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SOME INEQUALITIES FOR LOGARITHM VIA TAYLOR'S EXPANSION WITH INTEGRAL REMAINDER

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Abstract. In this paper we establish several inequalities for logarithm by the use of Taylor's expansion with integral remainder. The case of two positive numbers and an analysis of which bound is better are also considered.

1. Introduction

In the recent paper [2] we established the following result:

$$(0\leq)\,(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\leq\nu\,(1-\nu)\,(b-a)\,(\ln b-\ln a) \tag{1.1}$$

for any a, $b>0$ and $\nu\in(0,1)$.

If we take in (1.1) $b=x+1$, $x>0$ and $a=1$, then we get

$$\ln\,(x+1)\geq\frac{1-\nu+\nu\,(x+1)-(x+1)^{\nu}}{\nu\,(1-\nu)\,x}\,(\geq0) \tag{1.2}$$

for any $\nu\in(0,1)$ and, in particular

$$\ln\,(x+1)\geq\frac{2\left(\sqrt{x+1}-1\right)^{2}}{x} \tag{1.3}$$

for any $x>0$ and $\nu\in(0,1)$.

If we take in (1.1) $b=x$ and $a=1$ we also have

$$\frac{\ln x}{x-1}\geq\frac{1-\nu+\nu x-x^{\nu}}{\nu\,(1-\nu)\,(x-1)^{2}} \tag{1.4}$$

for any $x>0$, $x\neq1$ and $\nu\in(0,1)$.

Further, by choosing in (1.4) $\nu=\frac{1}{2}$ and perform the calculations, we get

$$\frac{\ln x}{x-1}\geq\frac{2}{\left(\sqrt{x}+1\right)^{2}} \tag{1.5}$$

for any $x>0$, $x\neq1$.

In the recent paper [4] we obtained the following inequalities for logarithm as well

$$\begin{aligned}0 &\leq \frac{x-1}{x}\leq\frac{2\,(x-1)}{x+1}\leq\ln x\leq\frac{x-1}{\sqrt{x}}\\ &\leq \frac{x-1}{x+1}+\frac{x^{2}-1}{4x}\leq\frac{x^{2}-1}{2x}\leq x-1\end{aligned} \tag{1.6}$$

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where $x \geq 1$ and

$$0 \leq \frac{x^2-1}{2x} - \ln x \leq \frac{1}{8}\frac{(x-1)^3(x+1)}{x^2}, \tag{1.7}$$

$$0 \leq \ln x - \frac{2(x-1)}{x+1} \leq \frac{1}{8}\frac{(x-1)^3(x+1)}{x^2}, \tag{1.8}$$

where $x \geq 1$.

There are also a number of inequalities for logarithm that are well know and widely used in literature, such as:

$$\frac{x-1}{x} \leq \ln x \leq x-1 \text{ for } x>0, \tag{1.9}$$

$$\frac{2x}{2+x} \leq \ln(1+x) \leq \frac{x}{\sqrt{x+1}} \text{ for } x \geq 0, \tag{1.10}$$

$$x \leq -\ln(1-x) \leq \frac{x}{1-x}, \text{ for } x<1,$$

$$\ln x \leq n\left(x^{1/n}-1\right) \text{ for } n>0 \text{ and } x>0,$$

$$\ln(1-|x|) \leq \ln(x+1) \leq -\ln(1-|x|) \text{ for } |x|<1,$$

and

$$-\frac{3}{2}x \leq \ln(1-x) \leq \frac{3}{2}x \text{ for } 0<x\leq 0.5838,$$

see for instance

`http://functions.wolfram.com/ElementaryFunctions/Log/29/`

and [5].

A simple proof of the first inequality in (1.10) may be found, for instance, in [6], see also [7] where the following rational bounds are provided as well:

$$\frac{x\left(1+\frac{5}{6}x\right)}{(1+x)\left(1+\frac{1}{3}x\right)} \leq \ln(1+x) \leq \frac{x\left(1+\frac{1}{6}x\right)}{1+\frac{2}{3}x} \text{ for } x \geq 0.$$

In this paper we establish some bounds for the quantities

$$\ln b - \ln a - \sum_{k=1}^{2n-1}\frac{(b-a)^k}{kb^k} \text{ and } \sum_{k=1}^{2n-1}(-1)^{k-1}\frac{(b-a)^k}{ka^k} - \ln b + \ln a$$

when a, $b>0$. The case $a=1$ and $b=x$ is explored and various local and global bounds in the case that x is located in a bounded interval are also provided. By performing some numerical experiments it is also shown that the obtained upper bounds can not be compared in general, meaning that some time one is better than the other.

