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NEW INEQUALITIES OF OPIAL TYPE FOR CONFORMABLE FRACTIONAL INTEGRALS

MEHMET ZEKI SARIKAYA AND HÜSEYIN BUDAK

Abstract. In this paper, some Opial type inequalities for conformable fractional integral are obtained using the remainder function of Taylor Theorem for conformable integral.

1. Introduction

In the year 1960, Opial established the following interesting integral inequality[10]:

Theorem 1. *Let $x(t) \in C^{(1)}[0,h]$ be such that $x(0)=x(h)=0$, and $x(t)>0$ in $(0,h)$. Then, the following inequality holds*

$$\int_0^h |x(t)x'(t)|\,dt \leq \frac{h}{4}\int_0^h (x'(t))^2\,dt \tag{1.1}$$

The constant $h/4$ is best possible

Opial's inequality and its generalizations, extensions and discretizations, play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations. Over the last twenty years a large number of papers have been appeared in the literature which deals with the simple proofs, various generalizations and discrete analogues of Opial inequality and its generalizations, see [2],[4],[5], [11]-[14],[16],[17].

The purpose of this paper is to establish some Opial type inequalities for conformable integral. The structure of this paper is as follows:. In Section 2, we give the definitions of the conformable derivatives and conformable integral and introduce several useful notations conformable integral used our main results. In Section 3, the main result is presented. Using the remainder function of Taylor Theorem for conformable integral, we establish several Opial type inequalities.

2. Definitions and properties of conformable fractional derivative and integral

The following definitions and theorems with respect to conformable fractional derivative and integral were referred in (see, [1], [3], [6]-[9]).

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Definition 1 (Conformable fractional derivative)**.** *Given a function $f:[0,\infty)\to\mathbb{R}$. Then the "conformable fractional derivative" of f of order α is defined by*

$$D_{\alpha}(f)(t)=\lim_{\epsilon\to 0}\frac{f\left(t+\epsilon t^{1-\alpha}\right)-f(t)}{\epsilon} \tag{2.1}$$

for all $t>0,\ \alpha\in(0,1)$. If f is $\alpha-$differentiable in some $(0,a)$, $\alpha>0$, $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exist, then define

$$f^{(\alpha)}(0)=\lim_{t\to 0^+} f^{(\alpha)}(t). \tag{2.2}$$

We can write $f^{(\alpha)}(t)$ for $D_{\alpha}(f)(t)$ to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is $\alpha-$differentiable.

Theorem 2. *Let $\alpha\in(0,1]$ and f,g be $\alpha-$differentiable at a point $t>0$. Then*

i. $D_{\alpha}(af+bg)=aD_{\alpha}(f)+bD_{\alpha}(g)$, for all $a,b\in\mathbb{R}$,

ii. $D_{\alpha}(\lambda)=0$, for all constant functions $f(t)=\lambda$,

iii. $D_{\alpha}(fg)=fD_{\alpha}(g)+gD_{\alpha}(f)$,

iv. $D_{\alpha}\left(\frac{f}{g}\right)=\frac{fD_{\alpha}(g)-gD_{\alpha}(f)}{g^2}$.

If f is differentiable, then

$$D_{\alpha}(f)(t)=t^{1-\alpha}\frac{df}{dt}(t). \tag{2.3}$$

Definition 2 (Conformable fractional integral)**.** *Let $\alpha\in(0,1]$ and $0\leq a<b$. A function $f:[a,b]\to\mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral*

$$\int_a^b f(x)\,d_{\alpha}x:=\int_a^b f(x)\,x^{\alpha-1}dx \tag{2.4}$$

exists and is finite. All α-fractional integrable on $[a,b]$ is indicated by $L_{\alpha}^{1}([a,b])$.

Remark 1.

$$I_{\alpha}^{a}(f)(t)=I_{1}^{a}\left(t^{\alpha-1}f\right)=\int_a^t\frac{f(x)}{x^{1-\alpha}}dx,$$

where the integral is the usual Riemann improper integral, and $\alpha\in(0,1]$.

