Received 06/06/16

HOLDER-TYPE AND QI-TYPE INEQUALITIES FOR ISOTONIC LINEAR FUNCTIONALS

LOREDANA CIURDARIU

Abstract. In this paper are given new variants for subdividing of Holder-type inequality and Minkowski-type inequality in the case of isotonic linear functionals and then some variants of Qi's inequality for isotonic linear functionals using a new Young-type inequality. Also are presented some applications for integral on time scales.

1. Introduction

In [1] are given new results which extend many generalizations of Young's inequality given before. We recal now these results below in order to use them in the next sections.

Theorem 1. *Let* λ, ν *and* τ *be real numbers with* $\lambda \geq 1$ *and* $0 < \nu < \tau < 1$. *Then*

$$\left(\frac{\nu}{\tau}\right)^{\lambda} < \frac{A_{\nu}(a,b)^{\lambda} - G_{\nu}(a,b)^{\lambda}}{A_{\tau}(a,b)^{\lambda} - G_{\tau}(a,b)^{\lambda}} < \left(\frac{1-\nu}{1-\tau}\right)^{\lambda},$$

for all positive and distinct real numbers a *and* b. *Moreover, both bounds are sharp.*

The following important definition is given in [3], [5] and we will recall it here.

Let E be a nonempty set and L be a class of real-valued functions $f : E \to \mathbf{R}$ having the following properties:

(L1) If $f, g \in L$ and $a, b \in \mathbf{R}$, then $(af + bg) \in L$.

(L2) If $f(t) = 1$ for all $t \in E$, then $f \in L$.

An *isotonic linear functional* is a functional $A : L \to \mathbf{R}$ having the following properties:

(A1) If $f, g \in L$ and $a, b \in \mathbf{R}$, then $A(af + bg) = aA(f) + bA(g)$.

(A2) If $f \in L$ and $f(t) \geq 0$ for all $t \in E$, then $A(f) \geq 0$.

The mapping A is said to be *normalised* if

(A3) $A(\mathbf{1}) = 1$.

The following Holder-type inequalities are obtained from Theorem 1.1 which is given in [1] and will be used in the next sections as an important tool in our demonstrations.

Date: May 31, 2016.

2000 *Mathematics Subject Classification.* 26D20.

Key words and phrases. Young's inequality, Holder's inequality, Qi's inequality, time scales integrals.

RGMIA Res. Rep. Coll. 16 (2016), Art. 101

Theorem 2. *If L satisfy conditions L1, L2 and A satisfy A1, A2 on the set E. If f^p, g^q, fg, $f^{p\tau}$, $g^{q(1-\tau)} \in L$, $A(f^p) > 0$, $A(g^q) > 0$, $p\tau > 1$, $\tau < 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and f and g are positive functions then:*

$$\frac{1}{p\tau}\left[1 - \frac{A(f^{p\tau})A(g^{q(1-\tau)})}{A^{\tau}(f^p)A^{1-\tau}(g^q)}\right] < 1 - \frac{A(fg)}{A^{\frac{1}{p}}(f^p)A^{\frac{1}{q}}(g^q)} <$$

$$< \frac{1}{q(1-\tau)}\left[1 - \frac{A(f^{p\tau})A(g^{q(1-\tau)})}{A^{\tau}(f^p)A^{1-\tau}(g^q)}\right].$$

The following two generalizations of reversed Holder's inequality for isotonic linear functionals were given in [7] and [8] and will be also used in the next section.

Theorem 3. *([7]) Let $A : L \to \mathbf{R}$ be an normalised isotonic linear functional and $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f,\ g : E \to \mathbf{R}$ are such that $fg,\ f^p,\ g^q \in L$ and $0 < m_1 \le f \le M_1 < \infty$, $0 < m_2 \le g \le M_2 < \infty$ for some constants $m_1,\ M_1,\ m_2,\ M_2$ then*

$$A^{\frac{1}{p}}(f^p)A^{\frac{1}{q}}(g^q) \le K^U\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)A(fg)$$

where $U = \max\{\frac{1}{p}, \frac{1}{q}\}$ and $K(h) = \frac{(h+1)^2}{4h}$, $h > 0$ is the Kantorovich's ratio.