2. Logarithmic Inequalities for Two Positive Numbers

The following theorem is well known in the literature as Taylor's theorem with the integral remainder.

Theorem 1. *Let $I \subset \mathbb{R}$ be a closed interval, $a \in I$ and let m be a positive integer. If $f : I \longrightarrow \mathbb{R}$ is such that $f^{(m)}$ is absolutely continuous on I, then for each $x \in I$*

$$f(x) = T_m(f; a, x) + R_m(f; a, x) \tag{2.1}$$

where $T_m(f; a, x)$ is Taylor's polynomial, i.e.,

$$T_m(f; a, x) := \sum_{k=0}^{m} \frac{(x-a)^k}{k!} f^{(k)}(a).$$

(Note that $f^{(0)} := f$ and $0! := 1$), and the remainder is given by

$$R_m(f; a, x) := \frac{1}{m!} \int_a^x (x-t)^m f^{(m+1)}(t)\, dt.$$

The following result holds [1]:

Lemma 1. *For any a, $b > 0$ we have for $m \geq 1$ that*

$$\ln b - \ln a + \sum_{k=1}^{m} \frac{(-1)^k (b-a)^k}{k a^k} = (-1)^m \int_a^b \frac{(b-t)^m}{t^{m+1}} dt. \tag{2.2}$$

Proof. For the sake of completeness, we give a short proof here.

Consider the function $f : (0, \infty) \longrightarrow \mathbb{R}$, $f(x) = \ln x$, then,

$$f^{(m)}(x) = \frac{(-1)^{m-1}(m-1)!}{x^m}, \; m \geq 1, \; x > 0,$$

$$T_m(f; a, x) = \ln a + \sum_{k=1}^{m} \frac{(-1)^{k-1}(x-a)^k}{k a^k}, \; a > 0$$

and

$$R_m(f; a, x) = (-1)^m \int_a^x \frac{(x-t)^m}{t^{m+1}} dt.$$

Now, using (2.1) we have the equality,

$$\ln x = \ln a + \sum_{k=1}^{m} \frac{(-1)^{k-1}(x-a)^k}{k a^k} + (-1)^m \int_a^x \frac{(x-t)^m}{t^{m+1}} dt,$$

i.e.,

$$\ln x - \ln a + \sum_{k=1}^{m} \frac{(-1)^k (x-a)^k}{k a^k} = (-1)^m \int_a^x \frac{(x-t)^m}{t^{m+1}} dt, \quad x, \; a > 0.$$

Choosing in the last equality $x = b$, we get (2.2). □

We have:

Theorem 2. *For any $a,\ b>0$ we have for $n\geq 1$ that*

$$\frac{(b-a)^{2n}}{2n\max^{2n}\{a,b\}}\leq\sum_{k=1}^{2n-1}(-1)^{k-1}\frac{(b-a)^k}{ka^k}-\ln b+\ln a\leq\frac{(b-a)^{2n}}{2n\min^{2n}\{a,b\}} \tag{2.3}$$

and

$$\frac{(b-a)^{2n}}{2n\max^{2n}\{a,b\}}\leq\ln b-\ln a-\sum_{k=1}^{2n-1}\frac{(b-a)^k}{kb^k}\leq\frac{(b-a)^{2n}}{2n\min^{2n}\{a,b\}}. \tag{2.4}$$

Proof. For $m=2n-1$ with $n\geq 1$, then from (2.3) we have

$$\ln b-\ln a+\sum_{k=1}^{2n-1}\frac{(-1)^k(b-a)^k}{ka^k}=-\int_a^b\frac{(b-t)^{2n-1}}{t^{2n}}dt, \tag{2.5}$$

for any $a,\ b>0$, giving that

$$\int_a^b\frac{(b-t)^{2n-1}}{t^{2n}}dt=\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}(b-a)^k}{ka^k}-\ln b+\ln a. \tag{2.6}$$

If $b>a>0$, then

$$\int_a^b\frac{(b-t)^{2n-1}}{t^{2n}}dt\geq\frac{1}{b^{2n}}\int_a^b(b-t)^{2n-1}dt=\frac{(b-a)^{2n}}{2nb^{2n}} \tag{2.7}$$

and

$$\int_a^b\frac{(b-t)^{2n-1}}{t^{2n}}dt\leq\frac{1}{a^{2n}}\int_a^b(b-t)^{2n-1}dt=\frac{(b-a)^{2n}}{2na^{2n}}. \tag{2.8}$$

