq \frac{1}{2\gamma} \nu(1-\nu) (\langle f^2(A)x, x \rangle - 2f(\langle Ax, x \rangle) \langle f(A)x, x \rangle + f^2(\langle Ax, x \rangle))$$

for any $\nu \in [0, 1]$ and for any $x \in H$ with $\|x\| = 1$.

If f is concave on $[m, M]$, then

$$(3.4) \quad (0 \leq) \frac{1}{2\Gamma} \nu(1-\nu) (f(\langle Ax, x \rangle) - \langle f(A)x, x \rangle)^2 \\ \leq f^{1-\nu}(\langle Ax, x \rangle) [f^\nu(\langle Ax, x \rangle) - \langle f^\nu(A)x, x \rangle]$$

for any $\nu \in [0, 1]$ and for any $x \in H$ with $\|x\| = 1$.

Proof. Let $t \in [m, M]$ and $x \in H$ with $\|x\| = 1$. Then $f(t), f(\langle Ax, x \rangle) \in [\gamma, \Gamma]$ and by (1.3) we have

$$(3.5) \quad 0 \leq \frac{1}{2\Gamma} \nu(1-\nu) (f^2(t) - 2f(\langle Ax, x \rangle) f(t) + f^2(\langle Ax, x \rangle)) \\ \leq (1-\nu) f(\langle Ax, x \rangle) + \nu f(t) - f^{1-\nu}(\langle Ax, x \rangle) f^\nu(t) \\ \leq \frac{1}{2\gamma} \nu(1-\nu) (f^2(t) - 2f(\langle Ax, x \rangle) f(t) + f^2(\langle Ax, x \rangle))$$

for any $t \in [m, M]$ and $x \in H$ with $\|x\| = 1$.

If we use the functional calculus for the operator A with $m1_H \leq A \leq M1_H$, then by (3.5) we get

$$(3.6) \quad 0 \leq \frac{1}{2\Gamma} \nu(1-\nu) (f^2(A) - 2f(\langle Ax, x \rangle) f(A) + f^2(\langle Ax, x \rangle) 1_H) \\ \leq (1-\nu) f(\langle Ax, x \rangle) 1_H + \nu f(A) - f^{1-\nu}(\langle Ax, x \rangle) f^\nu(A) \\ \leq \frac{1}{2\gamma} \nu(1-\nu) (f^2(A) - 2f(\langle Ax, x \rangle) f(A) + f^2(\langle Ax, x \rangle) 1_H)$$

for any $x \in H$ with $\|x\| = 1$.

If we take the inner product in (3.6) over $y \in H$ with $\|y\| = 1$, then we get

$$(3.7) \quad 0 \leq \frac{1}{2\Gamma} \nu(1-\nu) (\langle f^2(A)y, y \rangle - 2f(\langle Ax, x \rangle) \langle f(A)y, y \rangle + f^2(\langle Ax, x \rangle)) \\ \leq (1-\nu) f(\langle Ax, x \rangle) + \nu \langle f(A)y, y \rangle - f^{1-\nu}(\langle Ax, x \rangle) \langle f^\nu(A)y, y \rangle \\ \leq \frac{1}{2\gamma} \nu(1-\nu) (\langle f^2(A)y, y \rangle - 2f(\langle Ax, x \rangle) \langle f(A)y, y \rangle + f^2(\langle Ax, x \rangle)),$$

which by taking $y = x$, produces the second, third and fourth inequalities in (3.1).

By Hölder-McCarthy inequality we have

$$\langle f^2(A)x, x \rangle \geq \langle f(A)x, x \rangle^2,$$

for any $x \in H$ with $\|x\| = 1$, which implies that

$$\langle f^2(A)x, x \rangle - 2f(\langle Ax, x \rangle) \langle f(A)x, x \rangle + f^2(\langle Ax, x \rangle) \\ \geq \langle f(A)x, x \rangle^2 - 2f(\langle Ax, x \rangle) \langle f(A)x, x \rangle + f^2(\langle Ax, x \rangle) \\ = (\langle f(A)x, x \rangle - f(\langle Ax, x \rangle))^2$$

proving the first inequality in (3.1).