Theorem 4. *([8]) Let $A : L \to \mathbf{R}$ be an normalised isotonic linear functional and $p,\ q > 1$, $p,\ q \ne 2$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f,\ g : E \to \mathbf{R}$ are so that $fg,\ f^p,\ g^q,\ f^{\frac{p}{2}}g^{\frac{q}{2}} \in L$ and $0 < m_1 \le f \le M_1 < \infty$, $0 < m_2 \le g \le M_2 < \infty$ for some constants $m_1,\ M_1,\ m_2,\ M_2$ then*

$$\frac{A(f^{\frac{p}{2}}g^{\frac{q}{2}})}{\sqrt{A(f^p)A(g^q)}} \le \frac{(1-2s)\sqrt{A(f^p)A(g^q)} + 2sA(f^{\frac{p}{2}}g^{\frac{q}{2}})}{\sqrt{A(f^p)A(g^q)}} \le$$

$$\le D_{p,q}\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}\right)\frac{A(fg)}{A^{\frac{1}{p}}(f^p)A^{\frac{1}{q}}(g^q)},$$

where $s = \min\{\frac{1}{p}, \frac{1}{q}\}$, $U = \max\{2s, 1-2s\}$ and $D_{p,q} = \min\{S(h), K^U(h)\}$, $b > 0$, and $S(h)$ is the Specht's ratio, i. e.

$$S(h) = \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln(h^{\frac{1}{h-1}})}, & \text{if } h \in (0,1)\bigcup(1,\infty); \\ 1, & \text{if } h=1. \end{cases}$$

2. **Subdividing of Holder's inequalities for isotonic linear functional**

In this section will be given several generalizations of some inequalities from [11] in Theorem 5, 6 and 7 using new Young-type inequality and reverse of Young's inequality given in [1], [7] and [8]. Then a consequence for integral on time scales will be presented.

Theorem 5. *Let $s,t \in \mathbb{R}$, $p = \frac{s-t}{1-t}$, and $q = \frac{s-t}{s-1}$ such that $s < 1 < t$ or $s > 1 > t$, and L satisfy conditions L1, L2 and A satisfy conditions A1, A2 on the set E.*

If f^{sp}, f^{tp}, g^{sq}, g^{tq}, fg, $(fg)^t$, $(fg)^s$, $f^{\frac{sp}{p_1}} g^{\frac{sq}{q_1}}$, $f^{\frac{tp}{p_1}} g^{\frac{tq}{q_1}}$, $(fg)^{\frac{s}{p_1}+\frac{t}{q_1}} \in L$ and f,g are positive functions then

$$A(fg) < A^{\frac{1}{p^2}}(f^{sp})A^{\frac{1}{q^2}}(g^{tq})\left[A(f^{tp})A(g^{sq})\right]^{\frac{1}{pq}} \cdot \left[1 - \frac{p_1}{p}\left(1 - \frac{A(f^{\frac{sp}{p_1}} g^{\frac{sq}{q_1}})}{A^{\frac{1}{p_1}}(f^{sp})A^{\frac{1}{q_1}}(g^{sq})}\right)\right]^{\frac{1}{p}} \cdot$$

$$\cdot \left[1 - \frac{p_1}{p}\left(1 - \frac{A(f^{\frac{tp}{p_1}} g^{\frac{tq}{q_1}})}{A^{\frac{1}{p_1}}(f^{tp})A^{\frac{1}{q_1}}(g^{tq})}\right)\right]^{\frac{1}{q}} \cdot \left[1 - \frac{p_1}{p}\left(1 - \frac{A((fg)^{\frac{s}{p_1}+\frac{t}{q_1}})}{A^{\frac{1}{p_1}}((fg)^t)A^{\frac{1}{q_1}}((fg)^s)}\right)\right],$$

