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REVERSES AND REFINEMENTS OF SEVERAL INEQUALITIES FOR RELATIVE OPERATOR ENTROPY

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Abstract. In this paper we obtain new inequalities for relative operator entropy $S(A|B)$ in the case of operators satisfying the condition $mA \leq B \leq MA$, with $0 < m < M$. Applications for the operator entropy are also given.

1. Introduction

Kamei and Fujii [5], [6] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B) := A^{\frac{1}{2}} \left(\ln \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) \right) A^{\frac{1}{2}}, \tag{1.1}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [11].

In general, we can define for positive operators A, B

$$S(A|B) := s - \lim_{\varepsilon \to 0+} S(A + \varepsilon 1_H | B)$$

if it exists, here 1_H is the identity operator.

For the entropy function $\eta(t) = -t \ln t$, the operator entropy has the following expression:

$$\eta(A) = -A \ln A = S(A|1_H) \geq 0$$

for positive contraction A. This shows that the relative operator entropy (1.1) is a relative version of the operator entropy.

Following [7, p. 149-p. 155], we recall some important properties of relative operator entropy for A and B positive invertible operators:

(i) We have the equalities

$$S(A|B) = -A^{1/2} \left(\ln A^{1/2} B^{-1} A^{1/2} \right) A^{1/2} = B^{1/2} \eta \left(B^{-1/2} A B^{-1/2} \right) B^{1/2}; \tag{1.2}$$

(ii) We have the inequalities

$$S(A|B) \leq A (\ln \|B\| - \ln A) \text{ and } S(A|B) \leq B - A; \tag{1.3}$$

(iii) For any C, D positive invertible operators we have that

$$S(A + B|C + D) \geq S(A|C) + S(B|D);$$

(iv) If $B \leq C$ then

$$S(A|B) \leq S(A|C);$$

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(v) If $B_n \downarrow B$ then

$$S(A|B_n) \downarrow S(A|B);$$

(vi) For $\alpha > 0$ we have

$$S(\alpha A|\alpha B) = \alpha S(A|B);$$

(vii) For every operator T we have

$$T^* S(A|B) T \leq S(T^* AT|T^* BT).$$

The relative operator entropy is jointly concave, namely, for any positive invertible operators A, B, C, D we have

$$S(tA + (1-t)B|tC + (1-t)D) \geq tS(A|C) + (1-t)S(B|D)$$

for any $t \in [0,1]$.

For other results on the relative operator entropy see [1], [3], [8], [9], [10] and [12].

Observe that, if we replace in (1.2) B with A, then we get

$$\begin{aligned} S(B|A) &= A^{1/2}\eta\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} \\ &= A^{1/2}\left(-A^{-1/2}BA^{-1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}, \end{aligned}$$

therefore we have

$$A^{1/2}\left(A^{-1/2}BA^{-1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2} = -S(B|A) \tag{1.4}$$

for positive invertible operators A and B.

It is well know that, in general $S(A|B)$ is not equal to $S(B|A)$.

In [14], A. Uhlmann has shown that the relative operator entropy $S(A|B)$ can be represented as the strong limit

$$S(A|B) = s\text{-}\lim_{t\to 0}\frac{A\sharp_t B - A}{t} \tag{1.5}$$

where

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2},\ \nu \in [0,1]$$

is the *weighted geometric mean* of positive invertible operators A and B. For $\nu = \frac{1}{2}$ we denote $A\sharp B$.

This definition of the weighted geometric mean can be extended for any real number ν with $\nu \neq 0$.

Motivated by the above results, in this paper we obtain new inequalities for the relative operator entropy in the case of operators satisfying the condition $mA \leq B \leq MA$, with $0 < m < M$. Applications for the operator entropy of a positive invertible operator C in the case that it satisfies the condition $p1_H \leq C \leq P1_H$, for some constants p, P with $0 < p < P$, are also given.

2. Upper and Lower Bounds

For $t > 0$ and the positive invertible operators A, B we define the *Tsallis relative operator entropy* (see also [2]) by

$$T_t(A|B) := \frac{A\sharp_t B - A}{t}.$$

We observe that, for the function

$$f(x)=\frac{1}{t}\left(1-x^{-t}\right)=\frac{x^{t}-1}{t}x^{-t},\ x>0,$$

we have

$$\begin{aligned}A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} &= -AT_{t}\left(A^{-1}|B^{-1}\right)A=T_{t}\left(A|B\right)\left(A^{-1}\sharp_{t}B^{-1}\right)A\\ &= T_{t}\left(A|B\right)\left(A\sharp_{t}B\right)^{-1}A\end{aligned}$$

for any positive invertible operators A, B and $t>0$.

