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REVERSES OF CALLEBAUT DISCRETE INEQUALITY VIA SOME RESULTS DUE TO ZHUANG

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Abstract. In this paper, by the use of Zhuang's inequalities, we establish some reverse inequalities for the celebrated refinement of the Cauchy-Bunyakovsky-Schwarz that was obtained by Callebaut in 1965. A numerical comparison is also provided.

1. Introduction

The following inequality

$$x^{1-\nu}y^{\nu}\leq(1-\nu)\,x+\nu y \tag{1.1}$$

is well known in literature as either *weighted Arithmetic mean-Geometric mean inequality* or as *Young's inequality*.

In 1991, Y.-D. Zhuang [13] established the following inequality for $0<m\leq x\leq M$, $0<k\leq y\leq K$, and $\nu\in[0,1]$

$$\nu x+(1-\nu)\,y\leq\max\left\{\frac{\nu M+(1-\nu)\,k}{M^{\nu}k^{1-\nu}},\frac{\nu m+(1-\nu)\,K}{m^{\nu}K^{1-\nu}}\right\}x^{\nu}y^{1-\nu} \tag{1.2}$$

or

$$x+y\leq\max\left\{\frac{M+k}{M^{\nu}k^{1-\nu}},\frac{m+K}{m^{\nu}K^{1-\nu}}\right\}x^{\nu}y^{1-\nu}. \tag{1.3}$$

The sign of equality in (1.2) and (1.3) holds if and only if either $(x,y)=(m,K)$ or $(x,y)=(M,k)$.

Moreover, if $m\geq K$, then

$$\frac{\nu m+(1-\nu)\,K}{m^{\nu}K^{1-\nu}}x^{\nu}y^{1-\nu}\leq\nu x+(1-\nu)\,y\leq\frac{\nu M+(1-\nu)\,k}{M^{\nu}k^{1-\nu}}. \tag{1.4}$$

The sign of equality on the right-hand side of (1.4) holds if and only if $(x,y)=(M,k)$ and the sign of equality on the left-hand side of (1.4) holds if and only if $(x,y)=(m,K)$. The sign of inequality in (1.4) is reversed if $k\geq M$.

Now, if we take $y=1$, then we have from the above inequalities for $x\in[m,M]\subset(0,\infty)$ and $\nu\in[0,1]$ that

$$\nu x+1-\nu\leq\max\left\{\frac{\nu M+1-\nu}{M^{\nu}},\frac{\nu m+1-\nu}{m^{\nu}}\right\}x^{\nu} \tag{1.5}$$

and

$$x+1\leq\max\left\{\frac{M+1}{M^{\nu}},\frac{m+1}{m^{\nu}}\right\}x^{\nu}. \tag{1.6}$$

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If $m \geq 1$, then we have

$$\frac{\nu m+1-\nu}{m^{\nu}}x^{\nu} \leq \nu x+1-\nu \leq \frac{\nu M+1-\nu}{M^{\nu}}x^{\nu} \tag{1.7}$$

for $x \in [m, M]$ and $\nu \in (0, 1)$.

If $M \leq 1$, then we have

$$\frac{\nu M+1-\nu}{M^{\nu}}x^{\nu} \leq \nu x+1-\nu \leq \frac{\nu m+1-\nu}{m^{\nu}}x^{\nu}, \tag{1.8}$$

for $x \in [m, M]$ and $\nu \in (0, 1)$.

The inequalities (1.5), (1.7) and (1.8) can be put together as

$$\begin{cases} \frac{\nu M+1-\nu}{M^{\nu}}x^{\nu} \text{ if } M<1, \\ 1 \text{ if } m \leq 1 \leq M, \\ \frac{\nu m+1-\nu}{m^{\nu}}x^{\nu} \text{ if } 1<m. \end{cases} \leq \frac{\nu x+1-\nu}{x^{\nu}} \tag{1.9}$$

$$\leq \begin{cases} \frac{\nu m+1-\nu}{m^{\nu}} \text{ if } M<1, \\ \max\left\{\frac{\nu M+1-\nu}{M^{\nu}}, \frac{\nu m+1-\nu}{m^{\nu}}\right\} \text{ if } m \leq 1 \leq M, \\ \frac{\nu M+1-\nu}{M^{\nu}} \text{ if } 1<m, \end{cases}$$

for $x \in [m, M] \subset (0, \infty)$ and $\nu \in [0, 1]$.