If $a>b>0$, then

$$\int_a^b\frac{(b-t)^{2n-1}}{t^{2n}}dt=-\int_b^a\frac{(b-t)^{2n-1}}{t^{2n}}dt=\int_b^a\frac{(t-b)^{2n-1}}{t^{2n}}dt.$$

Therefore

$$\int_b^a\frac{(t-b)^{2n-1}}{t^{2n}}dt\geq\frac{1}{a^{2n}}\int_b^a(t-b)^{2n-1}dt=\frac{(a-b)^{2n}}{2na^{2n}} \tag{2.9}$$

and

$$\int_b^a\frac{(t-b)^{2n-1}}{t^{2n}}dt\leq\frac{1}{b^{2n}}\int_b^a(t-b)^{2n-1}dt=\frac{(a-b)^{2n}}{2nb^{2n}}. \tag{2.10}$$

By making use of (2.7)-(2.10), we deduce the desired result (2.3).

Now, if we replace a with b in (2.3) we get (2.4). □

Corollary 1. *For any $a,\ b>0$ we have*

$$\frac{(b-a)^2}{2\max^2\{a,b\}}\leq\frac{b-a}{a}-\ln b+\ln a\leq\frac{(b-a)^2}{2\min^2\{a,b\}}, \tag{2.11}$$

$$\frac{(b-a)^2}{2\max^2\{a,b\}}\leq\ln b-\ln a-\frac{b-a}{b}\leq\frac{(b-a)^2}{2\min^2\{a,b\}}, \tag{2.12}$$

$$\frac{(b-a)^4}{4\max^4\{a,b\}} \leq \frac{b-a}{a} - \frac{(b-a)^2}{2a^2} + \frac{(b-a)^3}{3a^3} - \ln b + \ln a \leq \frac{(b-a)^4}{4\min^4\{a,b\}} \tag{2.13}$$

and

$$\frac{(b-a)^4}{4\max^4\{a,b\}} \leq \ln b - \ln a - \frac{b-a}{b} - \frac{(b-a)^2}{2b^2} - \frac{(b-a)^3}{3b^3} \leq \frac{(b-a)^4}{4\min^4\{a,b\}}. \tag{2.14}$$

Remark 1. *Since the lower bounds in (2.3) and (2.4) are positive, then we have*

$$\sum_{k=1}^{2n-1} \frac{(b-a)^k}{kb^k} \leq \ln b - \ln a \leq \sum_{k=1}^{2n-1} (-1)^{k-1} \frac{(b-a)^k}{ka^k} \tag{2.15}$$

for any $a,\ b > 0$ and $n \geq 1$.

In particular, we have for $n = 1$ the well known inequality

$$\frac{b-a}{b} \leq \ln b - \ln a \leq \frac{b-a}{a} \tag{2.16}$$

and for $n = 2$

$$\frac{b-a}{b} + \frac{(b-a)^2}{2b^2} + \frac{(b-a)^3}{3b^3} \leq \ln b - \ln a \leq \frac{b-a}{a} - \frac{(b-a)^2}{2a^2} + \frac{(b-a)^3}{3a^3} \tag{2.17}$$

for any $a,\ b > 0$.

We have the following upper bounds:

Theorem 3. *For any $a,\ b > 0$ we have for $n \geq 1$ that*

$$(0 \leq) \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}(b-a)^k}{ka^k} - \ln b + \ln a \leq \frac{|b-a|^{2n-1}\left|b^{2n-1} - a^{2n-1}\right|}{(2n-1)\, b^{2n-1}a^{2n-1}} \tag{2.18}$$

and

$$(0 \leq) \ln b - \ln a - \sum_{k=1}^{2n-1} \frac{(b-a)^k}{kb^k} \leq \frac{|b-a|^{2n-1}\left|b^{2n-1} - a^{2n-1}\right|}{(2n-1)\, b^{2n-1}a^{2n-1}}. \tag{2.19}$$

