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A NOTE ON NUMERICAL COMPARISON OF SOME MULTIPLICATIVE BOUNDS RELATED TO WEIGHTED ARITHMETIC AND GEOMETRIC MEANS

S. S. DRAGOMIR[1,2] AND A. MCANDREW[1]

Abstract. In this note we provided some numerical comparison for the upper and lower bounds in some recent inequalities related to the famous Young's inequality for two positive numbers. We drew the conclusion that neither of the inequalities below is always best.

1. Introduction

We consider the weighted *Arithmetic and Geometric means* defined by

$$A_\nu(a,b) := (1-\nu)a + \nu b,\ G_\nu(a,b) := a^{1-\nu}b^\nu,$$

where $a, b > 0$ and $\nu \in [0,1]$. If $\nu = \frac{1}{2}$ we denote $A(a,b) := \frac{a+b}{2}$ and $G(a,b) := \sqrt{ab}$, for simplicity.

The following inequality

$$G_\nu(a,b) \le A_\nu(a,b) \tag{1.1}$$

is well known in literature as either *weighted Arithmetic mean-Geometric mean inequality* or as *Young's inequality*.

We recall that *Specht's ratio* is defined by

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \ne 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality (1.1)

$$S\left(\left(\frac{a}{b}\right)^r\right) \le \frac{A_\nu(a,b)}{G_\nu(a,b)} \le S\left(\frac{a}{b}\right), \tag{1.2}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$.

The second inequality in (1.2) is due to Tominaga [9] while the first one is due to Furuichi [6].

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We consider the *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{1.3}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's ratio holds

$$\left[K\left(\frac{a}{b}\right)\right]^r \leq \frac{A_\nu(a,b)}{G_\nu(a,b)} \leq \left[K\left(\frac{a}{b}\right)\right]^R \tag{1.4}$$

where a, $b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu,\nu\}$ and $R = \max\{1-\nu,\nu\}$.

The first inequality in (1.4) was obtained by Zou et al. in [10] while the second by Liao et al. [7].

In [10] the authors also showed that

$$S(h^r) \leq K^r(h) \text{ for } h > 0 \text{ and } r \in \left[0,\frac{1}{2}\right]$$

implying that the lower bound in (1.4) is better than the lower bound from (1.2).

In the recent paper [2] Dragomir obtained for any a, $b > 0$, $\nu \in [0,1]$,

$$\frac{A_\nu(a,b)}{G_\nu(a,b)} \leq \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{1.5}$$

while in [3] the first author obtained for any a, $b > 0$, $\nu \in [0,1]$,

$$\begin{aligned} \exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max^2\{a,b\}}\right] &\leq \frac{A_\nu(a,b)}{G_\nu(a,b)} \\ &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min^2\{a,b\}}\right], \end{aligned} \tag{1.6}$$

see also the equivalent result from [1] that has been stated only for $a < b$.

In [5] the following asymmetric bound has been proven for any a, $b > 0$, $\nu \in [0,1]$

$$\frac{A_\nu(a,b)}{G_\nu(a,b)} \leq \min\left\{\left(\frac{\exp\left(\frac{b}{a}-1\right)}{\frac{b}{a}}\right)^\nu, \left(\frac{\frac{b}{a}}{\exp\left(1-\frac{1}{\frac{b}{a}}\right)}\right)^{1-\nu}\right\}. \tag{1.7}$$

It is therefore a natural question to compare the upper and lower bounds provided above for the quotient $\frac{A_\nu(a,b)}{G_\nu(a,b)}$.

In this note we provide some numerical comparison for the upper and lower bounds in the above inequalities related to the famous Young's inequality for two positive numbers.

2. Numerical Comparison for Weighted Bounds

In order to compare the upper bounds provided by the inequalities (1.4) and (1.5) we consider the difference

$$D_1(b,\nu) := (K(b))^{\max\{\nu,1-\nu\}} - \exp[4\nu(1-\nu)(K(b)-1)]$$

for $b > 0$ and $v \in [0,1]$.

The plot of this difference in the $[0,2]\times[0,0.4]$ is depicted in Figure 1 showing that neither of the upper bounds in (1.4) and (1.5) is better always.