We notice that the inequality (1.9) has been also obtained in [5] by a direct approach in studying the margins of the function $g(x) := \frac{\nu x+1-\nu}{x^{\nu}}$ with $x \in [m, M] \subset (0, \infty)$ and $\nu \in [0, 1]$.

For other similar results, see [1] and [3]-[12].

The following refinement of the Cauchy-Bunyakovsky-Schwarz was obtained by Callebaut [2] in 1965:

$$\left(\sum_{i=1}^{n} p_i a_i b_i\right)^2 \leq \sum_{i=1}^{n} p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{i=1}^{n} p_i a_i^{2\nu} b_i^{2(1-\nu)} \leq \sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2. \tag{1.10}$$

In this paper, by the use of Zhuang's inequalities (1.2) and (1.3) we establish some reverse inequalities for the quotient

$$\frac{\sum_{i\in I} p_i b_i^2 \sum_{i\in I} p a_i^2}{\sum_{i\in I} p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{i\in I} p_i a_i^{2\nu} b_i^{2(1-\nu)}}$$

under suitable conditions for the sequences a_k, $b_k > 0$, $k \in \mathbb{N}$ and $p_k \geq 0$, $k \in \mathbb{N}$.

2. Discrete Inequalities

We start with the following result:

Theorem 1. *Let a_k, $b_k > 0$, $k \in \mathbb{N}$ and I, J be finite sets of indices such that*

$$m \leq \frac{b_i}{a_i} \leq M \text{ and } k \leq \frac{b_j}{a_j} \leq K \tag{2.1}$$

for some constants $0 < m < M$, $0 < k < K$, *for any* $i \in I$ *and* $j \in J$. *If* $p_i \geq 0$ *for* $i \in I$, $q_j \geq 0$ *for* $j \in J$ *and* $\nu \in [0,1]$, *then we have the inequality*

$$\begin{aligned} &\nu \sum_{i \in I} p_i b_i^2 \sum_{j \in J} q_j a_j^2 + (1-\nu) \sum_{i \in I} p_i a_i^2 \sum_{j \in J} q_j b_j^2 \\ &\leq \max\left\{ \frac{\nu M^2 + (1-\nu) k^2}{M^{2\nu} k^{2(1-\nu)}}, \frac{\nu m^2 + (1-\nu) K^2}{m^{2\nu} K^{2(1-\nu)}} \right\} \\ &\times \sum_{i \in I} p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{j \in J} q_j a_j^{2\nu} b_j^{2(1-\nu)} \end{aligned} \tag{2.2}$$

and the inequality

$$\begin{aligned} &\sum_{i \in I} p_i b_i^2 \sum_{j \in J} q_j a_j^2 + \sum_{i \in I} p_i a_i^2 \sum_{j \in J} q_j b_j^2 \\ &\leq \max\left\{ \frac{M^2 + k^2}{M^{2\nu} k^{2(1-\nu)}}, \frac{m^2 + K^2}{m^{2\nu} K^{2(1-\nu)}} \right\} \sum_{i \in I} p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{j \in J} q_j a_j^{2\nu} b_j^{2(1-\nu)}. \end{aligned} \tag{2.3}$$

Proof. If we write the inequality (1.2) for $x = \left(\frac{b_i}{a_i}\right)^2$ and $y = \left(\frac{b_j}{a_j}\right)^2$, then we get

$$\begin{aligned} &\nu \left(\frac{b_i}{a_i}\right)^2 + (1-\nu) \left(\frac{b_j}{a_j}\right)^2 \\ &\leq \max\left\{ \frac{\nu M^2 + (1-\nu) k^2}{M^{2\nu} k^{2(1-\nu)}}, \frac{\nu m^2 + (1-\nu) K^2}{m^{2\nu} K^{2(1-\nu)}} \right\} \left(\frac{b_i}{a_i}\right)^{2\nu} \left(\frac{b_j}{a_j}\right)^{2(1-\nu)} \end{aligned} \tag{2.4}$$

for any $i \in I$ and $j \in J$.