Proof. Let $b > a > 0$, then

$$\begin{aligned}
\int_a^b \frac{(b-t)^{2n-1}}{t^{2n}} dt &\leq (b-a)^{2n-1} \int_a^b t^{-2n} dt \\
&= \frac{1}{2n-1} (b-a)^{2n-1} \frac{b^{2n-1} - a^{2n-1}}{b^{2n-1}a^{2n-1}} \\
&= \frac{1}{2n-1} |b-a|^{2n-1} \frac{\left|b^{2n-1} - a^{2n-1}\right|}{b^{2n-1}a^{2n-1}}
\end{aligned}$$

and for $a > b > a$ we also have

$$\begin{aligned}\int_b^a \frac{(t-b)^{2n-1}}{t^{2n}} dt &\le (a-b)^{2n-1} \int_b^a t^{-2n} dt \\ &= \frac{1}{2n-1} (a-b)^{2n-1} \frac{a^{2n-1} - b^{2n-1}}{b^{2n-1} a^{2n-1}} \\ &= \frac{1}{2n-1} |b-a|^{2n-1} \frac{\left|b^{2n-1} - a^{2n-1}\right|}{b^{2n-1} a^{2n-1}},\end{aligned}$$

which, by the identity (2.6) proves the inequality (2.18). □

Corollary 2. *For any a, $b > 0$ we have the known inequalities*

$$(0 \le) \frac{b-a}{a} - \ln b + \ln a \le \frac{(b-a)^2}{ab} \tag{2.20}$$

and

$$(0 \le) \ln b - \ln a - \frac{b-a}{b} \le \frac{(b-a)^2}{ab}. \tag{2.21}$$

We also have

$$(0 \le) \frac{b-a}{a} - \frac{(b-a)^2}{2a^2} + \frac{(b-a)^3}{3a^3} - \ln b + \ln a \le \frac{1}{3} (b-a)^4 \frac{b^2 + ba + a^2}{b^3 a^3} \tag{2.22}$$

and

$$(0 \le) \ln b - \ln a - \frac{b-a}{b} - \frac{(b-a)^2}{2b^2} - \frac{(b-a)^3}{3b^3} \le \frac{1}{3} (b-a)^4 \frac{b^2 + ba + a^2}{b^3 a^3} \tag{2.23}$$

for any a, $b > 0$.

We also have

Theorem 4. *Let p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. For any a, $b > 0$ we have for $n \ge 1$ that*

$$\begin{aligned}&(0 \le) \sum_{k=1}^{2n-1} \frac{(-1)^{k-1} (b-a)^k}{k a^k} - \ln b + \ln a \\ &\le \frac{|b-a|^{2n-1+1/p} \left|b^{2nq-1} - a^{2nq-1}\right|^{1/q}}{[(2n-1) p + 1]^{1/p} (2nq-1)^{1/q} (ba)^{2n-1/q}}\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}&(0 \le) \ln b - \ln a - \sum_{k=1}^{2n-1} \frac{(b-a)^k}{k b^k} \\ &\le \frac{|b-a|^{2n-1+1/p} \left|b^{2nq-1} - a^{2nq-1}\right|^{1/q}}{[(2n-1) p + 1]^{1/p} (2nq-1)^{1/q} (ba)^{2n-1/q}}.\end{aligned} \tag{2.25}$$

Proof. Using Hölder's integral inequality for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have for $b>a$ that

$$\begin{aligned}\int_a^b \frac{(b-t)^{2n-1}}{t^{2n}}dt &\le \left(\int_a^b (b-t)^{(2n-1)p}\,dt\right)^{1/p}\left(\int_a^b t^{-2nq}dt\right)^{1/q}\\ &= \left(\frac{(b-a)^{(2n-1)p+1}}{(2n-1)\,p+1}\right)^{1/p}\left(\frac{b^{-2nq+1}-a^{-2nq+1}}{-2nq+1}\right)^{1/q}\\ &= \frac{(b-a)^{2n-1+1/p}}{[(2n-1)\,p+1]^{1/p}}\frac{\left(b^{2nq-1}-a^{2nq-1}\right)^{1/q}}{(2nq-1)^{1/q}(ba)^{2n-1/q}}\\ &= \frac{(b-a)^{2n-1+1/p}\left(b^{2nq-1}-a^{2nq-1}\right)^{1/q}}{[(2n-1)\,p+1]^{1/p}(2nq-1)^{1/q}(ba)^{2n-1/q}}.\end{aligned}$$

If $a>b>0$, then in a similar way, we also have

$$\begin{aligned}\int_a^b \frac{(b-t)^{2n-1}}{t^{2n}}dt &= \int_b^a \frac{(t-b)^{2n-1}}{t^{2n}}dt\\ &\le \frac{(a-b)^{2n-1+1/p}\left(a^{2nq-1}-b^{2nq-1}\right)^{1/q}}{[(2n-1)\,p+1]^{1/p}(2nq-1)^{1/q}(ba)^{2n-1/q}}.\end{aligned}$$