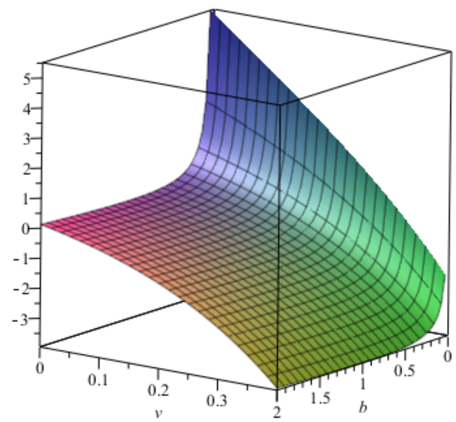

FIGURE 1. Plot of $D_1(b,v)$ on $[0,2]\times[0,0.4]$

For the inequalities (1.4) and (1.6) we consider the difference

$$D_2(b,\nu):=(K(b))^{\max\{\nu,1-\nu\}}-\exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-1)^2}{\min^2\{1,b\}}\right]$$

for $b>0$ and $v\in[0,1]$.

The plot of this difference in $[1,2]\times[0,0.5]$ is presented in Figure 2 showing that neither of the upper bounds in (1.4) and (1.6) is better always.

Now, if we want to compare the upper bounds provided by (1.4) and (1.7) we need to consider the difference

$$D_3(b,\nu):=(K(b))^{\max\{\nu,1-\nu\}}-\min\left\{\left(\frac{\exp(b-1)}{b}\right)^{\nu},\left(\frac{b}{\exp\left(1-\frac{1}{b}\right)}\right)^{1-\nu}\right\}$$

for $b>0$ and $v\in[0,1]$.

The graph of this difference in the box $[0,2]\times[0,1]$ is incorporated in Figure 3 proving that neither of the upper bounds in (1.4) and (1.7) is better always.

Now, for the inequalities (1.5) and (1.6) consider the difference

$$D_4(b,\nu):=\exp[4\nu(1-\nu)(K(b)-1)]-\exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-1)^2}{\min^2\{1,b\}}\right]$$

for $b>0$ and $v\in[0,1]$.

The graph of this difference in the box $[0,0.2]\times[0,0.4]$ is incorporated in Figure 4 and shows that neither of the upper bounds in (1.5) and (1.6) is always best.

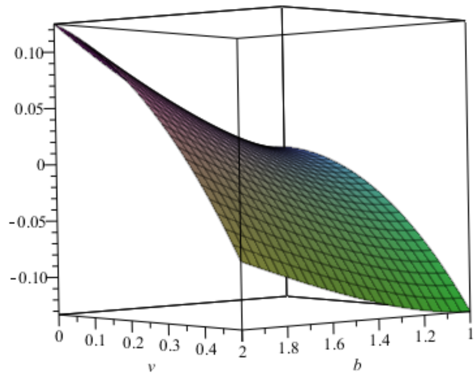

FIGURE 2. Plot of $D_2(b,v)$ on $[1,2]\times[0,0.5]$

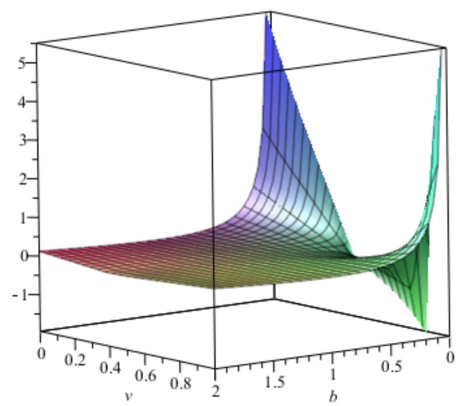

FIGURE 3. Plot of $D_3(b,v)$ on $[0,2]\times[0,1]$

Further, we consider for the inequalities (1.6) and (1.7) the difference

$$\begin{aligned} D_5(b,\nu) &:= \exp\left[4\nu(1-\nu)(K(b)-1)\right] \\ &\quad - \min\left\{\left(\frac{\exp(b-1)}{b}\right)^{\nu}, \left(\frac{b}{\exp\left(1-\frac{1}{b}\right)}\right)^{1-\nu}\right\} \end{aligned}$$

for $b>0$ and $v\in[0,1]$.

