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SOME INEQUALITIES FOR LOGARITHM WITH APPLICATIONS TO WEIGHTED MEANS

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Abstract. In this paper we establish several inequalities for logarithm and apply them to obtain some new inequalities involving weighted arithmetic mean, geometric mean and harmonic mean of n-tuples of positive sequences. The case of two positive numbers and an analysis of which bound is better and when are also considered.

1. Introduction

There are a number of inequalities for logarithm, see for instance

`http://functions.wolfram.com/ElementaryFunctions/Log/29/`

and [5] that are well know and widely used in literature, such as:

$$\frac{x-1}{x} \le \ln x \le x-1 \text{ for } x>0, \tag{1.1}$$

$$\frac{2x}{2+x} \le \ln(1+x) \le \frac{x}{\sqrt{x+1}} \text{ for } x \ge 0, \tag{1.2}$$

$$x \le -\ln(1-x) \le \frac{x}{1-x}, \text{ for } x<1,$$

$$\ln x \le n\left(x^{1/n}-1\right) \text{ for } n>0 \text{ and } x>0,$$

$$\ln(1-|x|) \le \ln(x+1) \le -\ln(1-|x|) \text{ for } |x|<1,$$

and

$$-\frac{3}{2}x \le \ln(1-x) \le \frac{3}{2}x \text{ for } 0<x\le 0.5838.$$

A simple proof of the first inequality in (1.2) may be found, for instance, in [6], see also [7] where the following rational bounds are provided as well:

$$\frac{x\left(1+\frac{5}{6}x\right)}{(1+x)\left(1+\frac{1}{3}x\right)} \le \ln(1+x) \le \frac{x\left(1+\frac{1}{6}x\right)}{1+\frac{2}{3}x} \text{ for } x \ge 0.$$

In the recent paper [3] we established the following result:

$$(0 \le)\,(1-\nu)a+\nu b-a^{1-\nu}b^{\nu} \le \nu(1-\nu)(b-a)(\ln b-\ln a) \tag{1.3}$$

for any $a, b>0$ and $\nu \in (0,1)$.

If we take in (1.3) $b=x+1$, $x>0$ and $a=1$, then we get

$$\ln(x+1) \ge \frac{1-\nu+\nu(x+1)-(x+1)^{\nu}}{\nu(1-\nu)x}\ (\ge 0) \tag{1.4}$$

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for any $\nu \in (0,1)$ and, in particular

$$\ln(x+1) \geq \frac{2\left(\sqrt{x+1}-1\right)^2}{x} \ (\geq 0) \tag{1.5}$$

for any $x > 0$ and $\nu \in (0,1)$.

In this paper we establish some inequalities for the quantity

$$\frac{b-a}{a} - \ln b + \ln a$$

when a, $b > 0$ and apply them to obtain some new inequalities involving weighted arithmetic mean, geometric mean and harmonic mean of n-tuples of positive numbers. The case of two positive numbers and an analysis of which bound is better and when are also considered.

2. Logarithmic Inequalities

The following theorem is well known in the literature as Taylor's theorem with the integral remainder.

Theorem 1. *Let $I \subset \mathbb{R}$ be a closed interval, $a \in I$ and let n be a positive integer. If $f : I \longrightarrow \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous on I, then for each $x \in I$*

$$f(x) = T_n(f;a,x) + R_n(f;a,x), \tag{2.1}$$

where $T_n(f;a,x)$ is Taylor's polynomial, i.e.,

$$T_n(f;a,x) := \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a).$$

(Note that $f^{(0)} := f$ and $0! := 1$), and the remainder is given by

$$R_n(f;a,x) := \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt.$$

The following result holds [2]:

Lemma 1. *For any a, $b > 0$ we have for $n \geq 1$ that*

$$\ln b - \ln a + \sum_{k=1}^{n} \frac{(-1)^k (b-a)^k}{ka^k} = (-1)^n \int_a^b \frac{(b-t)^n}{t^{n+1}}dt. \tag{2.2}$$

Proof. Consider the function $f : (0,\infty) \longrightarrow \mathbb{R}$, $f(x) = \ln x$, then