Therefore, we have

$$\int_a^b \frac{(b-t)^{2n-1}}{t^{2n}}dt \le \frac{|b-a|^{2n-1+1/p}\left|b^{2nq-1}-a^{2nq-1}\right|^{1/q}}{[(2n-1)\,p+1]^{1/p}(2nq-1)^{1/q}(ba)^{2n-1/q}}$$

for any $a,b>0$ and by using the representation (2.6) we get (2.24). □

Corollary 3. *Let p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. For any a, $b>0$ we have*

$$(0\le)\,\frac{b-a}{a}-\ln b+\ln a \le \frac{|b-a|^{1+1/p}\left|b^{2q-1}-a^{2q-1}\right|^{1/q}}{[(p+1)]^{1/p}(2q-1)^{1/q}(ba)^{2-1/q}} \tag{2.26}$$

and

$$(0\le)\,\ln b-\ln a-\frac{b-a}{b} \le \frac{|b-a|^{1+1/p}\left|b^{2q-1}-a^{2q-1}\right|^{1/q}}{[(p+1)]^{1/p}(2q-1)^{1/q}(ba)^{2-1/q}}. \tag{2.27}$$

We also have

$$\begin{aligned}&(0\le)\,\frac{b-a}{a}-\frac{(b-a)^2}{2a^2}+\frac{(b-a)^3}{3a^3}-\ln b+\ln a\\ &\le \frac{|b-a|^{3+1/p}\left|b^{4q-1}-a^{4q-1}\right|^{1/q}}{(3p+1)^{1/p}(4q-1)^{1/q}(ba)^{4-1/q}}\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}&(0\le)\,\ln b-\ln a-\frac{b-a}{b}-\frac{(b-a)^2}{2b^2}-\frac{(b-a)^3}{3b^3}\\ &\le \frac{|b-a|^{3+1/p}\left|b^{4q-1}-a^{4q-1}\right|^{1/q}}{(3p+1)^{1/p}(4q-1)^{1/q}(ba)^{4-1/q}}\end{aligned} \tag{2.29}$$

for any a, $b>0$.

Remark 2. *The Euclidean case, namely when $p = q$ produces in Theorem 4*

$$(0 \leq) \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}(b-a)^k}{ka^k} - \ln b + \ln a \leq \frac{|b-a|^{2n-1/2}\left|b^{4n-1} - a^{4n-1}\right|^{1/2}}{(4n-1)(ba)^{2n-1/2}} \tag{2.30}$$

and

$$(0 \leq) \ln b - \ln a - \sum_{k=1}^{2n-1} \frac{(b-a)^k}{kb^k} \leq \frac{|b-a|^{2n-1/2}\left|b^{4n-1} - a^{4n-1}\right|^{1/2}}{(4n-1)(ba)^{2n-1/2}}. \tag{2.31}$$

If we take in these inequalities $n = 1$, then we get

$$(0 \leq) \frac{b-a}{a} - \ln b + \ln a \leq \frac{1}{3}(b-a)^2 \sqrt{\frac{b^2 + ba + a^2}{ba}} \tag{2.32}$$

and

$$(0 \leq) \ln b - \ln a - \frac{b-a}{b} \leq \frac{1}{3}(b-a)^2 \sqrt{\frac{b^2 + ba + a^2}{ba}} \tag{2.33}$$

for any a, $b > 0$.

If we take in (2.30) and (2.31) $n = 2$, then we get

$$(0 \leq) \frac{b-a}{a} - \frac{(b-a)^2}{2a^2} + \frac{(b-a)^3}{3a^3} - \ln b + \ln a \leq \frac{|b-a|^{7/2}\left|b^7 - a^7\right|^{1/2}}{7(ba)^{7/2}} \tag{2.34}$$

and

$$(0 \leq) \ln b - \ln a - \frac{b-a}{b} - \frac{(b-a)^2}{2b^2} - \frac{(b-a)^3}{3b^3} \leq \frac{|b-a|^{7/2}\left|b^7 - a^7\right|^{1/2}}{7(ba)^{7/2}}. \tag{2.35}$$

We observe that, by the inequalities (2.11), (2.20) and (2.32) the quantity

$$\frac{b-a}{a} - \ln b + \ln a \text{ or } \left(\ln b - \ln a - \frac{b-a}{b}\right)$$

has as upper bounds

$$A_1(a,b) := \frac{(b-a)^2}{2\min^2\{a,b\}},\ A_2(a,b) = \frac{(b-a)^2}{ab}$$

and

$$A_3(a,b) := \frac{1}{3}(b-a)^2 \sqrt{\frac{b^2 + ba + a^2}{ba}},$$

for a, $b > 0$.