If we plot this difference on the box $[0,10]\times[0,0.1]$, see Figure 5 then we can

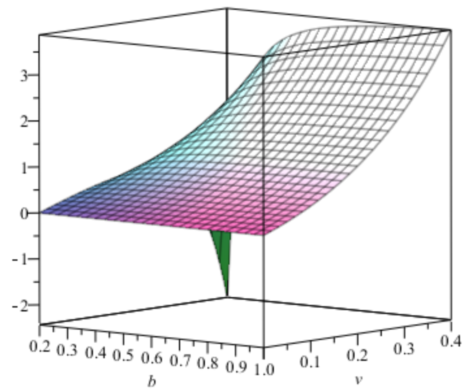

FIGURE 4. Plot of $D_4(b, v)$ on $[0, 0.2] \times [0, 0.4]$

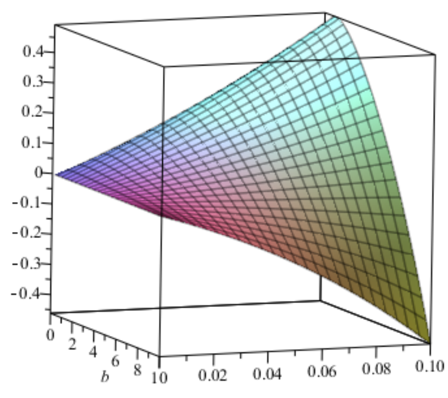

FIGURE 5. Plot of $D_5(b, v)$ on $[0, 10] \times [0, 0.1]$

conclude that neither of the upper bounds in (1.6) and (1.7) is always best.

Moreover, consider the difference

$$D_6(b, \nu) := \exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-1)^2}{\min^2\{1, b\}}\right] - \min\left\{\left(\frac{\exp(b-1)}{b}\right)^{\nu}, \left(\frac{b}{\exp\left(1-\frac{1}{b}\right)}\right)^{1-\nu}\right\}$$

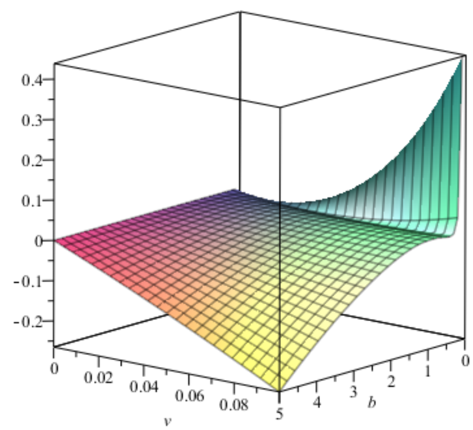

FIGURE 6. Plot of $D_6(b,v)$ on $[0,5]\times[0,0.1]$

for $b>0$ and $v\in[0,1]$.

The plot of this difference on the box $[0,5]\times[0,0.1]$ is incorporated in Figure 6 and shows that neither of the upper bounds in (1.5) and (1.7) is better always.

For the lower bounds, we need to study the differences

$$D_7(b,\nu):=S\left(b^{\min\{\nu,1-\nu\}}\right)-\exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-1)^2}{\max^2\{1,b\}}\right]$$

and

$$D_8(b,\nu):=(K(b))^{\min\{\nu,1-\nu\}}-\exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-1)^2}{\max^2\{1,b\}}\right]$$

for $b>0$ and $v\in[0,1]$.

The plot of the difference $D_7(b,\nu)$ in the box $[0,1]\times[0,1]$ is depicted in Figure 7 showing that neither of the lower bounds in (1.2) and (1.6) is always best.

Finally, if we plot the difference $D_8(b,\nu)$ on the box $[0,1]\times[0,1]$, see Figure 8 we conclude that neither of the lower bounds from (1.4) and (1.6) is always best.