The following result providing upper and lower bounds for relative operator entropy in terms of $T_{t}(\cdot|\cdot)$ has been obtained in [5] for $0<t\leq 1$. However, it hods for any $t>0$.

Theorem 1. *Let A, B be two positive invertible operators, then for any $t>0$ we have*

$$T_{t}\left(A|B\right)\left(A\sharp_{t}B\right)^{-1}A\leq S\left(A|B\right)\leq T_{t}\left(A|B\right). \tag{2.1}$$

In particular, we have

$$A-AB^{-1}A\leq S\left(A|B\right)\leq B-A,\ [5] \tag{2.2}$$

and

$$\frac{1}{2}A\left(1_{H}-\left(B^{-1}A\right)^{2}\right)\leq S\left(A|B\right)\leq\frac{1}{2}\left(BA^{-1}B-A\right). \tag{2.3}$$

The case $t=\frac{1}{2}$ is of interest as well. Since in this case

$$D_{\frac{1}{2}}\left(A|B\right):=2\left(A\sharp B-A\right)$$

and

$$T_{1/2}\left(A|B\right)\left(A\sharp_{1/2}B\right)^{-1}A=-2\left(A\left(A\sharp B\right)^{-1}A-A\right),$$

hence by (2.1) we get

$$2\left(A-A\left(A\sharp B\right)^{-1}A\right)\leq S\left(A|B\right)\leq 2\left(A\sharp B-A\right)\leq B-A. \tag{2.4}$$

For any positive invertible operator C and any $t>0$ we have

$$\frac{1}{t}C\left(1_{H}-C^{t}\right)\leq\eta\left(C\right)\leq\frac{C^{1-t}-C}{t}. \tag{2.5}$$

In particular, we have

$$C\left(1_{H}-C\right)\leq\eta\left(C\right)\leq 1_{H}-C, \tag{2.6}$$

$$\frac{1}{2}C\left(1_{H}-C^{2}\right)\leq\eta\left(C\right)\leq\frac{1}{2}\left(C^{-1}-C\right) \tag{2.7}$$

and

$$2C\left(1_{H}-C^{1/2}\right)\leq\eta\left(C\right)\leq 2C^{1/2}\left(1_{H}-C^{1/2}\right). \tag{2.8}$$

In order to prove our main result, we need the following lemma that is of interest in itself:

Lemma 1. *For any a, $b > 0$ we have*

$$\frac{1}{2}\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2 = \frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}} \le \frac{b-a}{a} - \ln b + \ln a \le \frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}} = \frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2. \tag{2.9}$$

Proof. Integrating by parts, we have

$$\int_a^b \frac{b-t}{t^2}dt = \int_a^b (t-b)\,d\left(\frac{1}{t}\right) = \left.\frac{t-b}{t}\right|_a^b - \int_a^b \frac{1}{t}dt = \frac{b-a}{a} - \ln b + \ln a \tag{2.10}$$

for any a, $b > 0$.

If $b > a$, then

$$\frac{1}{2}\frac{(b-a)^2}{a^2} \ge \int_a^b \frac{b-t}{t^2}dt \ge \frac{1}{2}\frac{(b-a)^2}{b^2}. \tag{2.11}$$

If $a > b$ then

$$\int_a^b \frac{b-t}{t^2}dt = -\int_b^a \frac{b-t}{t^2}dt = \int_b^a \frac{t-b}{t^2}dt$$

and

$$\frac{1}{2}\frac{(b-a)^2}{b^2} \ge \int_b^a \frac{t-b}{t^2}dt \ge \frac{1}{2}\frac{(b-a)^2}{a^2}. \tag{2.12}$$

Therefore, by (2.11) and (2.12) we have for any a, $b > 0$ that

$$\int_a^b \frac{b-t}{t^2}dt \ge \frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}} = \frac{1}{2}\left(\frac{\min\{a,b\}}{\max\{a,b\}}-1\right)^2$$

and

$$\int_a^b \frac{b-t}{t^2}dt \le \frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}} = \frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2.$$