By multiplying (2.4) with $a_i^2 a_j^2 \geq 0$ we get

$$\begin{aligned} &\nu b_i^2 a_j^2 + (1-\nu) a_i^2 b_j^2 \\ &\leq \max\left\{ \frac{\nu M^2 + (1-\nu) k^2}{M^{2\nu} k^{2(1-\nu)}}, \frac{\nu m^2 + (1-\nu) K^2}{m^{2\nu} K^{2(1-\nu)}} \right\} a_i^{2(1-\nu)} b_i^{2\nu} a_j^{2\nu} b_j^{2(1-\nu)} \end{aligned} \tag{2.5}$$

for any $i \in I$ and $j \in J$.

Multiply the inequality (2.5) by $q_j \geq 0$ and sum over $j \in J$ to get

$$\begin{aligned} &\nu b_i^2 \sum_{j \in J} q_j a_j^2 + (1-\nu) a_i^2 \sum_{j \in J} q_j b_j^2 \\ &\leq \max\left\{ \frac{\nu M^2 + (1-\nu) k^2}{M^{2\nu} k^{2(1-\nu)}}, \frac{\nu m^2 + (1-\nu) K^2}{m^{2\nu} K^{2(1-\nu)}} \right\} a_i^{2(1-\nu)} b_i^{2\nu} \sum_{j \in J} q_j a_j^{2\nu} b_j^{2(1-\nu)} \end{aligned} \tag{2.6}$$

for any $i \in I$.

If we multiply (2.6) by $p_i \geq 0$ and sum over $i \in I$ we get the desired inequality (2.2).

By the inequality (1.3) for $x = \left(\frac{b_i}{a_i}\right)^2$ and $y = \left(\frac{b_j}{a_j}\right)^2$ we have

$$\left(\frac{b_i}{a_i}\right)^2 + \left(\frac{b_j}{a_j}\right)^2 \leq \max\left\{ \frac{M^2 + k^2}{M^{2\nu} k^{2(1-\nu)}}, \frac{m^2 + K^2}{m^{2\nu} K^{2(1-\nu)}} \right\} \left(\frac{b_i}{a_i}\right)^{2\nu} \left(\frac{b_j}{a_j}\right)^{2(1-\nu)}, \tag{2.7}$$

for any $i \in I$ and $j \in J$. On making use of a similar argument as above, we deduce (2.3). □

Corollary 1. *Let a_k, $b_k > 0$, $k \in \mathbb{N}$ and I a finite set of indices such that*

$$m \leq \frac{b_i}{a_i} \leq M \tag{2.8}$$

for some constants $0 < m < M$, for any $i \in I$. If $p_i \geq 0$ for $i \in I$ and $\nu \in [0,1]$, then we have the inequality

$$\begin{aligned} &\frac{\sum_{i\in I} p_i b_i^2 \sum_{i\in I} p a_i^2}{\sum_{i\in I} p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{i\in I} p_i a_i^{2\nu} b_i^{2(1-\nu)}} \\ &\leq \max\left\{\frac{\nu M^2 + (1-\nu) m^2}{M^{2\nu} m^{2(1-\nu)}}, \frac{\nu m^2 + (1-\nu) M^2}{m^{2\nu} M^{2(1-\nu)}}\right\} \end{aligned} \tag{2.9}$$

and the inequality

$$\begin{aligned} &\frac{\sum_{i\in I} p_i b_i^2 \sum_{i\in I} p a_i^2}{\sum_{i\in I} p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{i\in I} p_i a_i^{2\nu} b_i^{2(1-\nu)}} \\ &\leq \frac{M^2 + m^2}{2} \max\left\{\frac{1}{M^{2\nu} m^{2(1-\nu)}}, \frac{1}{m^{2\nu} M^{2(1-\nu)}}\right\}. \end{aligned} \tag{2.10}$$

The inequalities (2.9) and (2.10) therefore provide multiplicative reverses of the second Callebaut inequality (1.10).