It is therefore natural the to ask *how these bounds do compare*?

In order to answer this question we consider the simpler functions

$$B_1(a,b) := \frac{1}{2\min^2\{a,b\}},\ B_2(a,b) = \frac{1}{ab}$$

and

$$B_3(a,b) := \frac{1}{3}\sqrt{\frac{b^2 + ba + a^2}{ba}}$$

for a, $b > 0$ and define the differences $D_1(a,b) := B_1(a,b) - B_2(a,b)$, $D_2(a,b) := B_2(a,b) - B_3(a,b)$ and $D_3(a,b) := B_1(a,b) - B_3(a,b)$.

The plot of the function $D_1(a,b)$ on the box $[0.5, 2] \times [0.5, 2]$ is depicted in Figure 1, the plot of $D_2(a,b)$ on the box $[1, 2] \times [1, 2]$ is depicted in Figure 2 and the plot of $D_3(a,b)$ on $[0.5, 2] \times [0.5, 2]$ in Figure 3

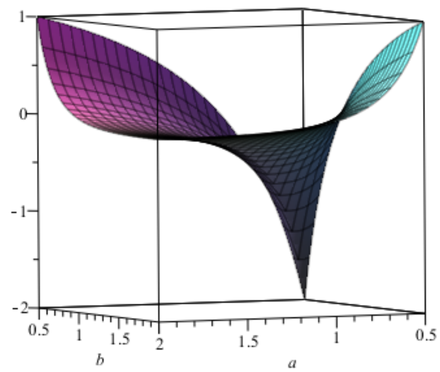

FIGURE 1. Plot of $D_1(a,b)$ on $[0.5,2]\times[0.5,2]$

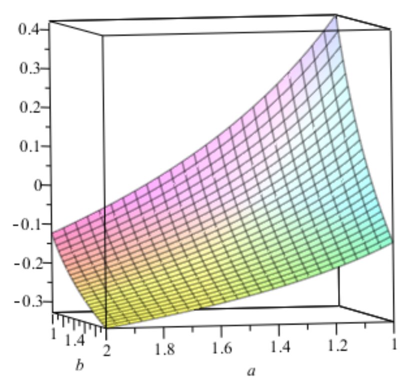

FIGURE 2. Plot of $D_2(a,b)$ on $[1,2]\times[1,2]$

3. Logarithmic Inequalities for a Positive Number

Since for $x>0$ we have:

$$\frac{(x-1)^{2n}}{\max^{2n}\{1,x\}}=\left(1-\frac{\min\{1,x\}}{\max\{1,x\}}\right)^{2n},$$

$$\frac{(x-1)^{2n}}{\min^{2n}\{1,x\}}=\left(\frac{\max\{1,x\}}{\min\{1,x\}}-1\right)^{2n},$$

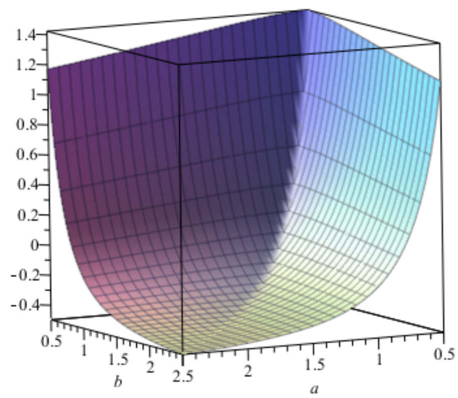

FIGURE 3. Plot of $D_3(a,b)$ on $[0.5, 2.5]^2$

$$\frac{(x-1)^{2n}}{\max^{2n}\{1,x\}} = (\min\{1,x\})^{2n}\left(\frac{x-1}{x}\right)^{2n}$$

and

$$\frac{(x-1)^{2n}}{\min^{2n}\{1,x\}} = (\max\{1,x\})^{2n}\left(\frac{x-1}{x}\right)^{2n},$$

hence, by taking in Theorem 2 $b = x \in (0,\infty)$ and $a = 1$, we get the inequalities

$$B_1(x,n) \le \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k}(x-1)^k - \ln x \le B_2(x,n) \tag{3.1}$$

and

$$B_1(x,n) \le \ln x - \sum_{k=1}^{2n-1} \frac{1}{k}\left(\frac{x-1}{x}\right)^k \le B_2(x,n) \tag{3.2}$$