By the representation (2.10) we then get the desired result (2.9). □

Remark 1. *If we take in (2.9) $a = 1$ and $b = y \in (0,\infty)$, then we get*

$$\frac{1}{2}\left(1-\frac{\min\{1,y\}}{\max\{1,y\}}\right)^2 = \frac{1}{2}\frac{(y-1)^2}{\max^2\{1,y\}} \le y - 1 - \ln y \le \frac{1}{2}\frac{(y-1)^2}{\min^2\{1,y\}} = \frac{1}{2}\left(\frac{\max\{1,y\}}{\min\{1,y\}}-1\right)^2 \tag{2.13}$$

and if we take $a=y$ *and* $b=1$, *then we also get*

$$\frac{1}{2}\left(1-\frac{\min\{1,y\}}{\max\{1,y\}}\right)^2=\frac{1}{2}\frac{(y-1)^2}{\max^2\{1,y\}}\leq\ln y-\frac{y-1}{y}\leq\frac{1}{2}\frac{(y-1)^2}{\min^2\{1,y\}}=\frac{1}{2}\left(\frac{\max\{1,y\}}{\min\{1,y\}}-1\right)^2. \tag{2.14}$$

If $y\in[k,K]\subset(0,\infty)$, then by analyzing all possible locations of the interval $[k,K]$ and 1 we have

$$\min\{1,k\}\leq\min\{1,y\}\leq\min\{1,K\}$$

and

$$\max\{1,k\}\leq\max\{1,y\}\leq\max\{1,K\}.$$

By (2.13) and (2.14) we get the *local bounds*

$$\frac{1}{2}\frac{(y-1)^2}{\max^2\{1,K\}}\leq y-1-\ln y\leq\frac{1}{2}\frac{(y-1)^2}{\min^2\{1,k\}} \tag{2.15}$$

and

$$\frac{1}{2}\frac{(y-1)^2}{\max^2\{1,K\}}\leq\ln y-\frac{y-1}{y}\leq\frac{1}{2}\frac{(y-1)^2}{\min^2\{1,k\}} \tag{2.16}$$

for any $y\in[k,K]$.

Observe also that for $y\in[k,K]$ we have

$$1-\frac{\min\{1,y\}}{\max\{1,y\}}\geq1-\frac{\min\{1,K\}}{\max\{1,k\}}\geq0$$

and

$$0\leq\frac{\max\{1,y\}}{\min\{1,y\}}-1\leq\frac{\max\{1,K\}}{\min\{1,k\}}-1.$$

Now, by (2.13) and (2.14) we get the *global bounds*

$$\frac{1}{2}\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^2\leq y-1-\ln y\leq\frac{1}{2}\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^2 \tag{2.17}$$

and

$$\frac{1}{2}\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^2\leq\ln y-\frac{y-1}{y}\leq\frac{1}{2}\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^2 \tag{2.18}$$

for any $y\in[k,K]$.

We also have:

Lemma 2. *For any* a, $b>0$, *the following inequalities are valid*

$$(0\leq)\,\frac{b-a}{a}-\ln b+\ln a\leq\frac{(b-a)^2}{ab} \tag{2.19}$$

and

$$(0\leq)\,\ln b-\ln a-\frac{b-a}{b}\leq\frac{(b-a)^2}{ab}. \tag{2.20}$$

Proof. If $b > a$, then

$$\int_a^b \frac{b-t}{t^2} dt \le (b-a) \int_a^b \frac{1}{t^2} dt = (b-a) \frac{b-a}{ab} = \frac{(b-a)^2}{ab}.$$

If $a > b$, then

$$\int_a^b \frac{b-t}{t^2} dt = \int_b^a \frac{t-b}{t^2} dt \le (a-b) \int_b^a \frac{1}{t^2} dt = (a-b) \frac{a-b}{ab} = \frac{(b-a)^2}{ab}.$$

Therefore,

$$\int_a^b \frac{b-t}{t^2} dt \le \frac{(b-a)^2}{ab}$$

for any $a, b > 0$ and by the representation (2.10) we get the desired result (2.19). □

It is natural to ask, which of the upper bounds for the quantity

$$\frac{b-a}{a} - \ln b + \ln a$$

as provided by (2.9) and (2.19) is better?

Consider the difference

$$\Delta(a,b) := \frac{1}{2} \frac{(b-a)^2}{\min^2\{a,b\}} - \frac{(b-a)^2}{ab}, \ a, \ b > 0.$$

We observe that for $b > a$ we get

$$\Delta(a,b) := \frac{1}{2} \frac{(b-a)^2}{a^2} - \frac{(b-a)^2}{ab} = \frac{(b-a)^2}{2a^2 b}(b-2a).$$

Therefore $\Delta(a,b) > 0$ if $b > 2a$ and $\Delta(a,b) < 0$ if $a < b < 2a$, meaning that neither of the upper bounds in (2.9) and (2.19) is always best.

If we take in (2.19) and (2.20) $a = 1$ and $b = y \in (0,\infty)$, then we get

$$(0 \le)\, y - 1 - \ln y \le \frac{(y-1)^2}{y} \tag{2.21}$$

and

$$(0 \le) \ln y - \frac{y-1}{y} \le \frac{(y-1)^2}{y} \tag{2.22}$$

for any $y > 0$.

If $y \in [k, K]$, then we have the global upper bounds

$$(0 \le)\, y - 1 - \ln y \le U(k,K) \tag{2.23}$$

and

$$(0 \le) \ln y - \frac{y-1}{y} \le U(k,K), \tag{2.24}$$

where

$$U(k,K) := \begin{cases} \frac{(1-k)^2}{k} \text{ if } K < 1, \\ \max\left\{\frac{(1-k)^2}{k}, \frac{(K-1)^2}{K}\right\} \text{ if } k \le 1 \le K, \\ \frac{(K-1)^2}{K} \text{ if } 1 < k. \end{cases} \tag{2.25}$$

Indeed, if we consider the function $f(y) = \frac{(y-1)^2}{y}$, $y > 0$, then we observe that

$$f'(y) = \frac{y^2-1}{y^2} \text{ and } f''(y) = \frac{2}{y^3},$$

which shows that f is strictly decreasing on $(0,1)$, strictly increasing on $[1,\infty)$ and strictly convex for $y > 0$. We also have $f\left(\frac{1}{y}\right) = f(y)$ for $y > 0$.

By (2.21) and by the properties of f we then have that for any $y \in [k, K]$

$$\begin{aligned} y - 1 - \ln y &\le \max_{y\in[k,K]} \frac{(y-1)^2}{y} \\ &= \begin{cases} \frac{(k-1)^2}{k} \text{ if } K < 1, \\ \max\left\{\frac{(k-1)^2}{k}, \frac{(K-1)^2}{K}\right\} \text{ if } k \le 1 \le K, \\ \frac{(K-1)^2}{K} \text{ if } 1 < k. \end{cases} \\ &= U(k,K). \end{aligned} \tag{2.26}$$

Let $y = \frac{1}{y}$ with $y \in [k, K]$. Then $y \in \left[\frac{1}{K}, \frac{1}{k}\right]$ and we have like in (2.26) that

$$\begin{aligned} y - 1 - \ln y &\le \max_{y\in[K^{-1},k^{-1}]} \frac{(y-1)^2}{y} \\ &= \begin{cases} \frac{\left(K^{-1}-1\right)^2}{K^{-1}} \text{ if } k^{-1} < 1, \\ \max\left\{\frac{\left(K^{-1}-1\right)^2}{K^{-1}}, \frac{\left(\frac{1}{k^{-1}}-1\right)^2}{k^{-1}}\right\} \text{ if } k \le 1 \le K^{-1}, \\ \frac{\left(\frac{1}{k^{-1}}-1\right)^2}{k^{-1}} \text{ if } 1 < \frac{1}{K^{-1}}. \end{cases} \\ &= U(k,K), \end{aligned}$$

which implies (2.24).

Now, let

$$V(k,K) := \frac{1}{2}\left(\frac{\max\{1,K\}}{\min\{1,k\}} - 1\right)^2 = \frac{1}{2}\begin{cases} \left(\frac{1-k}{k}\right)^2 \text{ if } K < 1, \\ \left(\frac{K-k}{k}\right)^2 \text{ if } k \le 1 \le K, \\ (K-1)^2 \text{ if } 1 < k, \end{cases} \tag{2.27}$$

and

$$v(k,K) := \frac{1}{2}\left(1 - \frac{\min\{1,K\}}{\max\{1,k\}}\right)^2 = \frac{1}{2}\begin{cases} (1-K)^2 \text{ if } K < 1, \\ 0 \text{ if } k \le 1 \le K, \\ \left(\frac{k-1}{k}\right)^2 \text{ if } 1 < k, \end{cases} \tag{2.28}$$

then by (2.17) and (2.18) we have

$$v(k,K) \le y - 1 - \ln y \le V(k,K) \tag{2.29}$$

and

$$v(k,K) \le \ln y - \frac{y-1}{y} \le V(k,K) \tag{2.30}$$

for any $y \in [k, K]$.

Therefore, we have the double inequalities of interest:

$$v(k,K) \le y - 1 - \ln y \le \min\{V(k,K), U(k,K)\} \tag{2.31}$$

and

$$v(k,K) \le \ln y - \frac{y-1}{y} \le \min\{V(k,K), U(k,K)\} \tag{2.32}$$

for any $y \in [k,K]$.

Lemma 3. *Let $x \in [k,K]$ and $t > 0$, then we have*

$$\frac{1}{t}v\left(k^t, K^t\right) \le \frac{x^t - 1}{t} - \ln x \le \frac{1}{t}\min\left\{V\left(k^t,K^t\right), U\left(k^t,K^t\right)\right\} \tag{2.33}$$

and

$$\frac{1}{t}v\left(k^t, K^t\right) \le \ln x - \frac{1-x^{-t}}{t} \le \frac{1}{t}\min\left\{V\left(k^t,K^t\right), U\left(k^t,K^t\right)\right\}. \tag{2.34}$$

The proof follows by choosing $y = x^t \in [k^t, K^t]$ in the inequalities (2.31) and (2.32).

Theorem 2. *Let A, B be two positive invertible operators and the constants $M > m > 0$ with the property that*

$$mA \le B \le MA. \tag{2.35}$$

Then for any $t > 0$ we have

$$\begin{aligned}\frac{1}{t}v\left(m^t, M^t\right)A &\le T_t(A|B) - S(A|B) \\ &\le \frac{1}{t}\min\left\{V\left(m^t,M^t\right), U\left(m^t,M^t\right)\right\}A\end{aligned} \tag{2.36}$$

and

$$\begin{aligned}\frac{1}{t}v\left(m^t, M^t\right)A &\le S(A|B) - T_t(A|B)(A\sharp_t B)^{-1}A \\ &\le \frac{1}{t}\min\left\{V\left(m^t,M^t\right), U\left(m^t,M^t\right)\right\}A,\end{aligned} \tag{2.37}$$

where the functions v, V and U are defined by (2.28), (2.27) and (2.25), respectively.

Proof. Since $mA \le B \le MA$ and A is invertible, then by multiplying both sides with $A^{-1/2}$ we get $m1_H \le A^{-1/2}BA^{-1/2} \le M$. Denote $X = A^{-1/2}BA^{-1/2}$ and by using the functional calculus for X and the Lemma 3, we get

$$\begin{aligned}\frac{1}{t}v\left(k^t, K^t\right) &\le \frac{\left(A^{-1/2}BA^{-1/2}\right)^t - 1}{t} - \ln A^{-1/2}BA^{-1/2} \\ &\le \frac{1}{t}\min\left\{V\left(k^t,K^t\right), U\left(k^t,K^t\right)\right\}\end{aligned} \tag{2.38}$$

and

$$\begin{aligned}\frac{1}{t}v\left(k^t, K^t\right) &\le \ln A^{-1/2}BA^{-1/2} - \frac{1-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t} \\ &\le \frac{1}{t}\min\left\{V\left(k^t,K^t\right), U\left(k^t,K^t\right)\right\},\end{aligned} \tag{2.39}$$

for any $t > 0$.

Now, if we multiply both sides of (2.38) and (2.39) by $A^{1/2}$ we get the desired results (2.36) and (2.37). □

Assume that the operators A, B satisfy the condition (2.35) for the constants the constants $M > m > 0$. If we take $t = 1$ in (2.36) and (2.37), then we get

$$v(m,M)A \le B - A - S(A|B) \le \min\{V(m,M), U(m,M)\}A \tag{2.40}$$

and

$$v(m,M)A \le S(A|B) - A + AB^{-1}A \le \min\{V(m,M), U(m,M)\}A. \tag{2.41}$$

If we take $t = 2$ in the same inequalities, then we get

$$\frac{1}{2}v\left(m^2,M^2\right)A \le \frac{1}{2}\left(BA^{-1}B - A\right) - S(A|B) \le \frac{1}{2}\min\left\{V\left(m^2,M^2\right), U\left(m^2,M^2\right)\right\}A \tag{2.42}$$

and

$$\frac{1}{2}v\left(m^2,M^2\right)A \le S(A|B) - \frac{1}{2}A\left(1_H - \left(B^{-1}A\right)^2\right) \le \frac{1}{2}\min\left\{V\left(m^2,M^2\right), U\left(m^2,M^2\right)\right\}A. \tag{2.43}$$

For $t = \frac{1}{2}$, we get from (2.36) and (2.37) that

$$2v\left(\sqrt{m},\sqrt{M}\right)A \le 2(A\sharp B - A) - S(A|B) \le 2\min\left\{V\left(\sqrt{m},\sqrt{M}\right), U\left(\sqrt{m},\sqrt{M}\right)\right\}A \tag{2.44}$$

and

$$2v\left(\sqrt{m},\sqrt{M}\right)A \le S(A|B) - 2\left(A - A(A\sharp B)^{-1}A\right) \le 2\min\left\{V\left(\sqrt{m},\sqrt{M}\right), U\left(\sqrt{m},\sqrt{M}\right)\right\}A. \tag{2.45}$$

Corollary 1. *Let C be a positive operator such that*

$$p1_H \le C \le P1_H \tag{2.46}$$

for some constants p, P with $0 < p < P$.

Then for any $t > 0$ we have

$$\frac{1}{t}v\left(P^{-t},p^{-t}\right)C \le \frac{C^{1-t} - C}{t} - \eta(C) \le \frac{1}{t}\min\left\{V\left(P^{-t},p^{-t}\right), U\left(P^{-t},p^{-t}\right)\right\}C \tag{2.47}$$

and

$$\frac{1}{t}v\left(P^{-t},p^{-t}\right)C \le \eta(C) - \frac{1}{t}C\left(1_H - C^t\right) \le \frac{1}{t}\min\left\{V\left(P^{-t},p^{-t}\right), U\left(P^{-t},p^{-t}\right)\right\}C, \tag{2.48}$$

where the functions v, V and U are defined by (2.28), (2.27) and (2.25), respectively.

If C is as in Corollary 1, then by taking $t = 1$ in (2.47) and (2.48) we get

$$v\left(P^{-1},p^{-1}\right)C \le 1_H - C - \eta(C) \le \min\left\{V\left(P^{-1},p^{-1}\right), U\left(P^{-1},p^{-1}\right)\right\}C \tag{2.49}$$

and

$$v\left(P^{-1},p^{-1}\right)C \le \eta(C) - C(1_H - C) \le \min\left\{V\left(P^{-1},p^{-1}\right), U\left(P^{-1},p^{-1}\right)\right\}C. \tag{2.50}$$

For $t=2$ we get

$$\begin{aligned}(2.51)\qquad \frac{1}{2}v\left(P^{-2},p^{-2}\right)C &\leq \frac{1}{2}\left(C^{-1}-C\right)-\eta\left(C\right)\\ &\leq \frac{1}{2}\min\left\{V\left(P^{-2},p^{-2}\right),U\left(P^{-2},p^{-2}\right)\right\}C\end{aligned}$$

and

$$\begin{aligned}(2.52)\qquad \frac{1}{2}v\left(P^{-2},p^{-2}\right)C &\leq \eta\left(C\right)-\frac{1}{2}C\left(1_H-C^2\right)\\ &\leq \frac{1}{2}\min\left\{V\left(P^{-2},p^{-2}\right),U\left(P^{-2},p^{-2}\right)\right\}C.\end{aligned}$$

Finally, if we take $t=\frac{1}{2}$ in (2.47) and (2.48), then we get

$$\begin{aligned}(2.53)\qquad 2v\left(P^{-1/2},p^{-1/2}\right)C &\leq 2C^{1/2}\left(1_H-C^{1/2}\right)-\eta\left(C\right)\\ &\leq 2\min\left\{V\left(P^{-1/2},p^{-1/2}\right),U\left(P^{-1/2},p^{-1/2}\right)\right\}C\end{aligned}$$

and

$$\begin{aligned}(2.54)\qquad 2v\left(P^{-1/2},p^{-1/2}\right)C &\leq \eta\left(C\right)-2C\left(1_H-C^{1/2}\right)\\ &\leq 2\min\left\{V\left(P^{-1/2},p^{-1/2}\right),U\left(P^{-1/2},p^{-1/2}\right)\right\}C.\end{aligned}$$

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