The following result also holds:

Theorem 2. *Let a_k, $b_k > 0$, $k \in \mathbb{N}$ and I a finite set of indices such that*

$$a \leq a_i \leq A \text{ and } b \leq b_i \leq B \tag{2.11}$$

for some constants $0 < a < A$, $0 < b < B$ for any $i \in I$. If $w_i \geq 0$ for $i \in I$ with $\sum_{i\in I} w_i = 1$ and $\nu \in [0,1]$, then we have the inequality

$$\begin{aligned} &\frac{\left(\sum_{i\in I} w_i a_i^2\right)^\nu \left(\sum_{i\in I} w_i b_i^2\right)^{1-\nu}}{\sum_{j\in I} w_j a_j^{2\nu} b_j^{2(1-\nu)}} \\ &\leq \max\left\{\frac{\nu A^2 B^2 + (1-\nu) a^2 b^2}{A^{2\nu} a^{2(1-\nu)} B^{2\nu} b^{2(1-\nu)}}, \frac{\nu a^2 b^2 + (1-\nu) A^2 B^2}{A^{2(1-\nu)} a^{2\nu} B^{2(1-\nu)} b^{2\nu}}\right\} \end{aligned} \tag{2.12}$$

and the inequality

$$\begin{aligned} &\frac{\left(\sum_{i\in I} w_i a_i^2\right)^\nu \left(\sum_{i\in I} w_i b_i^2\right)^{1-\nu}}{\sum_{j\in I} w_j a_j^{2\nu} b_j^{2(1-\nu)}} \\ &\leq \frac{A^2 B^2 + a^2 b^2}{2} \\ &\times \max\left\{\frac{1}{A^{2\nu} a^{2(1-\nu)} B^{2\nu} b^{2(1-\nu)}}, \frac{1}{A^{2(1-\nu)} a^{2\nu} B^{2(1-\nu)} b^{2\nu}}\right\}. \end{aligned} \tag{2.13}$$

Proof. Let $x = \frac{a_j^2}{\sum_{i\in I} w_i a_i^2}$ and $y = \frac{b_j^2}{\sum_{i\in I} w_i b_i^2}$, for $j \in I$, then we get

$$\frac{a^2}{A^2} \leq x \leq \frac{A^2}{a^2},\ j \in I$$

and

$$\frac{b^2}{B^2} \leq y \leq \frac{B^2}{b^2},\ j \in I.$$

If we write the inequality (1.2) for x and y as above, then we get

$$\begin{aligned}(2.14)\qquad &\nu\frac{a_j^2}{\sum_{i\in I}w_ia_i^2}+(1-\nu)\frac{b_j^2}{\sum_{i\in I}w_ib_i^2}\\ &\leq\max\left\{\frac{\nu\frac{A^2}{a^2}+(1-\nu)\frac{b^2}{B^2}}{\left(\frac{A^2}{a^2}\right)^{\nu}\left(\frac{b^2}{B^2}\right)^{1-\nu}},\frac{\nu\frac{a^2}{A^2}+(1-\nu)\frac{B^2}{b^2}}{\left(\frac{a^2}{A^2}\right)^{\nu}\left(\frac{B^2}{b^2}\right)^{1-\nu}}\right\}\\ &\times\frac{a_j^{2\nu}}{\left(\sum_{i\in I}w_ia_i^2\right)^{\nu}}\frac{b_j^{2(1-\nu)}}{\left(\sum_{i\in I}w_ib_i^2\right)^{1-\nu}}\end{aligned}$$

for any $j\in I$.

Since

$$\frac{\nu\frac{A^2}{a^2}+(1-\nu)\frac{b^2}{B^2}}{\left(\frac{A^2}{a^2}\right)^{\nu}\left(\frac{b^2}{B^2}\right)^{1-\nu}}=\frac{\nu A^2B^2+(1-\nu)a^2b^2}{A^{2\nu}a^{2(1-\nu)}B^{2\nu}b^{2(1-\nu)}}$$

and

$$\frac{\nu\frac{a^2}{A^2}+(1-\nu)\frac{B^2}{b^2}}{\left(\frac{a^2}{A^2}\right)^{\nu}\left(\frac{B^2}{b^2}\right)^{1-\nu}}=\frac{\nu a^2b^2+(1-\nu)A^2B^2}{A^{2(1-\nu)}a^{2\nu}B^{2(1-\nu)}b^{2\nu}},$$