for $n \ge 1$, where the bounds

$$\begin{aligned} B_1(x,n) &:= \frac{(x-1)^{2n}}{2n\max^{2n}\{1,x\}} = \frac{1}{2n}(\min\{1,x\})^{2n}\left(\frac{x-1}{x}\right)^{2n} \\ &= \frac{1}{2n}\left(1 - \frac{\min\{1,x\}}{\max\{1,x\}}\right)^{2n} \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} B_2(x,n) &:= \frac{(x-1)^{2n}}{2n\min^{2n}\{1,x\}} = \frac{1}{2n}(\max\{1,x\})^{2n}\left(\frac{x-1}{x}\right)^{2n} \\ &= \frac{1}{2n}\left(\frac{\max\{1,x\}}{\min\{1,x\}} - 1\right)^{2n}. \end{aligned} \tag{3.4}$$

From the inequality (2.15) we have

$$\sum_{k=1}^{2n-1}\frac{1}{k}\left(\frac{x-1}{x}\right)^{k}\leq\ln x\leq\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}(x-1)^{k} \tag{3.5}$$

for any $x>0$, while from (2.18) and (2.19) we get

$$(0\leq)\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}(x-1)^{k}-\ln x\leq\frac{|x-1|^{2n-1}\left|x^{2n-1}-1\right|}{(2n-1)\,x^{2n-1}} \tag{3.6}$$

and

$$(0\leq)\ln x-\sum_{k=1}^{2n-1}\frac{1}{k}\left(\frac{x-1}{x}\right)^{k}\leq\frac{|x-1|^{2n-1}\left|x^{2n-1}-1\right|}{(2n-1)\,x^{2n-1}}, \tag{3.7}$$

for any $x>0$.

The inequalities (2.24) and (2.25) produce

$$(0\leq)\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}(x-1)^{k}-\ln x\leq\frac{|x-1|^{2n-1+1/p}\left|x^{2nq-1}-1\right|^{1/q}}{[(2n-1)\,p+1]^{1/p}\,(2nq-1)^{1/q}\,x^{2n-1/q}} \tag{3.8}$$

and

$$(0\leq)\ln x-\sum_{k=1}^{2n-1}\frac{1}{k}\left(\frac{x-1}{x}\right)^{k}\leq\frac{|x-1|^{2n-1+1/p}\left|x^{2nq-1}-1\right|^{1/q}}{[(2n-1)\,p+1]^{1/p}\,(2nq-1)^{1/q}\,x^{2n-1/q}}, \tag{3.9}$$

which in the case $p=q=2$ reduce to

$$(0\leq)\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}(x-1)^{k}-\ln x\leq\frac{|x-1|^{2n-1/2}\left|x^{4n-1}-1\right|^{1/2}}{(4n-1)\,x^{2n-1/2}} \tag{3.10}$$

and

$$(0\leq)\ln x-\sum_{k=1}^{2n-1}\frac{1}{k}\left(\frac{x-1}{x}\right)^{k}\leq\frac{|x-1|^{2n-1/2}\left|x^{4n-1}-1\right|^{1/2}}{(4n-1)\,x^{2n-1/2}}, \tag{3.11}$$

for any $x>0$.

Assume that $k,\,K>0$ with $k<K$. If $x\in[k,K]\subset(0,\infty)$, then by analyzing all possible locations of the interval $[k,K]$ and 1 we have

$$\min\{1,k\}\leq\min\{1,x\}\leq\min\{1,K\}$$

and

$$\max\{1,k\}\leq\max\{1,x\}\leq\max\{1,K\}.$$

By using the inequalities (3.1) and (3.2) and the first two equalities in (3.3) and (3.4) we have *the local bounds:*

$$\frac{(x-1)^{2n}}{2n\max^{2n}\{1,K\}}\leq\sum_{j=1}^{2n-1}\frac{(-1)^{j-1}}{j}(x-1)^{j}-\ln x\leq\frac{(x-1)^{2n}}{2n\min^{2n}\{1,k\}}, \tag{3.12}$$

$$\frac{(x-1)^{2n}}{2n\max^{2n}\{1,K\}}\leq\ln x-\sum_{j=1}^{2n-1}\frac{1}{j}\left(\frac{x-1}{x}\right)^{j}\leq\frac{(x-1)^{2n}}{2n\min^{2n}\{1,k\}}, \tag{3.13}$$