Theorem 3. *Let $f:(a,b)\to\mathbb{R}$ be differentiable and $0<\alpha\leq 1$. Then, for all $t>a$ we have*

$$I_{\alpha}^{a}D_{\alpha}^{a}f(t)=f(t)-f(a). \tag{2.5}$$

Theorem 4 (Integration by parts)**.** *Let $f,g:[a,b]\to\mathbb{R}$ be two functions such that fg is differentiable. Then*

$$\int_a^b f(x)\,D_{\alpha}^{a}(g)(x)\,d_{\alpha}x=\left.fg\right|_a^b-\int_a^b g(x)\,D_{\alpha}^{a}(f)(x)\,d_{\alpha}x. \tag{2.6}$$

Theorem 5. *Assume that $f:[a,\infty)\to\mathbb{R}$ such that $f^{(n)}(t)$ is continuous and $\alpha\in(n,n+1]$. Then, for all $t>a$ we have*

$$D_{\alpha}^{a}f(t)\,I_{\alpha}^{a}=f(t).$$

We can give the Hölder's inequality in conformable integral as follows:

Lemma 1. *Let $f, g \in C[a,b]$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\int_a^b |f(x)g(x)|\, d_\alpha x \leq \left(\int_a^b |f(x)|^p\, d_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\, d_\alpha x\right)^{\frac{1}{q}}.$$

Remark 2. *If we take $p = q = 2$ in Lemma 1 , the we have the Cauchy-Schwartz inequality for conformable integral.*

Theorem 6 (Taylor Formula)**.** [3] *Let $\alpha \in (0,1]$ and $n \in \mathbb{N}$. Suppose f is $n+1$ times $\alpha-$ fractional differentiable on $[0,\infty)$, and $s, t \in [0,\infty)$. Then we have*

$$f(t) = \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^k D_\alpha^k f(s) + \frac{1}{n!} \int_s^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha}\right)^n D_\alpha^{n+1} f(\tau) d_\alpha \tau.$$

Using the Taylor's Theorem, we define tthe remainder function by

$$R_{-1,f}(.,s) := f(s),$$

and for $n > -1$,

$$\begin{aligned} R_{n,f}(t,s) \quad : &= f(s) - \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^k D_\alpha^k f(s) \\ &= \frac{1}{n!} \int_s^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha}\right)^n D_\alpha^{n+1} f(\tau) d_\alpha \tau. \end{aligned} \tag{2.7}$$

Opial inequality can be represented for conformable fractional integral forms as follows [15]:

Theorem 7. *Let $\alpha \in (0,1]$ and u be an α-fractional differantiable function on $(0,h)$ with $u(0) = u(h) = 0$. Then, the following inequality for conformable fractional integral holds:*

$$\int_0^h |u(t) D_\alpha(u)(t)|\, d_\alpha t \leq \frac{h^\alpha}{4\alpha} \int_0^h |D_\alpha(u)(t)|^2\, d_\alpha t. \tag{2.8}$$

Now, we present the main results:

3. Opial type inequalities for conformable fractional integral

Let $\alpha \in (0,1]$. In the following we adapt to the α-fractional setting some results from [2] by applying the fractional Opial inequality.

Theorem 8. *Let $\alpha \in (0,1]$, $f : [a,b] \to \mathbb{R}$ be an $n+1$ times $\alpha-$ fractional differentiable function, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $t \geq x_0$, $t, x_0 \in [a,b]$. Then, we have the* *following inequality*

$$\int_{x_0}^{t} |R_{n,f}(x_0,\tau)| \left|D_\alpha^{n+1} f(\tau)\right| d_\alpha \tau \tag{3.1}$$

$$\leq \frac{(t^\alpha - x_0^\alpha)^{n+2/p}}{\alpha^{n+2/p} 2^{\frac{1}{q}} n! \left[(np+1)(np+2)\right]^{1/p}} \left(\int_{x_0}^{t} \left|D_\alpha^{n+1} f(\tau)\right|^q d_\alpha \tau \right)^{\frac{2}{q}}.$$