then by (2.14) we have

$$\begin{aligned}(2.15)\qquad &\nu\frac{a_j^2}{\sum_{i\in I}w_ia_i^2}+(1-\nu)\frac{b_j^2}{\sum_{i\in I}w_ib_i^2}\\ &\leq\max\left\{\frac{\nu A^2B^2+(1-\nu)a^2b^2}{A^{2\nu}a^{2(1-\nu)}B^{2\nu}b^{2(1-\nu)}},\frac{\nu a^2b^2+(1-\nu)A^2B^2}{A^{2(1-\nu)}a^{2\nu}B^{2(1-\nu)}b^{2\nu}}\right\}\\ &\times\frac{a_j^{2\nu}}{\left(\sum_{i\in I}w_ia_i^2\right)^{\nu}}\frac{b_j^{2(1-\nu)}}{\left(\sum_{i\in I}w_ib_i^2\right)^{1-\nu}}\end{aligned}$$

for any $j\in I$.

If we multiply (2.15) by w_j and sum, then we get

$$\begin{aligned}&\nu\frac{\sum_{j\in I}w_ja_j^2}{\sum_{i\in I}w_ia_i^2}+(1-\nu)\frac{\sum_{j\in I}w_jb_j^2}{\sum_{i\in I}w_ib_i^2}\\ &\leq\max\left\{\frac{\nu A^2B^2+(1-\nu)a^2b^2}{A^{2\nu}a^{2(1-\nu)}B^{2\nu}b^{2(1-\nu)}},\frac{\nu a^2b^2+(1-\nu)A^2B^2}{A^{2(1-\nu)}a^{2\nu}B^{2(1-\nu)}b^{2\nu}}\right\}\\ &\times\frac{\sum_{j\in I}w_ja_j^{2\nu}b_j^{2(1-\nu)}}{\left(\sum_{i\in I}w_ia_i^2\right)^{\nu}\left(\sum_{i\in I}w_ib_i^2\right)^{1-\nu}}\end{aligned}$$

that is equivalent to (2.12).

By the inequality (1.3) we also have

$$\begin{aligned}(2.16)\qquad &\frac{a_j^2}{\sum_{i\in I}w_ia_i^2}+\frac{b_j^2}{\sum_{i\in I}w_ib_i^2}\\ &\leq\max\left\{\frac{\frac{A^2}{a^2}+\frac{b^2}{B^2}}{\left(\frac{A^2}{a^2}\right)^{\nu}\left(\frac{b^2}{B^2}\right)^{1-\nu}},\frac{\frac{a^2}{A^2}+\frac{B^2}{b^2}}{\left(\frac{a^2}{A^2}\right)^{\nu}\left(\frac{B^2}{b^2}\right)^{1-\nu}}\right\}\\ &\times\frac{a_j^{2\nu}}{\left(\sum_{i\in I}w_ia_i^2\right)^{\nu}}\frac{b_j^{2(1-\nu)}}{\left(\sum_{i\in I}w_ib_i^2\right)^{1-\nu}}\end{aligned}$$

for any $j\in I$ and since

$$\begin{aligned}
&\max\left\{\frac{\frac{A^2}{a^2}+\frac{b^2}{B^2}}{\left(\frac{A^2}{a^2}\right)^{\nu}\left(\frac{b^2}{B^2}\right)^{1-\nu}},\frac{\frac{a^2}{A^2}+\frac{B^2}{b^2}}{\left(\frac{a^2}{A^2}\right)^{\nu}\left(\frac{B^2}{b^2}\right)^{1-\nu}}\right\}\\
&=\left(A^2B^2+a^2b^2\right)\\
&\times\max\left\{\frac{1}{A^{2\nu}a^{2(1-\nu)}B^{2\nu}b^{2(1-\nu)}},\frac{1}{A^{2(1-\nu)}a^{2\nu}B^{2(1-\nu)}b^{2\nu}}\right\},
\end{aligned}$$

then by (2.16) we get

$$\begin{aligned}
&\frac{a_j^2}{\sum_{i\in I}w_ia_i^2}+\frac{b_j^2}{\sum_{i\in I}w_ib_i^2}\\
&\le\left(A^2B^2+a^2b^2\right)\\
&\times\max\left\{\frac{1}{A^{2\nu}a^{2(1-\nu)}B^{2\nu}b^{2(1-\nu)}},\frac{1}{A^{2(1-\nu)}a^{2\nu}B^{2(1-\nu)}b^{2\nu}}\right\}\\
&\times\frac{a_j^{2\nu}}{\left(\sum_{i\in I}w_ia_i^2\right)^{\nu}}\frac{b_j^{2(1-\nu)}}{\left(\sum_{i\in I}w_ib_i^2\right)^{1-\nu}}
\end{aligned}\tag{2.17}$$

for any $j\in I$.