$$\frac{1}{2n}\left(\min\{1,k\}\right)^{2n}\left(\frac{x-1}{x}\right)^{2n} \leq \sum_{j=1}^{2n-1}\frac{(-1)^{j-1}}{j}(x-1)^{j}-\ln x \leq \frac{1}{2n}\left(\max\{1,K\}\right)^{2n}\left(\frac{x-1}{x}\right)^{2n}, \tag{3.14}$$

and

$$\frac{1}{2n}\left(\min\{1,k\}\right)^{2n}\left(\frac{x-1}{x}\right)^{2n} \leq \ln x-\sum_{j=1}^{2n-1}\frac{1}{j}\left(\frac{x-1}{x}\right)^{j} \leq \frac{1}{2n}\left(\max\{1,K\}\right)^{2n}\left(\frac{x-1}{x}\right)^{2n} \tag{3.15}$$

for any $x\in[k,K]$ and $n\geq 1$.

If $x\in[k,K]\subset(0,\infty)$, then by analyzing all possible locations of the interval $[k,K]$ and 1 we also have

$$1-\frac{\min\{1,x\}}{\max\{1,x\}}\geq 1-\frac{\min\{1,K\}}{\max\{1,k\}}\geq 0$$

and

$$0\leq\frac{\max\{1,x\}}{\min\{1,x\}}-1\leq\frac{\max\{1,K\}}{\min\{1,k\}}-1.$$

By using the inequalities (3.1) and (3.2) and the last equalities in (3.3) and (3.4) we have *the global bounds:*

$$\frac{1}{2n}\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^{2n} \leq \sum_{j=1}^{2n-1}\frac{(-1)^{j-1}}{j}(x-1)^{j}-\ln x \leq \frac{1}{2n}\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^{2n}, \tag{3.16}$$

and

$$\frac{1}{2n}\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^{2n} \leq \ln x-\sum_{j=1}^{2n-1}\frac{1}{j}\left(\frac{x-1}{x}\right)^{j} \leq \frac{1}{2n}\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^{2n}, \tag{3.17}$$

for any $x\in[k,K]$ and $n\geq 1$.

Observe that

$$\max_{x\in[m,M]}(x-1)^{2n}=\begin{cases}(1-k)^{2n} \text{ if } K<1,\\ \max\left\{(1-k)^{2n},(K-1)^{2n}\right\} \text{ if } k\leq 1\leq K,\\ (K-1)^{2n} \text{ if } 1<k\end{cases}$$

and

$$\min_{x\in[m,M]}(x-1)^{2n}=\begin{cases}(1-K)^{2n} \text{ if } K<1,\\ 0 \text{ if } k\leq 1\leq K,\\ (k-1)^{2n} \text{ if } 1<k\end{cases}$$

then by (3.12) and (3.13) we get the following *global bounds*:

$$\frac{1}{2n}\begin{cases}(1-K)^{2n} \text{ if } K<1,\\ 0 \text{ if } k\le 1\le K,\\ \left(\frac{k-1}{K}\right)^{2n} \text{ if } 1<k\end{cases} \le \sum_{j=1}^{2n-1}\frac{(-1)^{j-1}}{j}(x-1)^j-\ln x \le \frac{1}{2n}\begin{cases}\left(\frac{1-k}{k}\right)^{2n} \text{ if } K<1,\\ \max\left\{\left(\frac{1-k}{k}\right)^{2n},\left(\frac{K-1}{k}\right)^{2n}\right\} \text{ if } k\le 1\le K,\\ (K-1)^{2n} \text{ if } 1<k\end{cases} \tag{3.18}$$

and

$$\frac{1}{2n}\begin{cases}(1-K)^{2n} \text{ if } K<1,\\ 0 \text{ if } k\le 1\le K,\\ \left(\frac{k-1}{K}\right)^{2n} \text{ if } 1<k\end{cases} \le \ln x-\sum_{j=1}^{2n-1}\frac{1}{j}\left(\frac{x-1}{x}\right)^j \le \frac{1}{2n}\begin{cases}\left(\frac{1-k}{k}\right)^{2n} \text{ if } K<1,\\ \max\left\{\left(\frac{1-k}{k}\right)^{2n},\left(\frac{K-1}{k}\right)^{2n}\right\} \text{ if } k\le 1\le K,\\ (K-1)^{2n} \text{ if } 1<k\end{cases} \tag{3.19}$$

for any $x\in[k,K]$ and $n\ge 1$.

Some similar results may be obtained by utilising (3.6)-(3.11), however we do not present the details here.

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