Proof. From (2.7), we have

$$R_{n,f}(x_0,t) = \frac{1}{n!} \int_{x_0}^{t} \left(\frac{t^\alpha - \tau^\alpha}{\alpha} \right)^n D_\alpha^{n+1} f(\tau) d_\alpha \tau, \ x_0, t \in [a,b].$$

By using the Hölder inequality for conformable integral, it follows that

$$\begin{aligned} |R_{n,f}(x_0,t)| &\leq \frac{1}{\alpha^n n!} \int_{x_0}^{t} (t^\alpha - \tau^\alpha)^n \left|D_\alpha^{n+1} f(\tau)\right| d_\alpha \tau \\ &\leq \frac{1}{\alpha^n n!} \left(\int_{x_0}^{t} (t^\alpha - \tau^\alpha)^{np} d_\alpha \tau \right)^{\frac{1}{p}} \left(\int_{x_0}^{t} \left|D_\alpha^{n+1} f(\tau)\right|^q d_\alpha \tau \right)^{\frac{1}{q}} \\ &= \frac{1}{\alpha^{n+1/p} n!} \frac{(t^\alpha - x_0^\alpha)^{n+1/p}}{(np+1)^{1/p}} (z(t))^{\frac{1}{q}} \end{aligned} \tag{3.2}$$

where

$$z(t) = \int_{x_0}^{t} \left|D_\alpha^{n+1} f(\tau)\right|^q d_\alpha \tau, \ x_0 \leq t \leq b, \ z(x_0) = 0.$$

Thus,

$$D_\alpha z(t) = \left|D_\alpha^{n+1} f(t)\right|^q$$

and

$$\left|D_\alpha^{n+1} f(t)\right| = (D_\alpha z(t))^{1/q}. \tag{3.3}$$

By (3.2) and (3.3), we get

$$|R_{n,f}(x_0,t)| \left|D_\alpha^{n+1} f(t)\right| \leq \frac{1}{\alpha^{n+1/p} n!} \frac{(t^\alpha - x_0^\alpha)^{n+1/p}}{(np+1)^{1/p}} (z(t) D_\alpha z(t))^{\frac{1}{q}}. \tag{3.4}$$

Integrating the inequality (3.4) and using the Hölder inequality for conformable integral, we have

$$\begin{aligned}
&\int_{x_0}^{t} |R_{n,f}(x_0,\tau)| \left|D_\alpha^{n+1} f(\tau)\right| d_\alpha \tau \\
\leq\ & \frac{1}{\alpha^{n+1/p} n! (np+1)^{1/p}} \int_{x_0}^{t} \left(\tau^\alpha - x_0^\alpha\right)^{n+1/p} \left(z(\tau) D_\alpha z(\tau)\right)^{\frac{1}{q}} d_\alpha \tau \\
\leq\ & \frac{1}{\alpha^{n+1/p} n! (np+1)^{1/p}} \left(\int_{x_0}^{t} \left(\tau^\alpha - x_0^\alpha\right)^{np+1} d_\alpha \tau \right)^{\frac{1}{p}} \left(\int_{x_0}^{t} z(\tau) D_\alpha z(\tau) d_\alpha \tau \right)^{\frac{1}{q}} \\
=\ & \frac{\left(t^\alpha - x_0^\alpha\right)^{n+2/p}}{\alpha^{n+2/p} n! \left[(np+1)(np+2)\right]^{1/p}} \frac{(z(t))^{\frac{2}{q}}}{2^{\frac{1}{q}}}
\end{aligned}$$

which completes the proof. □

Corollary 1. *Under assumption of Theorem 8 with $p = q = 2$, we get*

$$\int_{x_0}^{t} |R_{n,f}(x_0,\tau)| \left|D_\alpha^{n+1} f(\tau)\right| d_\alpha \tau \leq \frac{\left(t^\alpha - x_0^\alpha\right)^{n+1}}{2\alpha^{n+1} n! \sqrt{(n+1)(2p+1)}} \int_{x_0}^{t} \left|D_\alpha^{n+1} f(\tau)\right|^2 d_\alpha \tau.$$