If we multiply (2.17) by w_j and sum, then we get the desired result (2.13). □

Remark 1. *With the assumptions of Theorem 2 we have the Callebaut reverse inequalities*

$$\begin{aligned}
&\frac{\sum_{i\in I}w_ia_i^2\sum_{i\in I}w_ib_i^2}{\sum_{j\in I}w_ja_j^{2\nu}b_j^{2(1-\nu)}\sum_{j\in I}w_ja_j^{2(1-\nu)}b_j^{2\nu}}\\
&\le\max\left\{\frac{\left(\nu A^2B^2+(1-\nu)a^2b^2\right)^2}{A^{4\nu}a^{4(1-\nu)}B^{4\nu}b^{4(1-\nu)}},\frac{\left(\nu a^2b^2+(1-\nu)A^2B^2\right)^2}{A^{4(1-\nu)}a^{4\nu}B^{4(1-\nu)}b^{4\nu}}\right\}
\end{aligned}\tag{2.18}$$

and

$$\begin{aligned}
&\frac{\sum_{i\in I}w_ia_i^2\sum_{i\in I}w_ib_i^2}{\sum_{j\in I}w_ja_j^{2\nu}b_j^{2(1-\nu)}\sum_{j\in I}w_ja_j^{2(1-\nu)}b_j^{2\nu}}\\
&\le\left(\frac{A^2B^2+a^2b^2}{2}\right)^2\\
&\times\max\left\{\frac{1}{A^{4\nu}a^{4(1-\nu)}B^{4\nu}b^{4(1-\nu)}},\frac{1}{A^{4(1-\nu)}a^{4\nu}B^{4(1-\nu)}b^{4\nu}}\right\}.
\end{aligned}\tag{2.19}$$

Indeed, by the inequality (2.12) for $1-\nu$ instead of ν we have

$$\begin{aligned}
&\frac{\left(\sum_{i\in I}w_ia_i^2\right)^{1-\nu}\left(\sum_{i\in I}w_ib_i^2\right)^{\nu}}{\sum_{j\in I}w_ja_j^{2(1-\nu)}b_j^{2\nu}}\\
&\le\max\left\{\frac{(1-\nu)A^2B^2+\nu a^2b^2}{A^{2(1-\nu)}a^{2\nu}B^{2(1-\nu)}b^{2\nu}},\frac{(1-\nu)a^2b^2+\nu A^2B^2}{A^{2\nu}a^{2(1-\nu)}B^{2\nu}b^{2(1-\nu)}}\right\}.
\end{aligned}\tag{2.20}$$

If we multiply (2.12) with (2.20) we obtain (2.18).

The inequality (2.19) follows in a similar way by (2.13) and the details are omitted.

The inequalities from (2.19) and (2.20) can be however improved as follows:

Theorem 3. *Let $a_k, b_k > 0$, $k \in \mathbb{N}$ and I a finite set of indices such that the inequality (2.11) is valid for some constants $0 < a < A$, $0 < b < B$ for any $i \in I$. If $w_i \geq 0$ for $i \in I$ with $\sum_{i\in I} w_i = 1$ and $\nu \in [0,1]$, then we have the inequalities*

$$\begin{aligned} &\frac{\sum_{i\in I} w_i a_i^2 \sum_{i\in I} w_i b_i^2}{\sum_{j\in I} w_j a_j^{2\nu} b_j^{2(1-\nu)} \sum_{j\in I} w_j a_j^{2(1-\nu)} b_j^{2\nu}} \\ &\leq \max\left\{ \frac{\nu A^2B^2 + (1-\nu)\, a^2b^2}{A^{2\nu}B^{2\nu}a^{2(1-\nu)}b^{2(1-\nu)}}, \frac{\nu a^2b^2 + (1-\nu)\, A^2B^2}{a^{2\nu}b^{2\nu}A^{2(1-\nu)}B^{2(1-\nu)}} \right\} \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} &\frac{\sum_{i\in I} w_i a_i^2 \sum_{i\in I} w_i b_i^2}{\sum_{j\in I} w_j a_j^{2\nu} b_j^{2(1-\nu)} \sum_{j\in I} w_j a_j^{2(1-\nu)} b_j^{2\nu}} \\ &\leq \frac{A^2B^2 + a^2b^2}{2} \\ &\times \max\left\{ \frac{1}{A^{2\nu}B^{2\nu}a^{2(1-\nu)}b^{2(1-\nu)}}, \frac{1}{a^{2\nu}b^{2\nu}A^{2(1-\nu)}B^{2(1-\nu)}} \right\}. \end{aligned} \tag{2.22}$$