If f is convex, then by Jensen's inequality we have $\langle f(A)x, x \rangle \geq f(\langle Ax, x \rangle)$ for any $x \in H$ with $\|x\| = 1$. Using the first two inequalities in (3.1) we get

$$\begin{aligned}
(3.8) \quad (0 \leq) & \frac{1}{2\Gamma} \nu(1-\nu) (\langle f(A)x, x \rangle - f(\langle Ax, x \rangle))^2 \\
& \leq \frac{1}{2\Gamma} \nu(1-\nu) (\langle f^2(A)x, x \rangle - 2f(\langle Ax, x \rangle) \langle f(A)x, x \rangle + f^2(\langle Ax, x \rangle)) \\
& \leq (1-\nu) f(\langle Ax, x \rangle) + \nu \langle f(A)x, x \rangle - f^{1-\nu}(\langle Ax, x \rangle) \langle f^\nu(A)x, x \rangle \\
& \leq \langle f(A)x, x \rangle - f^{1-\nu}(\langle Ax, x \rangle) \langle f^\nu(A)x, x \rangle
\end{aligned}$$

proving the inequality (3.2).

From the fourth inequality in (3.1) we have

$$\begin{aligned}
(3.9) \quad & f^{1-\nu}(\langle Ax, x \rangle) (f^\nu(\langle Ax, x \rangle) - \langle f^\nu(A)x, x \rangle) \\
& = f(\langle Ax, x \rangle) - f^{1-\nu}(\langle Ax, x \rangle) \langle f^\nu(A)x, x \rangle \\
& \leq (1-\nu) f(\langle Ax, x \rangle) + \nu \langle f(A)x, x \rangle - f^{1-\nu}(\langle Ax, x \rangle) \langle f^\nu(A)x, x \rangle \\
& \leq \frac{1}{2\gamma} \nu(1-\nu) (\langle f^2(A)x, x \rangle - 2f(\langle Ax, x \rangle) \langle f(A)x, x \rangle + f^2(\langle Ax, x \rangle))
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$, proving the inequality (3.3).

If f is concave on $[m, M]$, then by the first two inequalities in (3.1) we get

$$\begin{aligned}
(3.10) \quad (0 \leq) & \frac{1}{2\Gamma} \nu(1-\nu) (\langle f(A)x, x \rangle - f(\langle Ax, x \rangle))^2 \\
& \leq \frac{1}{2\Gamma} \nu(1-\nu) (\langle f^2(A)x, x \rangle - 2f(\langle Ax, x \rangle) \langle f(A)x, x \rangle + f^2(\langle Ax, x \rangle)) \\
& \leq (1-\nu) f(\langle Ax, x \rangle) + \nu \langle f(A)x, x \rangle - f^{1-\nu}(\langle Ax, x \rangle) \langle f^\nu(A)x, x \rangle \\
& \leq f(\langle Ax, x \rangle) - f^{1-\nu}(\langle Ax, x \rangle) \langle f^\nu(A)x, x \rangle \\
& = f^{1-\nu}(\langle Ax, x \rangle) [f^\nu(\langle Ax, x \rangle) - \langle f^\nu(A)x, x \rangle]
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$, proving the inequality (3.4). \square

Remark 2. The function $f(t) = t^r$, $r \geq 1$ is convex on \mathbb{R}_+ . Then for any $\nu \in [0, 1]$ and a selfadjoint operator $m1_H \leq A \leq M1_H$ with $0 < m < M$ we have from (3.2) that

$$\begin{aligned}
(3.11) \quad (0 \leq) & \frac{1}{2M^r} \nu(1-\nu) (\langle A^r x, x \rangle - \langle Ax, x \rangle^r)^2 \\
& \leq \langle A^r x, x \rangle - \langle Ax, x \rangle^{(1-\nu)r} \langle A^{\nu r} x, x \rangle
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

If we take in (3.11) $\nu = \frac{1}{2}$, then we get the inequality

$$(3.12) \quad (0 \leq) \frac{1}{8M^r} (\langle A^r x, x \rangle - \langle Ax, x \rangle^r)^2 \leq \langle A^r x, x \rangle - \langle Ax, x \rangle^{r/2} \langle A^{r/2} x, x \rangle$$

for any $x \in H$ with $\|x\| = 1$.

If we take $r = 2$ in (3.11), then we get

$$\begin{aligned}
(3.13) \quad (0 \leq) & \frac{1}{2M^2} \nu(1-\nu) (\langle A^2 x, x \rangle - \langle Ax, x \rangle^2)^2 \\
& \leq \langle A^2 x, x \rangle - \langle Ax, x \rangle^{2(1-\nu)} \langle A^{2\nu} x, x \rangle
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

The function $f(t) = t^q$, $q \in (0, 1)$ is concave on \mathbb{R}_+ . Then for any $\nu \in [0, 1]$ and a selfadjoint operator $m1_H \leq A \leq M1_H$ with $0 < m < M$ we have from (3.4)

$$(3.14) \quad \begin{aligned} (0 \leq) & \frac{1}{2M^q} \nu(1-\nu) (\langle Ax, x \rangle^q - \langle A^q x, x \rangle)^2 \\ & \leq \langle Ax, x \rangle^{(1-\nu)q} [\langle Ax, x \rangle^{\nu q} - \langle A^{\nu q} x, x \rangle] \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Since $\langle Ax, x \rangle^{(1-\nu)q} \leq M^{(1-\nu)q}$ for any $x \in H$ with $\|x\| = 1$, then by (3.14) we have

$$(3.15) \quad (0 \leq) \frac{1}{2M^{(2-\nu)q}} \nu(1-\nu) (\langle Ax, x \rangle^q - \langle A^q x, x \rangle)^2 \leq \langle Ax, x \rangle^{\nu q} - \langle A^{\nu q} x, x \rangle$$

for any $x \in H$ with $\|x\| = 1$.

If we use the second Cartwright-Field inequality that holds for any $a, b \in [\gamma, \Gamma]$ and $\nu \in [0, 1]$, namely

$$\begin{aligned} \frac{1}{2} \nu(1-\nu) \gamma \left(\frac{b}{a} + \frac{a}{b} - 2 \right) & \leq (1-\nu)a + \nu b - a^{1-\nu} b^\nu \\ & \leq \frac{1}{2} \nu(1-\nu) \Gamma \left(\frac{b}{a} + \frac{a}{b} - 2 \right) \end{aligned}$$

we can state the following result as well:

Theorem 6. *With the assumptions of Theorem 5 we have*

$$(3.16) \quad \begin{aligned} & \frac{1}{2} \gamma \nu(1-\nu) \\ & \times \left(\left\langle f^{1/2}(A)x, x \right\rangle f^{-1/2}(\langle Ax, x \rangle) - \left\langle f^{-1/2}(A)x, x \right\rangle f^{1/2}(\langle Ax, x \rangle) \right)^2 \\ & \leq \frac{1}{2} \gamma \nu(1-\nu) (\langle f(A)x, x \rangle f^{-1}(\langle Ax, x \rangle) + f(\langle Ax, x \rangle) \langle f^{-1}(A)x, x \rangle - 2) \\ & \leq (1-\nu) f(\langle Ax, x \rangle) + \nu \langle f(A)x, x \rangle - f^{1-\nu}(\langle Ax, x \rangle) \langle f^\nu(A)x, x \rangle \\ & \leq \frac{1}{2} \Gamma \nu(1-\nu) (\langle f(A)x, x \rangle f^{-1}(\langle Ax, x \rangle) + f(\langle Ax, x \rangle) \langle f^{-1}(A)x, x \rangle - 2) \end{aligned}$$

for any $\nu \in [0, 1]$ and for any $x \in H$ with $\|x\| = 1$.

Moreover, if f is convex on $[m, M]$, then

$$(3.17) \quad \begin{aligned} (0 \leq) & \frac{1}{2} \gamma \nu(1-\nu) \\ & \times \left(\left\langle f^{1/2}(A)x, x \right\rangle f^{-1/2}(\langle Ax, x \rangle) - \left\langle f^{-1/2}(A)x, x \right\rangle f^{1/2}(\langle Ax, x \rangle) \right)^2 \\ & \leq \langle f(A)x, x \rangle - f^{1-\nu}(\langle Ax, x \rangle) \langle f^\nu(A)x, x \rangle \end{aligned}$$

and

$$(3.18) \quad \begin{aligned} & f^{1-\nu}(\langle Ax, x \rangle) [f^\nu(\langle Ax, x \rangle) - \langle f^\nu(A)x, x \rangle] \\ & \leq \frac{1}{2} \Gamma \nu(1-\nu) (\langle f(A)x, x \rangle f^{-1}(\langle Ax, x \rangle) + f(\langle Ax, x \rangle) \langle f^{-1}(A)x, x \rangle - 2) \end{aligned}$$

for any $\nu \in [0, 1]$ and for any $x \in H$ with $\|x\| = 1$.