Theorem 9. *Let $\alpha \in (0,1]$, $f : [a,b] \to \mathbb{R}$ be an $n+1$ times $\alpha-$ fractional differentiable function, $p,q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $t \leq x_0$, $t, x_0 \in [a,b]$. Then, we have the following inequality*

$$\begin{aligned}
&\int_{t}^{x_0} |R_{n,f}(x_0,\tau)| \left|D_\alpha^{n+1} f(\tau)\right| d_\alpha \tau \\
\leq\ & \frac{\left(x_0^\alpha - t^\alpha\right)^{n+2/p}}{\alpha^{n+1+2/p} 2^{\frac{1}{q}} n! \left[(np+1)(np+2)\right]^{1/p}} \left(\int_{t}^{x_0} \left|D_\alpha^{n+1} f(\tau)\right|^q d_\alpha \tau \right)^{\frac{2}{q}}.
\end{aligned} \tag{3.5}$$

Proof. From (2.7), we have

$$\begin{aligned}
|R_{n,f}(x_0,t)| &= \frac{1}{\alpha^n n!} \left| \int_{x_0}^{t} \left(t^\alpha - \tau^\alpha\right)^n D_\alpha^{n+1} f(\tau) d_\alpha \tau \right| \\
&\leq \frac{1}{\alpha^n n!} \int_{t}^{x_0} \left(\tau^\alpha - t^\alpha\right)^n \left|D_\alpha^{n+1} f(\tau)\right| d_\alpha \tau \\
&\leq \frac{1}{\alpha^n n!} \left(\int_{t}^{x_0} \left(\tau^\alpha - t^\alpha\right)^{np} d_\alpha \tau \right)^{\frac{1}{p}} \left(\int_{t}^{x_0} \left|D_\alpha^{n+1} f(\tau)\right|^q d_\alpha \tau \right)^{\frac{1}{q}} \\
&= \frac{1}{\alpha^{n+1/p} n!} \frac{\left(x_0^\alpha - t^\alpha\right)^{n+1/p}}{(np+1)^{1/p}} (z(t))^{\frac{1}{q}}
\end{aligned} \tag{3.6}$$

where

$$z(t)=\int_{t}^{x_0}\left|D_{\alpha}^{n+1}f(\tau)\right|^{q}d_{\alpha}\tau,\ a\leq t\leq x_0,\ z(x_0)=0.$$

Therefore,

$$D_{\alpha}z(t)=-\left|D_{\alpha}^{n+1}f(t)\right|^{q}$$

and

$$\left|D_{\alpha}^{n+1}f(t)\right|=\left(-D_{\alpha}z(t)\right)^{1/q}. \tag{3.7}$$

From (3.6) and (3.7), it follows that

$$\left|R_{n,f}(x_0,t)\right|\left|D_{\alpha}^{n+1}f(t)\right|\leq\frac{1}{\alpha^{n+1/p}n!}\frac{\left(x_0^{\alpha}-t^{\alpha}\right)^{n+1/p}}{(np+1)^{1/p}}\left(-z(t)D_{\alpha}z(t)\right)^{\frac{1}{q}}. \tag{3.8}$$

Integrating the inequality (3.8) and using the Hölder inequality for conformable integral, we have

$$\begin{aligned}
&\int_{t}^{x_0}\left|R_{n,f}(x_0,\tau)\right|\left|D_{\alpha}^{n+1}f(\tau)\right|d_{\alpha}\tau\\
\leq\ &\frac{1}{\alpha^{n+1/p}n!(np+1)^{1/p}}\int_{t}^{x_0}\left(x_0^{\alpha}-\tau^{\alpha}\right)^{n+1/p}\left(z(\tau)D_{\alpha}z(\tau)\right)^{\frac{1}{q}}d_{\alpha}\tau\\
\leq\ &\frac{1}{\alpha^{n+1/p}n!(np+1)^{1/p}}\left(\int_{t}^{x_0}\left(x_0^{\alpha}-\tau^{\alpha}\right)^{np+1}d_{\alpha}\tau\right)^{\frac{1}{p}}\left(\int_{t}^{x_0}\left(-z(\tau)D_{\alpha}z(\tau)\right)d_{\alpha}\tau\right)^{\frac{1}{q}}\\
=\ &\frac{\left(x_0^{\alpha}-t^{\alpha}\right)^{n+2/p}}{\alpha^{n+2/p}n!\left[(np+1)(np+2)\right]^{1/p}}\frac{\left(z(t)\right)^{\frac{2}{q}}}{2^{\frac{1}{q}}}.
\end{aligned}$$