and

$$A(fg) > A^{\frac{1}{p^2}}(f^{sp})A^{\frac{1}{q^2}}(g^{tq})\left[A(f^{tp})A(g^{sq})\right]^{\frac{1}{pq}} \cdot \left[1 - \frac{q_1}{q}\left(1 - \frac{A(f^{\frac{sp}{p_1}} g^{\frac{sq}{q_1}})}{A^{\frac{1}{p_1}}(f^{sp})A^{\frac{1}{q_1}}(g^{sq})}\right)\right]^{\frac{1}{p}} \cdot$$

$$\cdot \left[1 - \frac{q_1}{q}\left(1 - \frac{A(f^{\frac{tp}{p_1}} g^{\frac{tq}{q_1}})}{A^{\frac{1}{p_1}}(f^{tp})A^{\frac{1}{q_1}}(g^{tq})}\right)\right]^{\frac{1}{q}} \cdot \left[1 - \frac{q_1}{q}\left(1 - \frac{A((fg)^{\frac{s}{p_1}+\frac{t}{q_1}})}{A^{\frac{1}{p_1}}((fg)^t)A^{\frac{1}{q_1}}((fg)^s)}\right)\right].$$

Proof. By inequality given in Theorem 2 applied for $p = \frac{s-t}{1-t} > 1,\ q = \frac{s-t}{s-1}$ we have

$$A(fg) = A\left([(fg)^s]^{\frac{1-t}{s-t}}[(fg)^t]^{\frac{s-1}{s-t}}\right) <$$

$$< A^{\frac{1-t}{s-t}}((fg)^s) \cdot A^{\frac{s-1}{s-t}}((fg)^t)\left[1 - \frac{p_1}{p}\left(1 - \frac{A((fg)^{\frac{s}{p_1}+\frac{t}{q_1}})}{A^{\frac{1}{p_1}}((fg)^s)A^{\frac{1}{q_1}}((fg)^t)}\right)\right].$$

Applying again Theorem 2 for $\frac{s-t}{1-t} > 1$ we get

$$A((fg)^s) \le A^{\frac{1-t}{s-t}}(f^{s\frac{s-t}{1-t}})A^{\frac{s-1}{s-t}}(g^{s\frac{s-t}{s-1}})\left[1 - \frac{p_1}{p}\left(1 - \frac{A(f^{\frac{sp}{p_1}} g^{\frac{sq}{q_1}})}{A^{\frac{1}{p_1}}(f^{sp})A^{\frac{1}{q_1}}(g^{sq})}\right)\right]$$

and

$$A((fg)^t) \le A^{\frac{1-t}{s-t}}(f^{t\frac{s-t}{1-t}})A^{\frac{s-1}{s-t}}(g^{t\frac{s-t}{s-1}})\left[1 - \frac{p_1}{p}\left(1 - \frac{A(f^{\frac{tp}{p_1}} g^{\frac{tq}{q_1}})}{A^{\frac{1}{p_1}}(f^{tp})A^{\frac{1}{q_1}}(g^{tq})}\right)\right].$$

Taking into account these three inequalities we obtain the desired inequality.

The second inequality will be checked by using the second inequality from Theorem 2 for $p = \frac{s-t}{1-t} > 1$ and $q = \frac{s-t}{s-1}$.

∎

Theorem 6. *Let $s,t \in \mathbb{R}$, $p = \frac{s-t}{1-t}$, $q = \frac{s-t}{s-1}$ such that $s < 1 < t$ or $s > 1 > t$, and L satisfy conditions L1, L2 and A a normalised isotonic linear functional $A : L \to \mathbf{R}$.*

If f^{sp}, f^{tp}, g^{sq}, g^{tq}, $fg \in L$ and f,g are positive functions with $0 < m_1 \le f \le M_1 < \infty$, $0 < m_2 \le g \le M_2 < \infty$ for some constants m_1, M_1, m_2, M_2 then

$$A(fg)K^U\left(\frac{M_1^{s+t}M_2^{s+t}}{m_1^{s+t}m_2^{s+t}}\right)K^{\frac{U}{p}}\left(\frac{M_1^{sp}M_2^{sq}}{m_1^{sp}m_2^{sq}}\right)K^{\frac{U}{q}}\left(\frac{M_1^{tp}M_2^{tq}}{m_1^{tp}m_2^{tq}}\right) \ge$$

$$\ge A^{\frac{1}{p^2}}(f^{sp})A^{\frac{1}{pq}}(g^{sq})A^{\frac{1}{pq}}(f^{tp})A^{\frac{1}{q^2}}(g^{tq}),$$

where $U = \max\{\frac{1}{p}, \frac{1}{q}\}$.