If f is concave on $[m, M]$, then

$$(3.19) \quad (0 \leq) \frac{1}{2} \gamma \nu (1 - \nu) \\ \times \left(\left\langle f^{1/2}(A)x, x \right\rangle f^{-1/2}(\langle Ax, x \rangle) - \left\langle f^{-1/2}(A)x, x \right\rangle f^{1/2}(\langle Ax, x \rangle) \right)^2 \\ \leq f^{1-\nu}(\langle Ax, x \rangle) [f^\nu(\langle Ax, x \rangle) - \langle f^\nu(A)x, x \rangle]$$

for any $\nu \in [0, 1]$ and for any $x \in H$ with $\|x\| = 1$.

The proof follows along the lines of the proof in Theorem 5 and we omit the details.

4. RELATED RESULTS

By the use of the Kittaneh-Manasrah inequality (1.8) we have:

Theorem 7. *With the assumptions of Theorem 5 we have*

$$(4.1) \quad r \left(\left\langle f^{1/2}(A)x, x \right\rangle - f^{1/2}(\langle Ax, x \rangle) \right)^2 \\ \leq r \left(\langle f(A)x, x \rangle + f(\langle Ax, x \rangle) - 2 \left\langle f^{1/2}(A)x, x \right\rangle f^{1/2}(\langle Ax, x \rangle) \right) \\ \leq (1 - \nu) f(\langle Ax, x \rangle) + \nu \langle f(A)x, x \rangle - f^{1-\nu}(\langle Ax, x \rangle) \langle f^\nu(A)x, x \rangle \\ \leq R \left(\langle f(A)x, x \rangle + f(\langle Ax, x \rangle) - 2 \left\langle f^{1/2}(A)x, x \right\rangle f^{1/2}(\langle Ax, x \rangle) \right)$$

for any $\nu \in [0, 1]$ and for any $x \in H$ with $\|x\| = 1$, where $r = \min\{1 - \nu, \nu\}$ and $R = \max\{1 - \nu, \nu\}$.

Moreover, if f is convex on $[m, M]$, then

$$(4.2) \quad r \left(\left\langle f^{1/2}(A)x, x \right\rangle - f^{1/2}(\langle Ax, x \rangle) \right)^2 \\ \leq \langle f(A)x, x \rangle - f^{1-\nu}(\langle Ax, x \rangle) \langle f^\nu(A)x, x \rangle$$

and

$$(4.3) \quad f^{1-\nu}(\langle Ax, x \rangle) [f^\nu(\langle Ax, x \rangle) - \langle f^\nu(A)x, x \rangle] \\ \leq R \left(\langle f(A)x, x \rangle + f(\langle Ax, x \rangle) - 2 \left\langle f^{1/2}(A)x, x \right\rangle f^{1/2}(\langle Ax, x \rangle) \right)$$

for any $\nu \in [0, 1]$ and for any $x \in H$ with $\|x\| = 1$.

If f is concave on $[m, M]$, then

$$(4.4) \quad r \left(\left\langle f^{1/2}(A)x, x \right\rangle - f^{1/2}(\langle Ax, x \rangle) \right)^2 \\ \leq f^{1-\nu}(\langle Ax, x \rangle) [f^\nu(\langle Ax, x \rangle) - \langle f^\nu(A)x, x \rangle]$$

for any $\nu \in [0, 1]$ and for any $x \in H$ with $\|x\| = 1$.

The function $f(t) = t^p$, $p \geq 1$ is convex on \mathbb{R}_+ . Then for any $\nu \in [0, 1]$ and a positive selfadjoint operator A we have from (4.2) that

$$(4.5) \quad r \left(\left\langle A^{p/2}x, x \right\rangle - \langle Ax, x \rangle^{p/2} \right)^2 \leq \langle A^p x, x \rangle - \langle Ax, x \rangle^{(1-\nu)p} \langle A^\nu x, x \rangle$$

for any $x \in H$ with $\|x\| = 1$, where $r = \min\{1 - \nu, \nu\}$ and $R = \max\{1 - \nu, \nu\}$.

If we take in (4.5) $\nu = \frac{1}{2}$, then we get the inequality

$$(4.6) \quad \frac{1}{2} \left(\langle A^{p/2}x, x \rangle - \langle Ax, x \rangle^{p/2} \right)^2 \leq \langle A^p x, x \rangle - \langle Ax, x \rangle^{p/2} \langle A^{p/2}x, x \rangle$$

for any $x \in H$ with $\|x\| = 1$.