This completes the proof. □

Corollary 2. *Under assumption of Theorem 9 with $p=q=2$, we get*

$$\int_{t}^{x_0}\left|R_{n,f}(x_0,\tau)\right|\left|D_{\alpha}^{n+1}f(\tau)\right|d_{\alpha}\tau\leq\frac{\left(x_0^{\alpha}-t^{\alpha}\right)^{n+1}}{2\alpha^{n+1}n!\sqrt{(n+1)(2p+1)}}\int_{t}^{x_0}\left|D_{\alpha}^{n+1}f(\tau)\right|^{2}d_{\alpha}\tau.$$

Theorem 10. *Let $\alpha\in(0,1]$, $f:[a,b]\to\mathbb{R}$ be an $n+1$ times $\alpha-$ fractional differentiable function, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, and $t,x_0\in[a,b]$. Then, we have the following inequality*

$$\begin{aligned}
&\left|\int_{x_0}^{t}\left|R_{n,f}(x_0,\tau)\right|\left|D_{\alpha}^{n+1}f(\tau)\right|d_{\alpha}\tau\right| \\
\leq\ &\frac{\left|t^{\alpha}-x_0^{\alpha}\right|^{n+2/p}}{\alpha^{n+2/p}2^{\frac{1}{q}}n!\left[(np+1)(np+2)\right]^{1/p}}\left|\int_{x_0}^{t}\left|D_{\alpha}^{n+1}f(\tau)\right|^{q}d_{\alpha}\tau\right|^{\frac{2}{q}}.
\end{aligned} \tag{3.9}$$

Proof. Combining Theorem 8 and Theorem 9, we can easily the required result. □

Corollary 3. *Under assumption of Theorem 10 with $p=q=2$, we get*

$$\left|\int_{x_0}^{t}|R_{n,f}(x_0,\tau)|\left|D_\alpha^{n+1}f(\tau)\right|d_\alpha\tau\right|\leq\frac{\left|t^\alpha-x_0^\alpha\right|^{n+1}}{2\alpha^{n+1}n!\sqrt{(n+1)(2p+1)}}\left|\int_{x_0}^{t}\left|D_\alpha^{n+1}f(\tau)\right|^2d_\alpha\tau\right|.$$

Using the Theorem 10 and Corollary 3, we obtain the following important inequality.

Corollary 4. *Let $\alpha\in(0,1]$, $f:[a,b]\to\mathbb{R}$ be an $n+1$ times $\alpha-$ fractional differentiable function, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, and $t,x_0\in[a,b]$. If $D_\alpha^k f(x_0)=0$, $k=0,1,...,n$, then we have the following Opial type inequality*

$$\begin{aligned}&\left|\int_{x_0}^{t}|f(\tau)|\left|D_\alpha^{n+1}f(\tau)\right|d_\alpha\tau\right|\\ \leq\;&\min\left\{\frac{\left|t^\alpha-x_0^\alpha\right|^{n+2/p}}{\alpha^{n+2/p}2^{\frac{1}{q}}n!\left[(np+1)(np+2)\right]^{1/p}}\left|\int_{x_0}^{t}\left|D_\alpha^{n+1}f(\tau)\right|^q d_\alpha\tau\right|^{\frac{2}{q}},\right.\\ &\left.\frac{\left|t^\alpha-x_0^\alpha\right|^{n+1}}{2\alpha^{n+1}n!\sqrt{(n+1)(2n+1)}}\left|\int_{x_0}^{t}\left|D_\alpha^{n+1}f(\tau)\right|^2d_\alpha\tau\right|\right\}.\end{aligned}$$

Corollary 5. *If we choose $n=0$ Corollary 4, then we have the following inequality*

$$\begin{aligned}&\left|\int_{x_0}^{t}|f(\tau)|\left|D_\alpha f(\tau)\right|d_\alpha\tau\right|\\ \leq\;&\frac{1}{2}\min\left\{\frac{\left|t^\alpha-x_0^\alpha\right|^{2/p}}{2\alpha^{2/p}}\left|\int_{x_0}^{t}\left|D_\alpha f(\tau)\right|^q d_\alpha\tau\right|^{\frac{2}{q}},\frac{\left|t^\alpha-x_0^\alpha\right|}{2\alpha}\left|\int_{x_0}^{t}\left|D_\alpha f(\tau)\right|^2d_\alpha\tau\right|\right\}.\end{aligned}$$

Theorem 11. *Let $\alpha\in(0,1]$, $f:[a,b]\to\mathbb{R}$ be an $n+1$ times $\alpha-$ fractional differentiable function, $p=1$, $q=\infty$ and $t\in[x_0,b]$. Then we have the inequality*

$$\int_{x_0}^{t}|R_{n,f}(x_0,\tau)|\left|D_\alpha^{n+1}f(\tau)\right|d_\alpha\tau\leq\frac{\left(t^\alpha-x_0^\alpha\right)^{n+2}}{\alpha^{n+2}(n+2)!}\left\|D_\alpha^{n+1}f\right\|_{\infty,[x_0,b]}^2.\tag{3.10}$$

Proof. From (2.7), we have

$$\begin{aligned}|R_{n,f}(x_0,t)|&\leq\frac{1}{\alpha^n n!}\int_{x_0}^{t}\left(t^\alpha-\tau^\alpha\right)^n\left|D_\alpha^{n+1}f(\tau)\right|d_\alpha\tau\\&\leq\frac{1}{\alpha^n n!}\left\|D_\alpha^{n+1}f\right\|_{\infty,[x_0,b]}\int_{x_0}^{t}\left(t^\alpha-\tau^\alpha\right)^n d_\alpha\tau\\&=\frac{\left\|D_\alpha^{n+1}f\right\|_{\infty,[x_0,b]}}{\alpha^{n+1}(n+1)!}\left(t^\alpha-x_0^\alpha\right)^{n+1}.\end{aligned}\tag{3.11}$$

Moreover, we get

$$\left|D_\alpha^{n+1}f(t)\right|\leq\left\|D_\alpha^{n+1}f\right\|_{\infty,[x_0,b]}$$

for all $t \in [x_0, b]$.

Therefore it follows that

$$|R_{n,f}(x_0,t)| \left|D_\alpha^{n+1} f(t)\right| \leq \frac{\left\|D_\alpha^{n+1} f\right\|_{\infty,[x_0,b]}^2}{\alpha^{n+1}(n+1)!} (t^\alpha - x_0^\alpha)^{n+1}. \tag{3.12}$$

Integrating the inequality (3.12), we have

$$\begin{aligned}
\int_{x_0}^{t} |R_{n,f}(x_0,\tau)| \left|D_\alpha^{n+1} f(\tau)\right| d_\alpha\tau &\leq \frac{\left\|D_\alpha^{n+1} f\right\|_{\infty,[x_0,b]}^2}{\alpha^{n+1}(n+1)!} \int_{x_0}^{t} (\tau^\alpha - x_0^\alpha)^{n+1} d_\alpha\tau \\
&= \frac{(t^\alpha - x_0^\alpha)^{n+2}}{\alpha^{n+2}(n+2)!} \left\|D_\alpha^{n+1} f\right\|_{\infty,[x_0,b]}^2
\end{aligned}$$

This completes the proof of the inequality (3.10). □

Theorem 12. *Let $p = 1$, $q = \infty$ and $t \in [a, x_0]$. Then we have the inequality*

$$\int_{x_0}^{t} |R_{n,f}(x_0,\tau)| \left|D_\alpha^{n+1} f(\tau)\right| d_\alpha\tau \leq \frac{(x_0^\alpha - t^\alpha)^{n+2}}{\alpha^{n+2}(n+2)!} \left\|D_\alpha^{n+1} f\right\|_{\infty,[a,x_0]}^2. \tag{3.13}$$