Proof. We use the same reason as in [10]. Therefore for $p=\frac{s-t}{1-t}>1,\ q=\frac{s-t}{s-1}$ we write

$$A(fg)=A\left([(fg)^s]^{\frac{1-t}{s-t}}[(fg)^t]^{\frac{s-1}{s-t}}\right)=A((fg)^{s\frac{1}{p}}(fg)^{t\frac{1}{q}})$$

and by Theorem 3 we obtain:

$$A(fg)K^U\left(\frac{M_1^{s+t}M_2^{s+t}}{m_1^{s+t}m_2^{s+t}}\right)\geq A^{\frac{1}{p}}((fg)^s)A^{\frac{1}{q}}((fg)^t).$$

But, by applying again Theorem 3 we have:

$$A((fg)^s)K^U\left(\frac{M_1^{sp}M_2^{sq}}{m_1^{sp}m_2^{sq}}\right)\geq A^{\frac{1}{p}}((fg)^{sp})A^{\frac{1}{q}}((fg)^{sq}),$$

and also

$$A((fg)^t)K^U\left(\frac{M_1^{tp}M_2^{tq}}{m_1^{tp}m_2^{tq}}\right)\geq A^{\frac{1}{p}}((fg)^{tp})A^{\frac{1}{q}}((fg)^{tq}).$$

Taking into account last three we will obtain the desired inequality.
∎

Theorem 7. *Let $s,t\in\mathbb{R}$, $p=\frac{s-t}{1-t}$, $q=\frac{s-t}{s-1}$ such that $s<1<t$ or $s>1>t$, $s+t\neq 2$ and L satisfy conditions L1, L2 and A a normalised isotonic linear functional $A:L\to\mathbf{R}$.*

If $f^{sp},\ f^{tp},\ g^{sq},\ g^{tq},\ \ fg,\ (fg)^s,\ (fg)^t,\ f^{\frac{sp}{2}}g^{\frac{sq}{2}},\ f^{\frac{tp}{2}}g^{\frac{tq}{2}},\ (fg)^{\frac{s+t}{2}}\in L$ and f,g are positive functions with $0<m_1\leq f\leq M_1<\infty,\ 0<m_2\leq g\leq M_2<\infty$ for some constants $m_1,\ M_1,\ m_2,\ M_2$ then

$$A(fg)D_{p,q}\left(\left(\frac{M_1M_2}{m_1m_2}\right)^{\frac{s+t}{2}}\right)D_{p,q}^{\frac{1}{p}-\frac{1}{2}}\left(\left(\frac{M_1}{m_1}\right)^{\frac{sp}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{sq}{2}}\right)D_{p,q}^{\frac{1}{q}-\frac{1}{2}}\left(\left(\frac{M_1}{m_1}\right)^{\frac{tp}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{tq}{2}}\right)\geq$$
$$\geq\left[(1-2s_1)\sqrt{A((fg)^s)A((fg)^t)}+2s_1A((fg)^{\frac{s+t}{2}})\right]\cdot$$
$$\cdot\left[(1-2s_1)\sqrt{A(f^{sp})A(g^{sq})}+2s_1A(f^{\frac{sp}{2}}g^{\frac{sq}{2}})\right]^{\frac{1}{p}-\frac{1}{2}}\cdot$$
$$\cdot\left[(1-2s_1)\sqrt{A(f^{tp})A(g^{tq})}+2s_1A(f^{\frac{tp}{2}}g^{\frac{tq}{2}})\right]^{\frac{1}{q}-\frac{1}{2}}\cdot$$
$$\cdot A^{(\frac{1}{p}-\frac{1}{2})^2}(f^{sp})A^{(\frac{1}{p}-\frac{1}{2})(\frac{1}{q}-\frac{1}{2})}(g^{sq})A^{(\frac{1}{p}-\frac{1}{2})(\frac{1}{q}-\frac{1}{2})}(f^{tp})A^{(\frac{1}{q}-\frac{1}{2})^2}(g^{tq}),$$

where $D_{p,q}$, $S(h)$, s_1 and U are like in Theorem 4.