Proof. Let $x = a_i^2 b_j^2$ and $y = a_j^2 b_i^2$ for i, $j \in I$. Then by the condition (2.11) we have

$$a^2b^2 \leq x \leq A^2B^2 \text{ and } a^2b^2 \leq y \leq A^2B^2.$$

By the inequalities (1.2) and (1.3) we have

$$\begin{aligned} &\nu a_i^2 b_j^2 + (1-\nu)\, a_j^2 b_i^2 \\ &\leq \max\left\{ \frac{\nu A^2B^2 + (1-\nu)\, a^2b^2}{\left(A^2B^2\right)^{\nu}\left(a^2b^2\right)^{1-\nu}}, \frac{\nu a^2b^2 + (1-\nu)\, A^2B^2}{\left(a^2b^2\right)^{\nu}\left(A^2B^2\right)^{1-\nu}} \right\} \\ &\times \left(a_i^2b_j^2\right)^{\nu}\left(a_j^2b_i^2\right)^{1-\nu} \\ &= \max\left\{ \frac{\nu A^2B^2 + (1-\nu)\, a^2b^2}{A^{2\nu}B^{2\nu}a^{2(1-\nu)}b^{2(1-\nu)}}, \frac{\nu a^2b^2 + (1-\nu)\, A^2B^2}{a^{2\nu}b^{2\nu}A^{2(1-\nu)}B^{2(1-\nu)}} \right\} \\ &\times a_i^{2\nu} b_i^{2(1-\nu)} a_j^{2(1-\nu)} b_j^{2\nu} \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} a_i^2b_j^2 + a_j^2b_i^2 &\leq \left(A^2B^2 + a^2b^2\right) \\ &\times \max\left\{ \frac{1}{A^{2\nu}B^{2\nu}a^{2(1-\nu)}b^{2(1-\nu)}}, \frac{1}{a^{2\nu}b^{2\nu}A^{2(1-\nu)}B^{2(1-\nu)}} \right\} \\ &\times a_i^{2\nu} b_i^{2(1-\nu)} a_j^{2(1-\nu)} b_j^{2\nu}, \end{aligned} \tag{2.24}$$

for i, $j \in I$.

If we multiply (2.23) and (2.24) by $w_i w_j$ and sum over i, $j \in I$ we get the desired inequalities (2.21) and (2.22). □

3. A Numerical Comparison

We consider the *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \ h > 0. \tag{3.1}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative reverse of Young inequality in terms of Kantorovich's ratio holds

$$(1-\nu)a + \nu b \leq K^R\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu}, \tag{3.2}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

This inequality was obtained by Liao et al. [10].

In [8] the first author obtained the following reverse of Callebaut inequality

$$\frac{\sum_{i\in I} p_i b_i^2 \sum_{i\in I} p a_i^2}{\sum_{i\in I} p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{i\in I} p_i a_i^{2\nu} b_i^{2(1-\nu)}} \leq K^{\max\{v,1-v\}}\left(\left(\frac{M}{m}\right)^2\right) \tag{3.3}$$

where a_k, $b_k > 0$, $k \in \mathbb{N}$ and I a finite set of indices such that the condition (2.11) is valid for some constants $0 < a < A$, $0 < b < B$ for any $i \in I$, $w_i \geq 0$ for $i \in I$ with $\sum_{i\in I} w_i = 1$ and $\nu \in [0,1]$.