Proof. From (2.7), we get

$$\begin{aligned}
|R_{n,f}(x_0,t)| &= \left|\frac{1}{\alpha^n n!}\int_{x_0}^{t} (t^\alpha - \tau^\alpha)^n D_\alpha^{n+1} f(\tau) d_\alpha\tau\right| \\
&\leq \frac{1}{\alpha^n n!}\int_{t}^{x_0} (\tau^\alpha - t^\alpha)^n \left|D_\alpha^{n+1} f(\tau)\right| d_\alpha\tau \\
&\leq \frac{1}{\alpha^n n!} \left\|D_\alpha^{n+1} f\right\|_{\infty,[x_0,b]} \int_{t}^{x_0} (\tau^\alpha - t^\alpha)^n d_\alpha\tau \\
&= \frac{\left\|D_\alpha^{n+1} f\right\|_{\infty,[x_0,b]}}{\alpha^{n+1}(n+1)!} (x_0^\alpha - t^\alpha)^{n+1}.
\end{aligned} \tag{3.14}$$

Furthermore, we have

$$\left|D_\alpha^{n+1} f(t)\right| \leq \left\|D_\alpha^{n+1} f\right\|_{\infty,[a,x_0]} \tag{3.15}$$

for all $t \in [a, x_0]$.

Thus, we obtain

$$\begin{aligned}
\int_{x_0}^{t} |R_{n,f}(x_0,\tau)| \left|D_\alpha^{n+1} f(\tau)\right| d_\alpha\tau &\leq \frac{\left\|D_\alpha^{n+1} f\right\|_{\infty,[a,x_0]}^2}{\alpha^{n+1}(n+1)!} \int_{x_0}^{t} (x_0^\alpha - \tau^\alpha)^{n+1} d_\alpha\tau \\
&= \frac{(x_0^\alpha - t^\alpha)^{n+2}}{\alpha^{n+2}(n+2)!} \left\|D_\alpha^{n+1} f\right\|_{\infty,[a,x_0]}^2
\end{aligned}$$

which completes the proof of the inequality (3.13). □

Combining Theorem 11 and Theorem 12, we have the following result.

Corollary 6. *Let $\alpha \in (0,1]$, $f : [a,b] \to \mathbb{R}$ be an $n+1$ times $\alpha-$ fractional differentiable function, $p = 1$, $q = \infty$ and $t \in [a,b]$. Then, the following inequality holds:*

$$\left| \int_{x_0}^{t} |R_{n,f}(x_0,\tau)| \left| D_\alpha^{n+1} f(\tau) \right| d_\alpha \tau \right| \leq \frac{|t^\alpha - x_0^\alpha|^{n+2}}{\alpha^{n+2}(n+2)!} \left\| D_\alpha^{n+1} f \right\|_\infty^2 .$$

Using the Corollary 6, we obtain the following important inequality.

Corollary 7. *Let $\alpha \in (0,1]$, $f : [a,b] \to \mathbb{R}$ be an $n+1$ times $\alpha-$ fractional differentiable function, $p = 1$, $q = \infty$ and $t \in [a,b]$. If $D_\alpha^k f(x_0) = 0$, $k = 0,1,...,n$, then we have the following Opial type inequality*

$$\left| \int_{x_0}^{t} |f(\tau)| \left| D_\alpha^{n+1} f(\tau) \right| d_\alpha \tau \right| \leq \frac{|t^\alpha - x_0^\alpha|^{n+2}}{\alpha^{n+2}(n+2)!} \left\| D_\alpha^{n+1} f \right\|_\infty^2 .$$

Corollary 8. *If we choose $n = 0$ Corollary 7, then we have the following inequality*

$$\left| \int_{x_0}^{t} |f(\tau)| \left| D_\alpha f(\tau) \right| d_\alpha \tau \right| \leq \frac{|t^\alpha - x_0^\alpha|^{2}}{2\alpha^{2}} \left\| D_\alpha f \right\|_\infty^2 .$$

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[Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey

E-mail address: sarikayamz@gmail.com

[Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey

E-mail address: hsyn.budak@gmail.com