Proof. For $p=\frac{s-t}{1-t}>1,\ q=\frac{s-t}{s-1}$ we write

$$A(fg)=A\left([(fg)^s]^{\frac{1-t}{s-t}}[(fg)^t]^{\frac{s-1}{s-t}}\right)=A((fg)^{s\frac{1}{p}}(fg)^{t\frac{1}{q}})$$

and by Theorem 4 we get:

$$A(fg)D_{p,q}\left(\left(\frac{M_1M_2}{m_1m_2}\right)^{\frac{s+t}{2}}\right)D_{p,q}^{\frac{1}{p}-\frac{1}{2}}\left(\left(\frac{M_1}{m_1}\right)^{\frac{sp}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{sq}{2}}\right)D_{p,q}^{\frac{1}{q}-\frac{1}{2}}\left(\left(\frac{M_1}{m_1}\right)^{\frac{tp}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{tq}{2}}\right)\geq$$
$$\geq\frac{(1-2s_1)\sqrt{A((fg)^s)A((fg)^t)}+2s_1A((fg)^{\frac{s+t}{2}})}{\sqrt{A((fg)^s)A((fg)^t)}}\cdot A^{\frac{1}{p}}((fg)^s)A^{\frac{1}{q}}((fg)^t)\cdot$$
$$\cdot D_{p,q}^{\frac{1}{p}-\frac{1}{2}}\left(\left(\frac{M_1}{m_1}\right)^{\frac{sp}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{sq}{2}}\right)D_{p,q}^{\frac{1}{q}-\frac{1}{2}}\left(\left(\frac{M_1}{m_1}\right)^{\frac{tp}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{tq}{2}}\right)=$$

$$[(1-2s_1)\sqrt{A((fg)^s)A((fg)^t)}+2s_1A((fg)^{\frac{s+t}{2}})]\cdot A^{\frac{1}{p}-\frac{1}{2}}((fg)^s)A^{\frac{1}{q}-\frac{1}{2}}((fg)^t)\cdot$$
$$\cdot D_{p,q}^{\frac{1}{p}-\frac{1}{2}}\left(\left(\frac{M_1}{m_1}\right)^{\frac{sp}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{sq}{2}}\right)D_{p,q}^{\frac{1}{q}-\frac{1}{2}}\left(\left(\frac{M_1}{m_1}\right)^{\frac{tp}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{tq}{2}}\right),$$

taking into account that $(m_1m_2)^{\frac{s}{p}}\le (fg)^{\frac{s}{p}}\le (M_1M_2)^{\frac{s}{p}}$ and $(m_1m_2)^{\frac{t}{q}}\le (fg)^{\frac{t}{q}}\le (M_1M_2)^{\frac{t}{q}}$.

Now applying again Theorem 4 for $\frac{s-t}{1-t}>1$ we find

$$A((fg)^s)D_{p,q}\left(\left(\frac{M_1}{m_1}\right)^{\frac{sp}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{sq}{2}}\right)\ge$$
$$\ge\frac{(1-2s_1)\sqrt{A(f^{sp})A(g^{sq})}+2s_1A(f^{\frac{sp}{2}}g^{\frac{sq}{2}})}{\sqrt{A(f^{sp})A(g^{sq})}}\cdot A^{\frac{1}{p}}(f^{sp})A^{\frac{1}{q}}(g^{sq})=$$
$$=[(1-2s_1)\sqrt{A(f^{sp})A(g^{sq})}+2s_1A(f^{\frac{sp}{2}}g^{\frac{sq}{2}})]A^{\frac{1}{p}-\frac{1}{2}}(f^{sp})A^{\frac{1}{q}-\frac{1}{2}}(g^{sq}),$$

and

$$A((fg)^t)D_{p,q}\left(\left(\frac{M_1}{m_1}\right)^{\frac{tp}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{tq}{2}}\right)\ge$$
$$\ge\frac{(1-2s_1)\sqrt{A(f^{tp})A(g^{tq})}+2s_1A(f^{\frac{tp}{2}}g^{\frac{tq}{2}})}{\sqrt{A(f^{tp})A(g^{tq})}}\cdot A^{\frac{1}{p}}(f^{tp})A^{\frac{1}{q}}(g^{tq})=$$
$$=[(1-2s_1)\sqrt{A(f^{tp})A(g^{tq})}+2s_1A(f^{\frac{tp}{2}}g^{\frac{tq}{2}})]\cdot A^{\frac{1}{p}-\frac{1}{2}}(f^{tp})A^{\frac{1}{q}-\frac{1}{2}}(g^{tq}),$$

using the hypothesis that $0<m_1\le f\le M_1<\infty,\ 0<m_2\le g\le M_2<\infty$.

Now, taking into account the last three inequalities, we get the desired inequality. ∎

Consequence 1. *Let $s,t\in\mathbb{R}$, $p=\frac{s-t}{1-t}$, and $q=\frac{s-t}{s-1}$ such that $s<1<t$ or $s>1>t$, and*

(i) Let $a,b\in\mathbb{T}$. If $f,g\in C_{rd}(\mathbb{T},\mathbb{R})$, are two positive functions then

$$\int_a^b f(x)g(x)\Delta x<\left[\int_a^b f^{sp}(x)\Delta x\right]^{\frac{1}{p^2}}\left[\int_a^b g^{tq}(x)\Delta x\right]^{\frac{1}{q^2}}\left[\int_a^b f^{tp}(x)\Delta x\int_a^b g^{sq}(x)\Delta x\right]^{\frac{1}{pq}}\cdot$$
$$\left[1-\frac{p_1}{p}\left(1-\frac{\int_a^b f^{\frac{sp}{p_1}}(x)g^{\frac{sq}{q_1}}(x)\Delta x}{\left[\int_a^b f^{sp}(x)\Delta x\right]^{\frac{1}{p_1}}\left[\int_a^b g^{sq}(x)\Delta x\right]^{\frac{1}{q_1}}}\right)\right]^{\frac{1}{p}}\cdot$$
$$\cdot\left[1-\frac{p_1}{p}\left(1-\frac{\int_a^b f^{\frac{tp}{p_1}}(x)g^{\frac{tq}{q_1}}(x)\Delta x}{\left[\int_a^b f^{tp}(x)\Delta x\right]^{\frac{1}{p_1}}\left[\int_a^b g^{tq}(x)\Delta x\right]^{\frac{1}{q_1}}}\right)\right]^{\frac{1}{q}}\cdot$$
$$\left[1-\frac{p_1}{p}\left(1-\frac{\int_a^b (f(x)g(x))^{\frac{s}{p_1}+\frac{t}{q_1}}\Delta x)}{\left[\int_a^b f(x)^tg(x)^t\Delta x\right]^{\frac{1}{p_1}}\left[\int_a^b f^t(x)g^t(x)\Delta x\right]^{\frac{1}{q_1}}}\right)\right],$$

(ii) Let $a, b \in \mathbb{T}$ and $f, g \in C_{ld}([a,b],\mathbb{R})$ be two positive functions as in Theorem 6. Then we have:

$$K^U\left(\frac{M_1^{s+t}M_2^{s+t}}{m_1^{s+t}m_2^{s+t}}\right)K^{\frac{U}{p}}\left(\frac{M_1^{sp}M_2^{sq}}{m_1^{sp}m_2^{sq}}\right)K^{\frac{U}{q}}\left(\frac{M_1^{tp}M_2^{tq}}{m_1^{tp}m_2^{tq}}\right)\int_a^b f(x)g(x)\nabla x \geq$$

$$\geq \left[\int_a^b f^{sp}(x)\nabla x\right]^{\frac{1}{p^2}}\left[\int_a^b g^{sq}(x)\nabla x\right]^{\frac{1}{pq}}\left[\int_a^b f^{tp}(x)\nabla x\right]^{\frac{1}{pq}}\left[\int_a^b g^{tq}(x)\nabla x\right]^{\frac{1}{q^2}},$$

3. **Qi's inequality for isotonic linear functionals**

Using new generalizations of reverse of Young's inequality given in [7] and [8] will be given new inequalities for isotonic linear functionals starting from some results given in [10].

Lemma 1. *Let $A : L \to \mathbf{R}$ be an normalised isotonic linear functional and $p,\ q > 1,\ p,\ q \neq 2$ with $\frac{1}{p}+\frac{1}{q}=1$. If $f,\ g : E \to \mathbf{R}$ are positive functions so that $f,\ g,\ \frac{f^{\frac{p}{2}}}{g^{\frac{q}{2}-1}},\ \frac{f^p}{g^{p-1}} \in L$ and $0 < m_1 \leq f \leq M_1 < \infty,\ 0 < m_2 \leq g \leq M_2 < \infty$ for some constants $m_1,\ M_1,\ m_2,\ M_2,\ A(f) > 0,\ A(g) > 0$ then*

$$\frac{A^{\frac{p}{2}-1}(g)}{A^p(f)}\left[(1-2s)\sqrt{A\left(\frac{f^p}{g^{p-1}}\right)A(g)}+2sA\left(\frac{f^{\frac{p}{2}}}{g^{\frac{p}{2}-1}}\right)\right]^p \leq$$

$$\leq D_{p,q}^p\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{p}{2}-1}\right)A^{\frac{p}{2}-1}\left(\frac{f^p}{g^{p-1}}\right),$$

where $s=\min\{\frac{1}{p},\frac{1}{q}\}$, $U=\max\{2s, 1-2s\}$ and $D_{p,q}=\min\{S(h), K^U(h)\},\ b>0$, and $S(h)$ is the Specht's ratio.

Proof. We apply Theorem 4 taking

$$A(f)=A\left(\frac{f}{g^{\frac{1}{q}}}g^{\frac{1}{q}}\right)$$

and we will obtain:

$$A(f)D_{p,q}\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{p}{2}-1}\right) \geq$$

$$\geq A^{\frac{1}{p}}\left(\left(\frac{f}{g^{\frac{1}{q}}}\right)^p\right)A^{\frac{1}{q}}(g)\frac{(1-2s)\sqrt{A\left(\frac{f^p}{g^{p-1}}\right)A(g)}+2sA\left(\frac{f^{\frac{p}{2}}}{g^{\frac{p}{2}-1}}\right)}{\sqrt{A\left(\frac{f^p}{g^{p-1}}\right)A(g)}}$$

or

$$A^{\frac{1}{p}}\left(\frac{f^p}{g^{p-1}}\right) \leq \frac{A(f)}{A^{\frac{1}{q}}(g)}\frac{D_{p,q}\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{p}{2}-1}\right)\sqrt{A\left(\frac{f^p}{g^{p-1}}\right)A(g)}}{(1-2s)\sqrt{A\left(\frac{f^p}{g^{p-1}}\right)A(g)}+2sA\left(\frac{f^{\frac{p}{2}}}{g^{\frac{p}{2}-1}}\right)}.$$

Then we take the p-power on both sides of the inequalities and have:

$$A\left(\frac{f^p}{g^{p-1}}\right) \leq \frac{A^p(f)}{A^{p-1}(g)} \frac{D^p_{p,q}\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{p}{2}-1}\right)\left(\sqrt{A\left(\frac{f^p}{g^{p-1}}\right)A(g)}\right)^p}{\left[(1-2s)\sqrt{A\left(\frac{f^p}{g^{p-1}}\right)A(g)}+2sA\left(\frac{f^{\frac{p}{2}}}{g^{\frac{p}{2}-1}}\right)\right]^p}.$$

Therefore the desired inequality takes place.

∎

Theorem 8. *Let $A : L \to \mathbf{R}$ be an normalised isotonic linear functional and $p,\ q > 1,\ p,\ q \neq 2$ with $\frac{1}{p}+\frac{1}{q}=1$. If $f : E \to \mathbf{R}$ is positive function so that $f,\ f^{\frac{p}{2}},\ f^p \in L$ and $0 < m_1 \leq f \leq M_1 < \infty$ for some constants $m_1,\ M_1,\ A(f) > 0$ and $A(f) < 1$ then*

$$\left[(1-2s)\sqrt{A\left(f^p\right)}+2sA\left(f^{\frac{p}{2}}\right)\right]^p \leq$$
$$\leq D^p_{p,q}\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\right)A^{\frac{p}{2}-1}\left(f^p\right)A^{\frac{p}{2}+1}(f),$$

where $s=\min\{\frac{1}{p},\frac{1}{q}\}$, $U=\max\{2s,1-2s\}$ and $D_{p,q}=\min\{S(h),K^U(h)\},\ b>0$, and $S(h)$ is the Specht's ratio.

Consequence 2. *Under conditions of Lemma 1, if $a,b \in \mathbf{R},\ a<b$ and $f,\ g \in C([a,b])$ are positive then we have:*

$$\frac{(\int_a^b g(x)\Delta x)^{\frac{p}{2}-1}}{(\int_a^b f(x)\Delta x)^p}\left[(1-2s)\sqrt{\int_a^b \frac{f^p(x)}{g^{p-1}(x)}\Delta x \int_a^b g(x)\Delta x}+2s\int_a^b \frac{f^{\frac{p}{2}}(x)}{g^{\frac{p}{2}-1}(x)}\right]^p \leq$$
$$\leq D^p_{p,q}\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{p}{2}-1}\right)\left(\int_a^b \frac{f^p(x)}{g^{p-1}(x)}\Delta x\right)^{\frac{p}{2}-1}.$$

References

[1] Alzer, H., Fonseca, C. M., Kovacec, A., Young-type inequalities and their matrix analogues, *Linear and Multilinear Algebra*, 63, 3, 622-635, (2015).

[2] Andrica D. and Badea C, Gruss'inequality for positive linear functionals, *Periodica Math. Hung.*, 19, 155-167, (1998).

[3] Anwar, M., Bibi, R., Bohner, M., and Pecaric, J., Integral Inequalities on Time Scales via the Theory of Isotonic Linear Functionals, *Abstract and Applied Analysis*, Article ID 483595, 16 pages, (2011).

[4] Bohner, M., Peterson, A., "Dynamic equations on time scales: an introduction with applications". Birkhauser, Boston (2001).

[5] Dragomir, S., S., A survey of Jessen's type inequalities for positive functionals, *RGMIA Res. Rep. Coll.*, 46 pp, (2011).

[6] Dragomir, S., S., A Gruss type inequality for isotonic linear functionals and applications, *RGMIA Res. Rep. Coll.*, 10 pp, (2002).

[7] Dragomir, S. S., Some results for isotonic functionals via an inequality due to Liao, Wu and Zhao, *RGMIA Res. Rep. Coll.*, 11 pp, (2015).

[8] Dragomir, S., S., Some results for isotonic functionals via two reverses of Young's inequality, *RGMIA Res. Rep. Coll.*, 13 pp, (2015).

[9] Guseinov, G. S., Integration on time scales, *J. Math. Anal. Appl*, 285, 107-127 (2003).
[10] Yin L, Qi, F., Some Integral Inequalities on Time Scales, *Results. Mah.*, 64, 371-381, (2013).
[11] Zhao, Chang-Jian and Cheung, W-S., On subdividing of Holder's inequality, *Far East Journal of Mathematical Sciences*, Vol. 60 , 1 (2012), 101-108.

Department of Mathematics, "Politehnica" University of Timisoara, P-ta. Victoriei, No.2, 300006-Timisoara