If we take $p = 1$ in (4.5), then we get

$$(4.7) \quad r \left(\langle Ax, x \rangle^{1/2} - \langle A^{1/2}x, x \rangle \right)^2 \leq \langle Ax, x \rangle - \langle Ax, x \rangle^{1-\nu} \langle A^\nu x, x \rangle$$

for any $x \in H$ with $\|x\| = 1$.

If $A \leq M1_H$, then $\langle Ax, x \rangle^{1-\nu} \leq M^{1-\nu}$ for $\nu \in (0, 1)$ and $x \in H$ with $\|x\| = 1$, and since

$$\langle Ax, x \rangle - \langle Ax, x \rangle^{1-\nu} \langle A^\nu x, x \rangle = \langle Ax, x \rangle^{1-\nu} (\langle Ax, x \rangle^\nu - \langle A^\nu x, x \rangle)$$

then by (4.7) we get

$$(4.8) \quad \frac{r}{M^{1-\nu}} \left(\langle Ax, x \rangle^{1/2} - \langle A^{1/2}x, x \rangle \right)^2 \leq \langle Ax, x \rangle^\nu - \langle A^\nu x, x \rangle$$

for any $x \in H$ with $\|x\| = 1$.

Finally, if we use the logarithmic inequality (1.13) we can state the following result as well:

Theorem 8. *With the assumptions of Theorem 5 we have*

$$(4.9) \quad \begin{aligned} & \frac{1}{2} \nu (1 - \nu) \gamma (\ln (\langle f(A)x, x \rangle) - \langle \ln f(A)x, x \rangle)^2 \\ & \leq \frac{1}{2} \nu (1 - \nu) \gamma \\ & \quad \times (\langle \ln^2 f(A)x, x \rangle - 2 \langle \ln f(A)x, x \rangle \ln (\langle f(A)x, x \rangle) + \ln^2 (\langle f(A)x, x \rangle)) \\ & \leq (1 - \nu) f(\langle Ax, x \rangle) + \nu \langle f(A)x, x \rangle - f^{1-\nu}(\langle Ax, x \rangle) \langle f^\nu(A)x, x \rangle \\ & \leq \frac{1}{2} \nu (1 - \nu) \Gamma \\ & \quad \times (\langle \ln^2 f(A)x, x \rangle - 2 \langle \ln f(A)x, x \rangle \ln (\langle f(A)x, x \rangle) + \ln^2 (\langle f(A)x, x \rangle)) \end{aligned}$$

for any $\nu \in [0, 1]$ and for any $x \in H$ with $\|x\| = 1$, where $r = \min\{1 - \nu, \nu\}$ and $R = \max\{1 - \nu, \nu\}$.

Moreover, if f is convex on $[m, M]$, then

$$(4.10) \quad \begin{aligned} & \frac{1}{2} \nu (1 - \nu) \gamma (\ln (\langle f(A)x, x \rangle) - \langle \ln f(A)x, x \rangle)^2 \\ & \leq \langle f(A)x, x \rangle - f^{1-\nu}(\langle Ax, x \rangle) \langle f^\nu(A)x, x \rangle \end{aligned}$$

and

$$(4.11) \quad \begin{aligned} & f^{1-\nu}(\langle Ax, x \rangle) [f^\nu(\langle Ax, x \rangle) - \langle f^\nu(A)x, x \rangle] \\ & \leq \frac{1}{2} \nu (1 - \nu) \Gamma \\ & \quad \times (\langle \ln^2 f(A)x, x \rangle - 2 \langle \ln f(A)x, x \rangle \ln (\langle f(A)x, x \rangle) + \ln^2 (\langle f(A)x, x \rangle)) \end{aligned}$$

for any $\nu \in [0, 1]$ and for any $x \in H$ with $\|x\| = 1$.

If f is concave on $[m, M]$, then

$$(4.12) \quad \begin{aligned} & \frac{1}{2} \nu (1 - \nu) \gamma (\ln (\langle f(A)x, x \rangle) - \langle \ln f(A)x, x \rangle)^2 \\ & \leq f^{1-\nu} (\langle Ax, x \rangle) [f^\nu (\langle Ax, x \rangle) - \langle f^\nu(A)x, x \rangle] \end{aligned}$$

for any $\nu \in [0, 1]$ and for any $x \in H$ with $\|x\| = 1$.

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