From (2.9), (2.10) and (3.3) we have the following upper bounds for the quotient

$$\begin{aligned} &\frac{\sum_{i\in I} p_i b_i^2 \sum_{i\in I} p a_i^2}{\sum_{i\in I} p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{i\in I} p_i a_i^{2\nu} b_i^{2(1-\nu)}} \\ &\leq B_1(m,M,\nu),\ B_2(m,M,\nu),\ B_3(m,M,\nu) \end{aligned} \tag{3.4}$$

where

$$B_1(m,M,\nu) := \max\left\{\frac{\nu M^2 + (1-\nu)m^2}{M^{2\nu} m^{2(1-\nu)}}, \frac{\nu m^2 + (1-\nu)M^2}{m^{2\nu} M^{2(1-\nu)}}\right\},$$

$$B_2(m,M,\nu) := \frac{M^2+m^2}{2}\max\left\{\frac{1}{M^{2\nu} m^{2(1-\nu)}}, \frac{1}{m^{2\nu} M^{2(1-\nu)}}\right\},$$

and

$$B_3(m,M,\nu) := K^{\max\{v,1-v\}}\left(\left(\frac{M}{m}\right)^2\right).$$

Here $0 < m \leq M < \infty$ and $v \in [0,1]$.

For $m = 1$, we consider the differences

$$\begin{aligned} D_1(M,\nu) &: = B_1(1,M,\nu) - B_2(1,M,\nu), \\ D_2(M,\nu) &: = B_3(1,M,\nu) - B_1(1,M,\nu), \\ D_3(M,\nu) &: = B_3(1,M,\nu) - B_2(1,M,\nu) \end{aligned}$$

for $M \geq 1$ and $v \in [0,1]$.

The plots of the differences $D_1(M,\nu)$, $D_2(M,\nu)$ and $D_3(M,\nu)$ in the box $[1,3]\times[0,1]$ are depicted in Figures 1, 2 and 3 below. They show that in (3.4) the bound B_1 is better than B_3 that is better than B_2.

Problem 1. *Is the following inequality*

$$B_1(m,M,\nu) \leq B_3(m,M,\nu) \leq B_2(m,M,\nu)$$

valid for any $0 < m \leq M < \infty$ *and* $v \in [0,1]$?

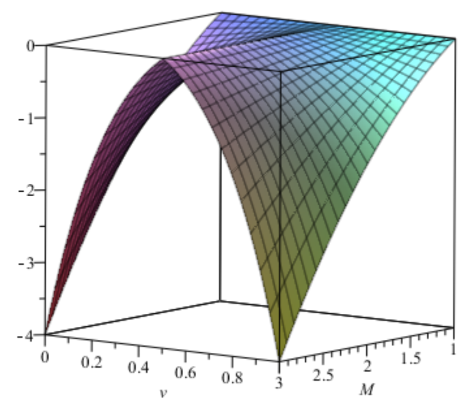


FIGURE 1. Plot of $D_1(M, v)$ in $[0,3] \times [0,1]$

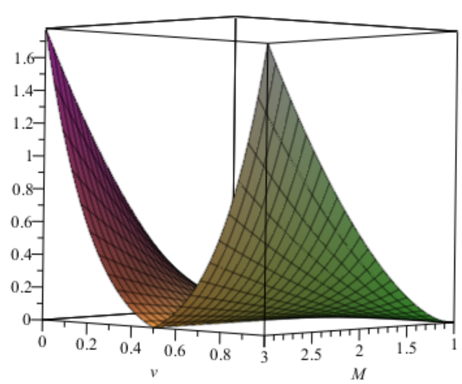
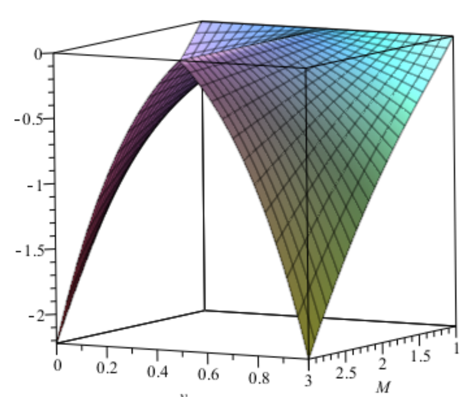


FIGURE 2. Plot of $D_2(M, \nu)$ on $[1,3] \times [0,1]$



Plot of $D_3(M,\nu)$ on $[1,3]\times[0,1]$

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