Received 22/06/16

SEVERAL INEQUALITIES FOR NORM

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Abstract. The aim of this paper is to present new forms of several inequalities for operators in Hilbert spaces, using the classical norm given by the scalar product.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space $(\mathcal{H}, <.,.>)$ and $A,\ B$ two positive operators in $\mathcal{B}(\mathcal{H})$ the space of linear bounded operators on $\mathcal{H}$. As in [8], we take into account the following notations for the weighted operator arithmetic mean and the weighted operator geometric mean:

$$A\nabla_{\mu}B = (1-\mu)A + \mu B$$

and

$$A\sharp_{\mu}B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\mu}A^{\frac{1}{2}}$$

when $\mu \in [0,1]$. If $\mu = \frac{1}{2}$ then we write only $A\nabla B,\ A\sharp B$.

In addition, let Φ be a continuous function defined on the interval J of real numbers, B a selfadjoint operator on the Hilbert space $\mathcal{H}$ and A a positive invertible operator on $\mathcal{H}$. We assume, like in [8] that the spectrum $Sp\{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\} \subset J$ and then by using the continuous functional calculus, we can use the noncommutative *perspective* $\mathcal{P}_{\Phi}(B,A)$ given by

$$\mathcal{P}_{\Phi}(B,A) = A^{\frac{1}{2}}\Phi\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}$$

in [8].It is clear that

$$\mathcal{P}_{\Phi}(B,A) = A\sharp_{\mu}B$$

when $\Phi(t) = t^{\mu}$. We can also mention that the *relative operator entropy* $S(A|B)$ for positive invertible operators A and B was defined by

$$S(A|B) = A^{\frac{1}{2}}\log\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}$$

in the papers of Kamei and Fuji, see [12] and [11].

It is necessary to recall, as in [7], that for selfadjoint operators $A, B \in B(H)$ we write $A \leq B$ (or $B \geq A$) if $< Ax, x > \leq < Bx, x >$ for every vector $x \in H$. We will consider for beginning A as being a selfadjoint linear operator on a complex Hilbert space $(H; <.,.>)$. The *Gelfand map* establishes a $*$- isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, and the C^*- algebra $C^*(A)$ generated by A and the identity

Date: June 09, 2016.
2000 *Mathematics Subject Classification.* 26D15.
Key words and phrases. selfadjoint operators, inner product, norm.

operator 1_H on H as follows: For any $f, f \in C(Sp(A))$ and for any $\alpha, \beta \in \mathbf{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;
(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f) = \Phi(f^*)$;
(iii) $||\Phi(f)|| = ||f|| := \sup_{t\in Sp(A)} |f(t)|$;
(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$ for $t \in Sp(A.)$

Using this notation, as in [7] for example, we define

$$f(A) := \Phi(f) \quad \text{for all} \quad f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A. It is known that if A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a *positive operator* on H. In addition, if and f and g are real valued functions on $Sp(A)$ then the following property holds:

$$f(t) \geq g(t) \quad \text{for any} \quad t \in Sp(A) \quad \text{implies that} \quad f(A) \geq g(A) \tag{1}$$

in the operator order of $B(H)$.

2. **Variant with norm of some inequalities**

In the first part of this section we reformulate inequalities from [4] in the case of the classical norm given by the scalar product in Hilbert spaces and then we extend some inequalities given in [13] for positive invertible operators on $\mathcal{H}$.

Theorem 1. *If $n \in \mathbf{N}$, $n \geq 2$, $N_1, N_2, ..., N_n \in \mathcal{B}(\mathcal{H})$ are commuting gramian normal operators on Hilbert space $\mathcal{H}$ and $a_1, ..., a_n \in \mathbf{R}^*$ with $\sum_{k=1}^n a_k \neq 0$ then we have:*

$$\sum_{i=1}^n \frac{||N_i x||^2}{a_i} - \frac{||(\sum_{i=1}^n N_i x||^2}{\sum_{i=1}^n a_i} = \frac{1}{\sum_{i=1}^n a_i} \sum_{1\leq i<j\leq n} \frac{||(a_i N_j - a_j N_i)x||^2}{a_i a_j},$$

where $|N| = (N^*N)^{\frac{1}{2}}$ is the modulus of N.