3. Numerical Comparison with Tominaga's Bound

In this section we provide a numerical comparison of the weighted upper bounds from (1.4), (1.5), (1.6) and (1.7) with Tominaga's upper bound (1.2) which does not depend on the weight $\nu\in[0,1]$.

Consider the difference $D_9(b,\nu)$ defined by

$$D_9(b,\nu):=(K(b))^{\max\{\nu,1-\nu\}}-S(b)$$

for $b>0$ and $\nu\in[0,1]$.

The plot of $D_9(b,\nu)$ in the box $[0,0.5]\times[0,1]$ is depicted in Figure 9 and shows that neither of the upper bounds (1.4) and (1.2) is always best.

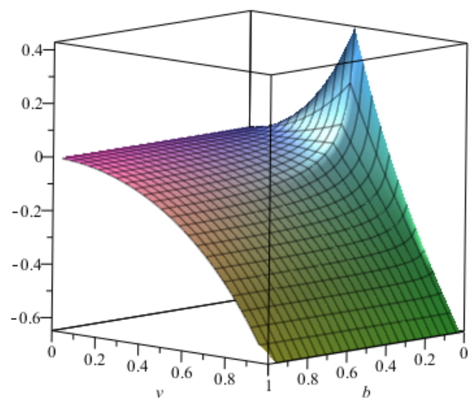

FIGURE 7. Plot of $D_7(b, v)$ on $[0,1] \times [0,1]$

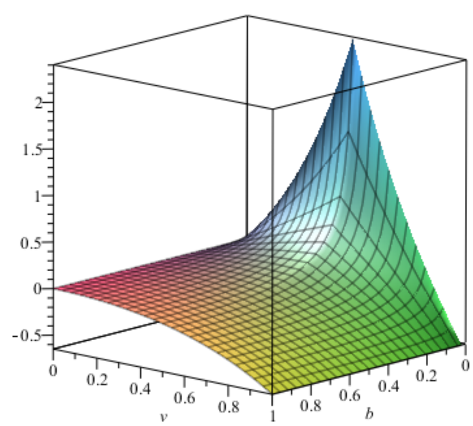

FIGURE 8. Plot of $D_8(b, v)$ on $[0,1] \times [0,1]$

Now, consider the difference $D_{10}(b, \nu)$ defined by

$$D_{10}(b, \nu) := \exp\left[4\nu(1-\nu)(K(b)-1)\right] - S(b)$$

for $b > 0$ and $\nu \in [0,1]$.

The graph of the difference $D_{10}(b, \nu)$ in the box $[0, 0.1] \times [0,1]$ is incorporated in Figure 10 showing that neither of the bounds (1.5) and (1.2) is always best.

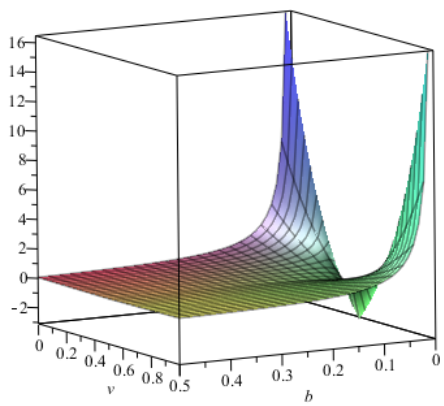

FIGURE 9. Plot of $D_9(b,v)$ on $[0,0.5]\times[0,1]$

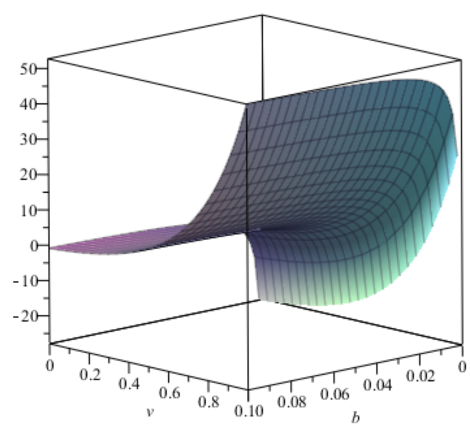

FIGURE 10. Plot of $D_{10}(b,v)$ on $[0,0.1]\times[0,1]$

Further, consider the difference

$$D_{11}(b,\nu) := \exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-1)^2}{\min^2\{1,b\}}\right] - S(b)$$

for $b>0$ and $\nu\in[0,1]$.