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n},\ n \geq 1,\ x > 0,$$

$$T_n(f;a,x) = \ln a + \sum_{k=1}^{n} \frac{(-1)^{k-1}(x-a)^k}{ka^k},\ a > 0$$

and

$$R_n(f;a,x) = (-1)^n \int_a^x \frac{(x-t)^n}{t^{n+1}}dt.$$

Now, using (2.1) we have the equality,

$$\ln x = \ln a + \sum_{k=1}^{n} \frac{(-1)^{k-1}(x-a)^k}{ka^k} + (-1)^n \int_a^x \frac{(x-t)^n}{t^{n+1}}dt,$$

i.e.,

$$\ln x - \ln a + \sum_{k=1}^{n} \frac{(-1)^k (x-a)^k}{k a^k} = (-1)^n \int_a^x \frac{(x-t)^n}{t^{n+1}} dt, \quad x, a > 0.$$

Choosing in the last equality $x = b$, we get (2.2). □

Theorem 2. *For any a, $b > 0$ we have*

$$\begin{aligned} \frac{1}{2}\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2 &= \frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}} \\ &\le \frac{b-a}{a} - \ln b + \ln a \\ &\le \frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}} = \frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2. \end{aligned} \tag{2.3}$$

Proof. For $n = 1$ we get from (2.2) that

$$\int_a^b \frac{b-t}{t^2} dt = \frac{b-a}{a} - \ln b + \ln a \tag{2.4}$$

for any a, $b > 0$.

If $b > a$, then

$$\frac{1}{2}\frac{(b-a)^2}{a^2} \ge \int_a^b \frac{b-t}{t^2} dt \ge \frac{1}{2}\frac{(b-a)^2}{b^2}. \tag{2.5}$$

If $a > b$ then

$$\int_a^b \frac{b-t}{t^2} dt = -\int_b^a \frac{b-t}{t^2} dt = \int_b^a \frac{t-b}{t^2} dt$$

and

$$\frac{1}{2}\frac{(b-a)^2}{b^2} \ge \int_b^a \frac{t-b}{t^2} dt \ge \frac{1}{2}\frac{(b-a)^2}{a^2}. \tag{2.6}$$

Therefore, by (2.5) and (2.6) we have for any a, $b > 0$ that

$$\int_a^b \frac{b-t}{t^2} dt \ge \frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}} = \frac{1}{2}\left(\frac{\min\{a,b\}}{\max\{a,b\}} - 1\right)^2$$

and

$$\int_a^b \frac{b-t}{t^2} dt \le \frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}} = \frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2.$$

By the representation (2.4) we then get the desired result (2.3). □

When some bounds for a, b are provided, then we have:

Corollary 1. *Assume that a, $b \in [m, M] \subset (0, \infty)$, then we have the local bounds*

$$\frac{1}{2}\frac{(b-a)^2}{M^2} \le \frac{b-a}{a} - \ln b + \ln a \le \frac{1}{2}\frac{(b-a)^2}{m^2} \tag{2.7}$$

and

$$\frac{1}{2}\frac{(b-a)^2}{M^2} \le \ln b - \ln a - \frac{b-a}{b} \le \frac{1}{2}\frac{(b-a)^2}{m^2}. \tag{2.8}$$

Remark 1. *If we take in (2.3) $a=1$ and $b=x\in(0,\infty)$, then we get*

$$\begin{aligned}\frac{1}{2}\left(1-\frac{\min\{1,x\}}{\max\{1,x\}}\right)^2 &= \frac{1}{2}\frac{(x-1)^2}{\max^2\{1,x\}} \\ &\le x-1-\ln x \\ &\le \frac{1}{2}\frac{(x-1)^2}{\min^2\{1,x\}}=\frac{1}{2}\left(\frac{\max\{1,x\}}{\min\{1,x\}}-1\right)^2\end{aligned} \tag{2.9}$$

and if we take $a=x$ and $b=1$, then we also get

$$\begin{aligned}\frac{1}{2}\left(1-\frac{\min\{1,x\}}{\max\{1,x\}}\right)^2 &= \frac{1}{2}\frac{(x-1)^2}{\max^2\{1,x\}} \\ &\le \ln x-\frac{x-1}{x} \\ &\le \frac{1}{2}\frac{(x-1)^2}{\min^2\{1,x\}}=\frac{1}{2}\left(\frac{\max\{1,x\}}{\min\{1,x\}}-1\right)^2.\end{aligned} \tag{2.10}$$

If $x\in[k,K]\subset(0,\infty)$, then by analyzing all possible locations of the interval $[k,K]$ and 1 we have

$$\min\{1,k\}\le\min\{1,x\}\le\min\{1,K\}$$

and

$$\max\{1,k\}\le\max\{1,x\}\le\max\{1,K\}.$$

By (2.9) and (2.10) we get the *local bounds*

$$\frac{1}{2}\frac{(x-1)^2}{\max^2\{1,K\}}\le x-1-\ln x\le\frac{1}{2}\frac{(x-1)^2}{\min^2\{1,k\}} \tag{2.11}$$

and

$$\frac{1}{2}\frac{(x-1)^2}{\max^2\{1,K\}}\le \ln x-\frac{x-1}{x}\le\frac{1}{2}\frac{(x-1)^2}{\min^2\{1,k\}} \tag{2.12}$$

for any $x\in[k,K]$.

We have by (2.11) and (2.12):

Corollary 2. *Let a, $b>0$ and such that $\frac{b}{a}\in[k,K]\subset(0,\infty)$. Then we have*

$$\frac{1}{2}\frac{(b-a)^2}{a^2\max^2\{1,K\}}\le\frac{b-a}{a}-\ln b+\ln a\le\frac{1}{2}\frac{(b-a)^2}{a^2\min^2\{1,k\}} \tag{2.13}$$

and

$$\frac{1}{2}\frac{(b-a)^2}{a^2\max^2\{1,K\}}\le\ln b-\ln a-\frac{b-a}{b}\le\frac{1}{2}\frac{(b-a)^2}{a^2\min^2\{1,k\}}. \tag{2.14}$$

If we assume that a, $b\in[m,M]\subset(0,\infty)$, then by taking $k=\frac{m}{M}<1<\frac{M}{m}=K$ in (2.13) and (2.14) we get

$$\begin{aligned}\frac{1}{2}\frac{m^2}{M^2}\left(\left(\frac{b}{a}\right)^2-2\frac{b}{a}+1\right) &\le\frac{b-a}{a}-\ln b+\ln a \\ &\le\frac{1}{2}\frac{M^2}{m^2}\left(\left(\frac{b}{a}\right)^2-2\frac{b}{a}+1\right)\end{aligned} \tag{2.15}$$

and

$$\frac{1}{2}\frac{m^2}{M^2}\left(\left(\frac{b}{a}\right)^2-2\frac{b}{a}+1\right)\leq \ln b-\ln a-\frac{b-a}{b} \leq \frac{1}{2}\frac{M^2}{m^2}\left(\left(\frac{b}{a}\right)^2-2\frac{b}{a}+1\right). \tag{2.16}$$

Observe also that for $x\in[k,K]$ we have

$$1-\frac{\min\{1,x\}}{\max\{1,x\}}\geq 1-\frac{\min\{1,K\}}{\max\{1,k\}}\geq 0$$

and

$$0\leq \frac{\max\{1,x\}}{\min\{1,x\}}-1\leq \frac{\max\{1,K\}}{\min\{1,k\}}-1.$$

Now, by (2.9) and (2.10) we get the *global bounds*

$$\frac{1}{2}\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^2\leq x-1-\ln x\leq \frac{1}{2}\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^2 \tag{2.17}$$

and

$$\frac{1}{2}\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^2\leq \ln x-\frac{x-1}{x}\leq \frac{1}{2}\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^2 \tag{2.18}$$

for any $x\in[k,K]$.

By (2.17) and (2.18) we have:

Corollary 3. *Let a, $b>0$ and such that $\frac{b}{a}\in[k,K]\subset(0,\infty)$. Then we have*

$$\frac{1}{2}\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^2\leq \frac{b-a}{a}-\ln b+\ln a\leq \frac{1}{2}\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^2 \tag{2.19}$$

and

$$\frac{1}{2}\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^2\leq \ln b-\ln a-\frac{b-a}{b}\leq \frac{1}{2}\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^2. \tag{2.20}$$

We observe that from (2.19) we actually have

$$\frac{1}{2}\begin{cases}(1-K)^2 \text{ if } K<1,\\ 0 \text{ if } k\leq 1\leq K,\\ \left(1-\frac{1}{k}\right)^2 \text{ if } 1<k,\end{cases} \leq \frac{b-a}{a}-\ln b+\ln a \leq \frac{1}{2}\begin{cases}\left(\frac{1}{k}-1\right)^2 \text{ if } K<1,\\ \left(\frac{K}{k}-1\right)^2 \text{ if } k\leq 1\leq K,\\ (K-1)^2 \text{ if } 1<k\end{cases} \tag{2.21}$$

and the same bounds for $\ln b-\ln a-\frac{b-a}{b}$.

We also have:

Theorem 3. *For any a, $b>0$ we have*

$$(0\leq)\ \frac{b-a}{a}-\ln b+\ln a\leq \frac{(b-a)^2}{ab} \tag{2.22}$$

and

$$(0 \leq) \ln b - \ln a - \frac{b-a}{b} \leq \frac{(b-a)^2}{ab}. \tag{2.23}$$

Proof. If $b > a$, then

$$\int_a^b \frac{b-t}{t^2} dt \leq (b-a) \int_a^b \frac{1}{t^2} dt = (b-a) \frac{b-a}{ab} = \frac{(b-a)^2}{ab}.$$

If $a > b$, then

$$\int_a^b \frac{b-t}{t^2} dt = \int_b^a \frac{t-b}{t^2} dt \leq (a-b) \int_b^a \frac{1}{t^2} dt = (a-b) \frac{a-b}{ab} = \frac{(b-a)^2}{ab}.$$

Therefore,

$$\int_a^b \frac{b-t}{t^2} dt \leq \frac{(b-a)^2}{ab}$$

for any a, $b > 0$ and by the representation (2.4) we get the desired result (2.22). $\square$

It is natural to ask, which of the upper bounds for the quantity

$$\frac{b-a}{a} - \ln b + \ln a$$

as provided by (2.3) and (2.22) is better?

Consider the difference

$$\Delta(a, b) := \frac{1}{2} \frac{(b-a)^2}{\min^2\{a, b\}} - \frac{(b-a)^2}{ab}, \ a, \ b > 0.$$

We observe that for $b > a$ we get

$$\Delta(a, b) := \frac{1}{2} \frac{(b-a)^2}{a^2} - \frac{(b-a)^2}{ab} = \frac{(b-a)^2}{2a^2 b} (b - 2a).$$

Therefore $\Delta(a, b) > 0$ if $b > 2a$ and $\Delta(a, b) < 0$ if $a < b < 2a$, meaning that neither of the upper bounds in (2.3) and (2.22) is always best.

If we take in (2.22) and (2.23) $a = 1$ and $b = x \in (0, \infty)$, then we get

$$(0 \leq) x - 1 - \ln x \leq \frac{(x-1)^2}{x} \tag{2.24}$$

and

$$(0 \leq) \ln x - \frac{x-1}{x} \leq \frac{(x-1)^2}{x} \tag{2.25}$$

for any $x > 0$.

Corollary 4. *Let a, $b > 0$ and such that $\frac{b}{a} \in [k, K] \subset (0, \infty)$. Then we have*

$$\frac{b-a}{a} - \ln b + \ln a \leq U(k, K) \tag{2.26}$$

and

$$\ln b - \ln a - \frac{b-a}{b} \leq U(k, K), \tag{2.27}$$

where

$$U(k,K) := \begin{cases} \frac{(k-1)^2}{k} \text{ if } K<1, \\ \max\left\{\frac{(k-1)^2}{k}, \frac{(K-1)^2}{K}\right\} \text{ if } k \le 1 \le K, \\ \frac{(K-1)^2}{K} \text{ if } 1<k. \end{cases}$$

Proof. Consider the function $f(x) = \frac{(x-1)^2}{x}$, $x>0$. We observe that

$$f'(x) = \frac{x^2-1}{x^2} \text{ and } f''(x) = \frac{2}{x^3},$$

which shows that f is strictly decreasing on $(0,1)$, strictly increasing on $[1,\infty)$ and strictly convex for $x>0$. We also have $f\left(\frac{1}{x}\right) = f(x)$ for $x>0$.

By (2.24) and by the properties of f we then have that for any $x \in [k,K]$

$$\begin{aligned} x-1-\ln x &\le \max_{x\in[k,K]} \frac{(x-1)^2}{x} \\ &= \begin{cases} \frac{(k-1)^2}{k} \text{ if } K<1, \\ \max\left\{\frac{(k-1)^2}{k}, \frac{(K-1)^2}{K}\right\} \text{ if } k \le 1 \le K, \\ \frac{(K-1)^2}{K} \text{ if } 1<k. \end{cases} \\ &= U(k,K). \end{aligned} \tag{2.28}$$

Now, put $x = \frac{b}{a} \in [k,K]$ in (2.28) to get the desired inequality (2.26).

Let $y = \frac{1}{x}$ with $x = \frac{b}{a} \in [k,K]$. Then $y \in \left[\frac{1}{K}, \frac{1}{k}\right]$ and we have like in (2.28) that

$$\begin{aligned} y-1-\ln y &\le \max_{y\in[K^{-1},k^{-1}]} \frac{(y-1)^2}{y} \\ &= \begin{cases} \frac{\left(K^{-1}-1\right)^2}{K^{-1}} \text{ if } k^{-1}<1, \\ \max\left\{\frac{\left(K^{-1}-1\right)^2}{K^{-1}}, \frac{\left(\frac{1}{k^{-1}}-1\right)^2}{k^{-1}}\right\} \text{ if } k \le 1 \le K^{-1}, \\ \frac{\left(\frac{1}{k^{-1}}-1\right)^2}{k^{-1}} \text{ if } 1 < \frac{1}{K^{-1}}, \end{cases} \\ &= U(k,K), \end{aligned}$$

which implies (2.27). $\square$

Now, by Corollary 1 we have the *global upper bound*

$$\frac{b-a}{a} - \ln b + \ln a \le \frac{1}{2}\frac{(M-m)^2}{m^2}, \tag{2.29}$$

for any $a, b \in [m,M]$. Moreover, if $a, b \in [m,M]$, then $K = \frac{M}{m}$ and $k = \frac{m}{M}$ and by Corollary 4 we also get

$$\frac{b-a}{a} - \ln b + \ln a \le \frac{(M-m)^2}{mM}, \tag{2.30}$$

which implies that

$$(0 \le) \frac{b-a}{a} - \ln b + \ln a \le \frac{(M-m)^2}{mM} \min\left\{\frac{M}{2m}, 1\right\} \tag{2.31}$$

for any $a, b \in [m,M]$.

We observe that, for $m<M<2m$, the inequality (2.29) is better than (2.30). If $M \ge 2m$, then the conclusion is the other way around.

From the above consideration, we can conclude that the following inequality is also valid

$$(0 \leq) \ln b - \ln a - \frac{b-a}{b} \leq \frac{(M-m)^2}{mM} \min\left\{\frac{M}{2m}, 1\right\} \tag{2.32}$$

for any $a, b \in [m, M]$.

3. Applications for Weighted AM-GM Inequality

Define the *weighted arithmetic mean* of the positive n-tuple $x = (x_1, ..., x_n)$ with the *probability distribution* $w = (w_1, ..., w_n)$ by

$$A_n(w, x) := \sum_{i=1}^{n} w_i x_i$$

and the *weighted geometric mean* of the same n-tuple, by

$$G_n(w, x) := \left(\prod_{i=1}^{n} x_i^{w_i}\right).$$

It is well know that the following arithmetic mean-geometric mean inequality holds

$$A_n(w, x) \geq G_n(w, x).$$

Define also

$$A_{n,2}(w, x) := \sum_{i=1}^{n} w_i\, x_i^2,$$

the *weighted harmonic mean*

$$H_n(w, x) := \frac{1}{\sum_{i=1}^{n} \frac{w_i}{x_i}} = A_n^{-1}\left(w, x^{-1}\right),$$

and the *dispersion*

$$D_n^2(w, x) := A_{n,2}(w, x) - A_n^2(w, x).$$

We have the following result:

Theorem 4. *Assume that the n-tuple $x = (x_1, ..., x_n)$ satisfies the condition*

$$0 < m \leq x_i \leq M < \infty \tag{3.1}$$

for any $i \in \{1, ..., n\}$, then for any probability distribution $w = (w_1, ..., w_n)$ we have

$$\begin{aligned} &\exp\left[A_n(w, x) H_n^{-1}(w, x) - 1 - \frac{1}{2m^2} D_n^2(w, x)\right] \\ &\leq \frac{A_n(w, x)}{G_n(w, x)} \\ &\leq \exp\left[A_n(w, x) H_n^{-1}(w, x) - 1 - \frac{1}{2M^2} D_n^2(w, x)\right] \end{aligned} \tag{3.2}$$

and

$$\exp\left[\frac{1}{2M^2} D_n^2(w, x)\right] \leq \frac{A_n(w, x)}{G_n(w, x)} \leq \exp\left[\frac{1}{2m^2} D_n^2(w, x)\right]. \tag{3.3}$$

Proof. We have that $A_n(w,x) \in [m,M]$ and by (2.7) we obtain

$$\frac{1}{2}\frac{(A_n(w,x)-a)^2}{M^2} \leq \frac{A_n(w,x)-a}{a} - \ln A_n(w,x) + \ln a \leq \frac{1}{2}\frac{(A_n(w,x)-a)^2}{m^2} \tag{3.4}$$

and

$$\frac{1}{2}\frac{(b-A_n(w,x))^2}{M^2} \leq \frac{b-A_n(w,x)}{A_n(w,x)} - \ln b + \ln A_n(w,x) \leq \frac{1}{2}\frac{(b-A_n(w,x))^2}{m^2} \tag{3.5}$$

for any $a, b \in [m,M]$.

Take in (3.4) $a = x_i$, multiply the obtained inequality by w_i and sum over $i \in \{1,...,n\}$ to get

$$\begin{aligned} &\frac{1}{2M^2}\sum_{i=1}^n w_i\ (A_n(w,x)-x_i)^2 \\ &\leq A_n(w,x)\sum_{i=1}^n \frac{w_i}{x_i} - 1 - \ln A_n(w,x) + \sum_{i=1}^n w_i\ \ln x_i \\ &\leq \frac{1}{2m^2}\sum_{i=1}^n w_i\ (A_n(w,x)-x_i)^2 . \end{aligned} \tag{3.6}$$

Since

$$\sum_{i=1}^n w_i\ (A_n(w,x)-x_i)^2 = A_{n,2}(w,x) - (A_n(w,x))^2 = D_n^2(w,x),$$

$$\sum_{i=1}^n \frac{w_i}{x_i} = H_n^{-1}(w,x)$$

and

$$\sum_{i=1}^n w_i\ \ln x_i = \ln G_n(w,x),$$

hence by (3.6) we have

$$\begin{aligned} &\frac{1}{2M^2}D_n^2(w,x) \\ &\leq A_n(w,x)H_n^{-1}(w,x) - 1 - \ln A_n(w,x) + \ln G_n(w,x) \\ &\leq \frac{1}{2m^2}D_n^2(w,x) \end{aligned} \tag{3.7}$$

that is equivalent to

$$\begin{aligned} &A_n(w,x)H_n^{-1}(w,x) - 1 - \frac{1}{2m^2}D_n^2(w,x) \\ &\leq \ln A_n(w,x) - \ln G_n(w,x) \\ &\leq A_n(w,x)H_n^{-1}(w,x) - 1 - \frac{1}{2M^2}D_n^2(w,x) \end{aligned}$$

and by taking the exponential, we get (3.2).

Further, take in (3.4) $b = x_i$, multiply the obtained inequality by w_i and sum over $i \in \{1, ..., n\}$ to get

$$\frac{1}{2M^2}\sum_{i=1}^{n} w_i \left(A_n(w,x) - x_i\right)^2 \le \ln A_n(w,x) - \ln G_n(w,x) \le \frac{1}{2m^2}\sum_{i=1}^{n} w_i \left(A_n(w,x) - x_i\right)^2 \tag{3.8}$$

and by taking the exponential, we deduce (3.3). □

Remark 2. *Choose $n = 2$ and let $w_1 = 1-\nu$, $w_2 = \nu$, $x_1 = a$, $x_2 = b$ with $\nu \in [0,1]$ and a, $b > 0$. Then*

$$A_2(w,x) = (1-\nu)a + \nu b,$$

$$H_2^{-1}(w,x) = (1-\nu)\frac{1}{a} + \nu\frac{1}{b} = \frac{(1-\nu)b + \nu a}{ab}$$

and

$$\begin{aligned} D_2^2(w,x) &= (1-\nu)a^2 + \nu b^2 - ((1-\nu)a + \nu b)^2 \\ &= (1-\nu)a^2 + \nu b^2 - (1-\nu)^2 a^2 - 2(1-\nu)\nu ab - \nu^2 b^2 \\ &= (1-\nu)\nu(b-a)^2 . \end{aligned}$$

Moreover,

$$\begin{aligned} &A_2(w,x) H_2^{-1}(w,x) - 1 \\ &= \frac{[(1-\nu)a + \nu b][(1-\nu)b + \nu a]}{ab} - 1 \\ &= \frac{(1-\nu)^2 ab + \nu(1-\nu)b^2 + \nu(1-\nu)a^2 + \nu^2 ab - ab}{ab} \\ &= \frac{\nu(1-\nu)(b-a)^2}{ab}. \end{aligned}$$

Then

$$\begin{aligned} &A_2(w,x) H_2^{-1}(w,x) - 1 - \frac{1}{2m^2}D_2^2(w,x) \\ &= \frac{\nu(1-\nu)(b-a)^2}{ab} - \frac{(1-\nu)\nu(b-a)^2}{2m^2} \\ &= \nu(1-\nu)(b-a)^2\left(\frac{1}{ab} - \frac{1}{2m^2}\right) \end{aligned}$$

and

$$\begin{aligned} &A_2(w,x) H_2^{-1}(w,x) - 1 - \frac{1}{2M^2}D_2^2(w,x) \\ &= \frac{\nu(1-\nu)(b-a)^2}{ab} - \frac{(1-\nu)\nu(b-a)^2}{2M^2} \\ &= \nu(1-\nu)(b-a)^2\left(\frac{1}{ab} - \frac{1}{2M^2}\right). \end{aligned}$$

Then by (3.2) and (3.3) we get

$$\exp\left[\nu\left(1-\nu\right)\left(b-a\right)^{2}\left(\frac{1}{ab}-\frac{1}{2m^{2}}\right)\right] \tag{3.9}$$
$$\leq \frac{A_{\nu}\left(a,b\right)}{G_{\nu}\left(a,v\right)} \leq \exp\left[\nu\left(1-\nu\right)\left(b-a\right)^{2}\left(\frac{1}{ab}-\frac{1}{2M^{2}}\right)\right]$$

and

$$\exp\left[\frac{1}{2M^{2}}\left(1-\nu\right)\nu\left(b-a\right)^{2}\right] \tag{3.10}$$
$$\leq \frac{A_{\nu}\left(a,b\right)}{G_{\nu}\left(a,v\right)} \leq \exp\left[\frac{1}{2m^{2}}\left(1-\nu\right)\nu\left(b-a\right)^{2}\right]$$

where

$$A_{\nu}\left(a,b\right) := \left(1-\nu\right)a+\nu b$$

is the weighted arithmetic mean of (a,b) *and*

$$G_{\nu}\left(a,b\right) := a^{1-\nu}b^{\nu}$$

is the weighted geometric mean of (a,b).

The inequality (3.10) has been obtained in different ways in either of the recent papers [1] *and* [4].

In order to compare the upper and lower bounds for the quotient $\frac{A_{\nu}(a,b)}{G_{\nu}(a,v)}$ *provided by (3.9) and (3.10) we consider the difference*

$$D_{m,M}\left(a,b\right) := \frac{1}{ab}-\frac{1}{2M^{2}}-\frac{1}{2m^{2}}$$

where $a,b\in[m,M]$.

We observe that

$$\lim_{a,b\to m} D_{m,M}\left(a,b\right) := \frac{1}{m^{2}}-\frac{1}{2M^{2}}-\frac{1}{2m^{2}} = \frac{M^{2}-m^{2}}{2m^{2}M^{2}} > 0$$

and

$$\lim_{a,b\to M} D_{m,M}\left(a,b\right) = \frac{1}{M^{2}}-\frac{1}{2M^{2}}-\frac{1}{2m^{2}} = \frac{m^{2}-M^{2}}{2m^{2}M^{2}} < 0,$$

which show that neither of the lower or upper bounds in (3.9) and (3.10) is always best.

We also have:

Theorem 5. *Assume that the n-tuple* $x=(x_1,...,x_n)$ *satisfies the condition (3.1) for any* $i\in\{1,...,n\}$ *then for any probability distribution* $w=(w_1,...,w_n)$ *we have*

$$\frac{\exp\left[A_{n}\left(w,x\right)H_{n}^{-1}\left(w,x\right)-1\right]}{\frac{A_{n}(w,x)}{G_{n}(w,x)}} \leq \exp\left[\frac{\left(M-m\right)^{2}}{mM}\min\left\{\frac{M}{2m},1\right\}\right] \tag{3.11}$$

and

$$\frac{A_{n}\left(w,x\right)}{G_{n}\left(w,x\right)} \leq \exp\left[\frac{\left(M-m\right)^{2}}{mM}\min\left\{\frac{M}{2m},1\right\}\right]. \tag{3.12}$$

Proof. From the inequalities (2.31) and (2.32) we have

$$\frac{A_n(w,x)-a}{a}-\ln A_n(w,x)+\ln a\leq\frac{(M-m)^2}{mM}\min\left\{\frac{M}{2m},1\right\} \tag{3.13}$$

and

$$\frac{b-A_n(w,x)}{A_n(w,x)}-\ln b+\ln A_n(w,x)\leq\frac{(M-m)^2}{mM}\min\left\{\frac{M}{2m},1\right\} \tag{3.14}$$

for any $a,\ b\in[m,M]$.

By a similar argument to the one in the proof of Theorem 4 we get

$$A_n(w,x)H_n^{-1}(w,x)-1-\ln A_n(w,x)+\ln G_n(w,x)\leq\frac{(M-m)^2}{mM}\min\left\{\frac{M}{2m},1\right\}$$

and

$$\ln A_n(w,x)-\ln G_n(w,x)\leq\frac{(M-m)^2}{mM}\min\left\{\frac{M}{2m},1\right\}$$

that are equivalent to the desired results (3.11) and (3.12). □

Now, we observe that since $\nu(1-\nu)\leq\frac{1}{4}$ for any $\nu\in[0,1]$, then by (3.10) we have

$$\frac{A_\nu(a,b)}{G_\nu(a,v)}\leq\exp\left[\frac{1}{8m^2}(M-m)^2\right] \tag{3.15}$$

while from (3.12) we get

$$\frac{A_\nu(a,b)}{G_\nu(a,v)}\leq\exp\left[\frac{(M-m)^2}{mM}\min\left\{\frac{M}{2m},1\right\}\right] \tag{3.16}$$

for any $\nu\in[0,1]$ and any $a,\ b\in[m,M]$.

Now, if $m<M<2m$, then $\frac{(M-m)^2}{mM}\min\left\{\frac{M}{2m},1\right\}=\frac{(M-m)^2}{2m^2}$, which shows that the upper bound from (3.15) is better than the one from (3.16). If $2m<M<8m$ then $\frac{(M-m)^2}{mM}\min\left\{\frac{M}{2m},1\right\}=\frac{(M-m)^2}{mM}$, which shows that still the upper bound from (3.15) is better than the one from (3.16). If $8m\leq M$, then the bound in (3.16) is better than the one in (3.15).

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