Proof. We take into account the inequality (1) from [4] and then for every $x \in \mathcal{H}$, considering the inner product, we have:

$$< \sum_{i=1}^n \frac{|N_i|^2}{a_i} - \frac{|\sum_{i=1}^n N_i|^2}{\sum_{i=1}^n a_i} x, x >=< \frac{1}{\sum_{i=1}^n a_i} \sum_{1\leq i<j\leq n} \frac{|a_i N_j - a_j N_i|^2}{a_i a_j} x, x > .$$

By calculus we obtain:

$$\sum_{i=1}^n < \frac{|N_i|^2}{a_i} x, x > - < \frac{|\sum_{i=1}^n N_i|^2}{\sum_{i=1}^n a_i} x, x >= \frac{1}{\sum_{i=1}^n a_i} < \sum_{1\leq i<j\leq n} \frac{|a_i N_j - a_j N_i|^2}{a_i a_j} x, x >$$

which leads to desired inequality.

∎

Theorem 2. *If $n \in \mathbf{N}$, $n \geq 2$, N_1, $N_2, ..., N_n \in \mathcal{B}(\mathcal{H})$ are gramian normal operators on Hilbert space $\mathcal{H}$ which commute as pairs and $a_1, ..., a_n \in \mathbf{R}^*$ with $\sum_{k=1}^n a_k \neq 0$ then we have:*

$$\frac{||\sum_{i=1}^n N_i x||^2}{\sum_{i=1}^n a_i} + \sum_{k=1}^n \frac{||N_k x||^2}{a_k} - \frac{2}{\sum_{k=1}^n a_k} \sum_{k=1}^n ||N_k x||^2 =$$

$$= \frac{1}{\sum_{i=1}^n a_i} \sum_{1 \leq i < j \leq n} \frac{||(a_i N_j + a_j N_i) x||^2}{a_i a_j}.$$

Proof. Starting with the inequality (6) from Theorem 6, see [4], by similary calculus we obtain the desired inequality.

∎

Proposition 1. *Let $N_1, ..., N_n$ $(n \geq 2)$ be a sequence of gramian normal operators in $\mathcal{B}(\mathcal{H})$ commuting as pairs so that $\sum_{k=1}^n a_k N_k$ is hermitian and $a_1, ..., a_n \in \mathbf{R}^*$ with $\sum_{k=1}^n a_k > 0$. Then we have:*

$$\sum_{1 \leq i < j \leq n} \frac{||(a_i N_j + a_j N_i) x||^2}{a_1 a_j} \leq$$

$$\leq \frac{n-4}{2} \sum_{k=1}^n ||N_k x||^2 + \left(\sum_{k=1}^n a_k \right) \sum_{k=1}^n \frac{||N_k x||^2}{a_k} + \frac{n}{2} || \, | \sum_{k=1}^n N_k^2 |^{\frac{1}{2}} x||^2.$$

Proof. We use the same method as in Theorem 1 and take into account the inequality from Proposition 2, see [4].

∎

Proposition 2. *Let $N_1, ..., N_n$ $(n \geq 2)$ be a sequence of gramian normal operators in $\mathcal{B}(\mathcal{H})$ commuting as pairs so that $\sum_{k=1}^n a_k N_k$ is hermitian and $a_1, ..., a_n \in \mathbf{R}^*$ with $\sum_{k=1}^n a_k > 0$. Then the following inequality takes place:*

$$\sum_{1 \leq i < j \leq n} \frac{||(a_i N_j - a_j N_i) x||^2}{a_1 a_j} \geq$$

$$\geq \left(\sum_{i=1}^n a_i \right) \sum_{k=1}^n \frac{||N_k x||^2}{a_k} - \frac{n}{2} \sum_{k=1}^n ||N_i x||^2 - \frac{n}{2} || \, | \sum_{k=1}^n N_k^2 |^{\frac{1}{2}} x||^2.$$

Proof. We use inequality from Proposition 3, see [4] and the same reason as in Theorem 1.

∎

Consequence 1. *Let $\lambda \in (0,1)$ and A and B two positive invertible operators on a complex Hilbert space. Then we have:*

$$r\left[B+2A-2A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+I\right)^{\frac{1}{2}}A^{\frac{1}{2}}\right] \leq \lambda B + A - A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+I\right)^{\lambda}A^{\frac{1}{2}} \leq$$

$$\leq (1-r)\left[B+2A-2A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+I\right)^{\frac{1}{2}}A^{\frac{1}{2}}\right],$$

where $r = \min\{\lambda, 1-\lambda\}$.

Proof. Using inequality (2.11) from Theorem 2.2, see [13], by the continuous functional calculus as in [8], we obtain for an operator $C > 0$ that

$$r(C+2I-2\sqrt{C+I}) \leq \lambda C + I - (C+I)^{\lambda} \leq (1-r)(C+2I-2\sqrt{C+I}).$$

Now we take $C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ obtaining:

$$r\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+2I-2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+I)^{\frac{1}{2}}\right) \leq \lambda A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + I - \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+I\right)^{\lambda} \leq$$

$$\leq (1-r)\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+2I-2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+I)^{\frac{1}{2}}\right).$$

If we multiply both sides of previous inequality with $A^{\frac{1}{2}}$ we find the desired inequality.
∎

Remark 1. *Under cunditions of Consequence 1 we have:*

$$r\left[||B^{\frac{1}{2}}x||^2 + 2||A^{\frac{1}{2}}x||^2 - 2||(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+I)^{\frac{1}{4}}A^{\frac{1}{2}}x||^2\right] \leq$$

$$\leq \lambda||B^{\frac{1}{2}}x||^2 + ||A^{\frac{1}{2}}x||^2 - ||(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+I)^{\frac{\lambda}{2}}A^{\frac{1}{2}}x||^2 \leq$$

$$\leq (1-r)\left[||B^{\frac{1}{2}}x||^2 + 2||A^{\frac{1}{2}}x||^2 - 2||(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+I)^{\frac{1}{4}}A^{\frac{1}{2}}x||^2\right].$$

Proof. Using inequality from Consequence 1, the proof will be as in Theorem 1. ∎

Theorem 3. *Let $\lambda \in (0,1)$ and A and B two positive invertible operators on a complex Hilbert space with $B \geq A$. Then the following inequality takes place:*

$$2r\left(A\nabla_{\frac{1}{2}}B - A\sharp_{\frac{1}{2}}B\right) + A_1(\lambda)A^{\frac{1}{2}}\log^2\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}} \leq$$

$$\leq A\nabla_{\lambda}B - A\sharp_{\lambda}B \leq$$

$$\leq 2(1-r)\left(A\nabla_{\frac{1}{2}}B - A\sharp_{\frac{1}{2}}B\right) + B_1(\lambda)A^{\frac{1}{2}}\log^2\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}},$$

or

$$2r\left(A\nabla_{\frac{1}{2}}B - A\sharp_{\frac{1}{2}}B\right) + A_1(\lambda)\mathcal{P}_{\log^2}(B,A) \leq$$

$$\leq A\nabla_{\lambda}B - A\sharp_{\lambda}B \leq$$

$$\leq 2(1-r)\left(A\nabla_{\frac{1}{2}}B - A\sharp_{\frac{1}{2}}B\right) + B_1(\lambda)\mathcal{P}_{\log^2}(B,A),$$

where $r = \min\{\lambda, 1-\lambda\}$, $A_1(\lambda) = \frac{\lambda(1-\lambda)}{1} - \frac{r}{4}$ and $B_1(\lambda) = \frac{\lambda(1-\lambda)}{1} - \frac{1-r}{4}$.

Proof. In this case we consider the inequality (2.9) from Theorem 2.1, see [13], taking $b = 1$. Thus we get:

$$r(\sqrt{a}-1)^2 + A_1(\lambda)\log^2 a \le \lambda a + 1 - \lambda - a^\lambda \le$$
$$\le (1-r)(\sqrt{a}-1)^2 + B_1(\lambda)\log^2 a.$$

Now if $C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ by continuous functional calculus as in [8] we have:

$$r(\sqrt{C}-I)^2 + A_1(\lambda)\log^2 C \le \lambda C + (1-\lambda)I - C^\lambda \le$$
$$\le (1-r)(\sqrt{C}-I)^2 + B_1(\lambda)\log^2 C$$

or

$$r((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}} - I)^2 + A_1(\lambda)\log^2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \le$$
$$\le \lambda A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + (1-\lambda)I - (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\lambda \le$$
$$\le (1-r)((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}} - I)^2 + B_1(\lambda)\log^2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}).$$

We multiply both sides of previous inequality with $A^{\frac{1}{2}}$ and we get

$$rA^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}} - I)^2A^{\frac{1}{2}} + A_1(\lambda)A^{\frac{1}{2}}\log^2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \le$$
$$\le \lambda B + (1-\lambda)A - A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\lambda A^{\frac{1}{2}} \le$$
$$\le (1-r)A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}} - I)^2A^{\frac{1}{2}} + B_1(\lambda)A^{\frac{1}{2}}\log^2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

∎

Proposition 3. *Under condition of Theorem 3 the following inequality holds:*

$$r\left[||B^{\frac{1}{2}}x||^2 + ||A^{\frac{1}{2}}x||^2 - 2||(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{4}}A^{\frac{1}{2}}x||^2\right] + A_1(\lambda)||\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}x||^2 \le$$
$$\le \lambda||B^{\frac{1}{2}}x||^2 + (1-\lambda)||A^{\frac{1}{2}}x||^2 - ||(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{\lambda}{2}}A^{\frac{1}{2}}x||^2 \le$$
$$\le (1-r)\left[||B^{\frac{1}{2}}x||^2 + ||A^{\frac{1}{2}}x||^2 - 2||(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{4}}A^{\frac{1}{2}}x||^2\right] +$$
$$+B_1(\lambda)||\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}x||^2.$$

Proof. We will use inequality from Theorem 3 and the same method as in Theorem 1.

∎

Proposition 4. *Let $\lambda \in (0,1)$ and A and B two positive invertible operators on a complex Hilbert space with $B \le A$. Then the following inequality holds:*

$$2r\left(A\nabla_{\frac{1}{2}}B - A\sharp_{\frac{1}{2}}B\right) + A_1(\lambda)BA^{-\frac{1}{2}}\log^2\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}} \le$$
$$\le A\nabla_\lambda B - A\sharp_\lambda B \le$$
$$\le 2(1-r)\left(A\nabla_{\frac{1}{2}}B - A\sharp_{\frac{1}{2}}B\right) + B_1(\lambda)BA^{-\frac{1}{2}}\log^2\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}},$$

where $r = \min\{\lambda, 1-\lambda\}$, $A_1(\lambda) = \frac{\lambda(1-\lambda)}{1} - \frac{r}{4}$ *and* $B_1(\lambda) = \frac{\lambda(1-\lambda)}{1} - \frac{1-r}{4}$.

Proof. For the beginning we take $b = 1$ in inequality (3.18) from Application 3.2, see [13], and we will obtain the following inequality:

$$r(\sqrt{a}-1)^2 + A(\lambda)a\log^2 a \leq \lambda a + (1-\lambda) - a^{\lambda} \leq$$

$$\leq (1-r)(\sqrt{a}-1)^2 + B(\lambda)a\log^2 a,$$

when $0 < a \leq 1$, $\lambda \in (0,1)$ and r, $A(\lambda)$, $B(\lambda)$ are as in Application 3.2. By continuous functional calculus if $C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ as in [8] we get:

$$r(\sqrt{C}-I)^2 + A_1(\lambda)C\log^2 C \leq \lambda C + (1-\lambda)I - C^{\lambda} \leq$$

$$\leq (1-r)(\sqrt{C}-I)^2 + B_1(\lambda)a\log^2 C,$$

or

$$r\left((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-\frac{1}{2}} - I\right)^2 + A_1(\lambda)A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\log^2\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) \leq$$

$$\leq \lambda A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + (1-\lambda)I - \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\lambda} \leq$$

$$\leq (1-r)\left((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-\frac{1}{2}} - I\right)^2 + B_1(\lambda)A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\log^2\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right),$$

if in addition we denote by $A_1(\lambda)$ and $B_1(\lambda)$ the quantities from Application 3.2 in last inequality. By multiplying both sides of previous inequality with $A^{\frac{1}{2}}$ we find the following:

$$rA^{\frac{1}{2}}\left((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-\frac{1}{2}} - I\right)^2 A^{\frac{1}{2}} + A_1(\lambda)BA^{-\frac{1}{2}}\log^2\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}} \leq$$

$$\leq \lambda B + (1-\lambda)A - A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\lambda}A^{\frac{1}{2}} \leq$$

$$\leq (1-r)A^{\frac{1}{2}}\left((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-\frac{1}{2}} - I\right)^2 A^{\frac{1}{2}} + B_1(\lambda)BA^{-\frac{1}{2}}\log^2\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}},$$

which leads to the desired inequality.
∎

Consequence 2. *Under conditions of Proposition 4, the following inequalities take place:*

$$r[||B^{\frac{1}{2}}x||^2 + ||A^{\frac{1}{2}}x||^2 - 2||(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{4}}A^{\frac{1}{2}}x||^2] + A_1(\lambda)||[(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\log^2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})]^{\frac{1}{2}}A^{\frac{1}{2}}x||^2 \leq$$

$$\leq \lambda[||B^{\frac{1}{2}}x||^2 + (1-\lambda)||A^{\frac{1}{2}}x||^2 - ||(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{\lambda}{2}}A^{\frac{1}{2}}x||^2 \leq$$

$$\leq (1-r)[||B^{\frac{1}{2}}x||^2 + ||A^{\frac{1}{2}}x||^2 - 2||(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{4}}A^{\frac{1}{2}}x||^2] + B_1(\lambda)||[(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\log^2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})]^{\frac{1}{2}}A^{\frac{1}{2}}x||^2.$$

3. **Bohr's inequality for norms**

As a particular case of Bohr's generalized inequality given in [10] we can obtain:

Remark 2. *Let A, B be two operators in $\mathcal{B}(\mathcal{H})$ and $a,b,c,d \in \mathbf{R}$, $a,b,c,d \neq 0$ such that $ab+cd > 0$ and $\frac{a}{b} > 0$ or $ab+cd < 0$ and $\frac{a}{b} < 0$. Then*

$$||(aA-bB)x||^2 + ||(cA-dB)x||^2 \geq c^2(1-\frac{ad}{cb})||Ax||^2 + d^2(1-\frac{cb}{ad})||Bx||^2.$$

This inequality also takes place if $ab+cd > 0$ and $\frac{c}{d} > 0$ or $ab+cd < 0$ and $\frac{c}{d} < 0$. The reverse inequality holds if $ab+cd < 0$ and $\frac{a}{b} > 0$ or $ab+cd > 0$ and $\frac{c}{d} < 0$.

Remark 3. *(i) For any $r \geq 2$, A and B two positive invertible operators on a complex Hilbert space with $B \geq A$ and $u,v > 0$ with the property $uv(u+v)^{r-2} = 1$ we have:*

$$(1+\frac{u}{v})BA^{-1}B + (1+\frac{v}{u})A \geq (uB+vA)A^{-1}(uB+vA) + (B-A)A^{-1}(B-A)$$

and therefore also

$$(1+\frac{u}{v})||A^{-\frac{1}{2}}Bx||^2 + (1+\frac{v}{u})||A^{\frac{1}{2}}x||^2 \geq ||A^{-\frac{1}{2}}(uB+vA)x||^2 + ||A^{-\frac{1}{2}}(B-A)x||^2.$$

(ii0 For any $1 \leq r \leq 2$, A and B two positive invertible operators on a complex Hilbert space with $B \geq A$ and $u,v > 0$ with the property $uv(u+v)^{r-2} = 1$ the reverse inequality takes place.

Proof. The demonstration will be by the same method as in Theorem 3 taking into account the inequalities from Theorem 2, see [5]. ∎

Consequence 3. *(i) Let (A_i) in $\mathcal{B}(\mathcal{H})$ with $A_i^* A_j = 0$, $1 \leq i < j \leq n$ and α_{ik}, $p_i \in \mathbf{R}$ for $i = 1,...,n$ and $k = 1,...,m$. Define $X = (x_{ij})$ where $x_{ij} = \sum_{k=1}^m \alpha_{ik}^4 - p_i$ if $i = j$ and $x_{ij} = \sum_{k=1}^m \alpha_{ik}^2 \alpha_{jk}^2$ if $i \neq j$. If $X \geq 0$ then*

$$\sum_{k=1}^m || \, |\sum_{i=1}^n a_{ik} A_i|^2 y||^2 \geq \sum_{i=1}^n |p_i| \cdot || \, |A_i|^2 y||^2.$$

(i) If $\Lambda(a) + \Lambda(b) \leq \Lambda(c)$ for $a,b,c \in \mathbf{R}^n$ as in [10] then

$$\left|\left|\left(\sum_{i=1}^n a_i A_i\right) x\right|\right|^2 + \left|\left|\left(\sum_{i=1}^n b_i A_i\right) x\right|\right|^2 \leq \sum_{i=1}^n c_i ||A_i x||^2$$

for arbitrary n-tuple (A_i) in $\mathcal{B}(\mathcal{H})$. If $\Lambda(a) + \Lambda(b) \geq \Lambda(c)$ for $a,b,c \in \mathbf{R}^n$ then the reverse inequality takes place.

Proof. (i) We use Proposition 2 from [5] and the same method as in Theorem 1. (ii) We use Theorem 3.1 from [10] and the same method as in Theorem 1.
∎

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