The plot of $D_{11}(b,\nu)$ in the box $[1,2]\times[0,0.3]$ is depicted in Figure 11 and shows that neither of the upper bounds (1.6) and (1.2) is always best.

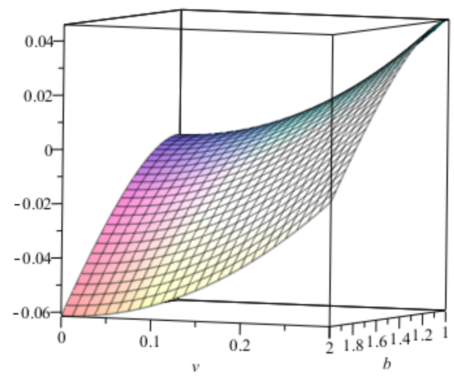

FIGURE 11. Plot of $D_{11}(b, v)$ on $[1,2] \times [0,0.3]$

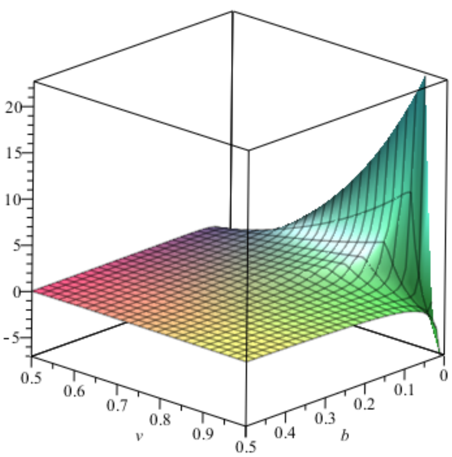

FIGURE 12. Plot of $D_{11}(b, v)$ on $[0,0.5] \times [0.5,1]$

Finally, define the difference

$$D_{12}\left(b,\nu\right) := \min\left\{\left(\frac{\exp\left(b-1\right)}{b}\right)^{\nu}, \left(\frac{b}{\exp\left(1-\frac{1}{b}\right)}\right)^{1-\nu}\right\} - S\left(b\right)$$

for $b > 0$ and $\nu \in [0,1]$.

The plot of this difference in the box $[0,0.5] \times [0.5,1]$ from Figure 12 proves that neither of the upper bounds (1.7) and (1.2) is always best.

We can draw the final

Conclusion 1. *Neither of the upper and lower bounds from the inequalities (1.2), (1.4), (1.5), (1.6) and (1.7) is always best.*

References

[1] H. Alzer, C. M. da Fonseca and A. Kovačec, Young-type inequalities and their matrix analogues, *Linear and Multilinear Algebra*, **63** (2015), Issue 3, 622-635.

[2] S. S. Dragomir, A Note on Young's Inequality, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 126. [http://rgmia.org/papers/v18/v18a126.pdf].

[3] S. S. Dragomir, A note on new refinements and reverses of Young's inequality, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 131. [http://rgmia.org/papers/v18/v18a131.pdf].

[4] S. S. Dragomir, New Refinements and Reverses of Hermite-Hadamard Inequality and Applications to Young's Operator Inequality, Preprint *RGMIA Res. Rep. Coll.* **19** (2016), Art. 42. [http://rgmia.org/papers/v19/v19a42.pdf].

[5] S. S. Dragomir, Some Asymmetric Reverses of Young's Scalar and Operator Inequalities with Applications, Preprint *RGMIA Res. Rep. Coll.* **19** (2016), Art. 44. [http://rgmia.org/papers/v19/v19a44.pdf].

[6] S. Furuichi, Refined Young inequalities with Specht's ratio, *J. Egyptian Math. Soc.* **20** (2012), 46–49.

[7] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.

[8] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.* **74** (1960), pp. 91-98.

[9] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.

[10] G. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.,* **5** (2011), 551-556.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: Sever.Dragomir@vu.edu.au; Alasdair.McAndrew@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa