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SOME GENERALIZED INEQUALITIES OF HERMITE-HADAMARD TYPE FOR STRONGLY $s-$CONVEX FUNCTIONS

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Abstract. In this paper, some new generalized results related to the left-hand and the right-hand of the Hermite-Hadamard inequalities for the class of funcitons whose derivatives are strongly $s-$convex functions in the second sense are established. Some previous results are also recaptured as a special case.

1. Introduction

In this section, we firstly list several definitions and some known results.

Definition 1. *A function $f: I \to \mathbb{R}$, $\varnothing \neq I \subset \mathbb{R}$,is said to be convex on I if the inequality*

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

holds for all $x.y \in I$ and $t \in [0,1]$.

Many inequalities have been established for convex functions but the most famous inequality is the Hermite-Hadamards inequality, due to its rich geometrical signicance and applications([12, p.137], [4]). These inequalities state that if $f: I \subseteq R \to R$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamards inequality may be regarded as a renement of the concept of convexity and it follows easily from Jensens inequality. Hadamards inequality for convex functions has received renewed attention in recent years and a remarkable variety of renements and generalizations have been found (see, for example, [2],[5],[11],[16],[18]) and the references cited therein.

In [6], Hudzik and Maligranda considered, among others, the class of functions which are s- convex in the second sense. This class is defined in the following way: a funciton $f: [0,\infty] \to \mathbb{R}$ is said to be s-convex in the second sense if

$$f(tx+(1-t)y) \leq t^s f(x)+(1-t)^s f(y)$$

holds for $x, y \in [0,\infty]$, $t \in [0,1]$ and for some fixed $s \in [0,1]$.This class of s-convex functions in the second sense is usually by K_S^2.

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It can be easily see that for $s=1$ s-convexity reduces to the ordinary convexity of functions defined on $[0,\infty)$.

Definition 2. [13] *A function $f: I\to\mathbb{R}$ is called strongly s-convex with modulus c if*

$$f(tx+(1-t)y)\leq t^s f(x)+(1-t)^s f(y)-ct(1-t)(b-a)^2$$

In [1], Angulo et al. proved the following Hermite-Hadamard type inequality for strongly $h-$convex function:

Theorem 1. *Let $h:(0,1)\to(0,\infty)$ be a given function. If a function $f: I\subset\mathbb{R}\to\mathbb{R}$ is Lebesgue integrable and strongly $h-$convex with modulus $c>0$, then*

$$\begin{aligned}(1.2)\qquad & \frac{1}{h\left(\frac{1}{2}\right)}\left[f\left(\frac{a+b}{2}\right)+\frac{c}{12}(b-a)^2\right] \\ \leq\; & \frac{1}{b-a}\int_a^b f(x)\,dx\leq (f(a)+f(b))\int_0^1 h(t)dt-\frac{c}{6}(b-a)^2\end{aligned}$$

for all $a,b\in I$, $a<b$.

Corollary 1. *Suppose that $f:[0,\infty]\to\mathbb{R}$ is a strongly $s-$convex function in the second sense with modulus $c>0$, where $s\in(0,1)$ (i.e $h(t)=t^s$ in (1.2)), then following inequalities hold;*

(1.3)
$$2^{s-1}\left[f\left(\frac{a+b}{2}\right)+\frac{c}{12}(b-a)^2\right]\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{s+1}-\frac{c}{6}(b-a)^2.$$

For more information and recent developments on inequalities for srongly convex function, please refer to ([1],[3],[8],[9],[10],[15],[17],[19],[20]).

To prove our main results, we consider the following Lemmas given by Sarikaya et al. in [14] and Kiriş and Sarikaya in [7], respectively:

Lemma 1. *Let $f: I\to\mathbb{R}$ be a differantiable mapping on I^0 where $a,b\in I$ with $a<b$. If $f'\in L[a,b]$, then we have*

$$\begin{aligned}(1.4)\qquad & \int_a^b p(x)f'(x)\,dx \\ =\; & (m-n)f\left(\frac{a+b}{2}\right)+(b-m)f(b)+(n-a)f(a)-\int_a^b f(x)\,dx\end{aligned}$$

where

$$p(x)=\left\{\begin{array}{ll} x-n, & x\in\left[a,\frac{a+b}{2}\right] \\ x-m, & x\in\left(\frac{a+b}{2},b\right]\end{array}\right\}$$

for $n\in\left[a,\frac{a+b}{2}\right]$ and $m\in\left[\frac{a+b}{2},b\right]$.

Lemma 2. *Let $f : I \to \mathbb{R}$ be a differantiable mapping on I^0 where $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then we have*

$$\frac{n}{b-a}\int_a^{\frac{a+b}{2}} f(x)\,dx + \frac{m}{b-a}\int_{\frac{a+b}{2}}^b f(x)\,dx - \frac{n+m}{2} f\left(\frac{a+b}{2}\right) \tag{1.5}$$
$$= (b-a)\left[\int_0^{\frac{1}{2}} mtf'((ta)+(1-t)b)\,dt + \int_{\frac{1}{2}}^1 n(1-t)f'((ta)+(1-t)b)\,dt\right]$$

where $n, m > 0$.

The aim of the paper is to establish some new generalized Hermite-Hadamard inequalities for function whose derivatives in absolute volue are strongly $s-$convex.

2. Main results

Firsty, we will give some calculated integrals which used our main results:

$$\int_a^{\frac{a+b}{2}} |x-n|(b-x)^s\,dx \tag{2.1}$$
$$= \frac{2(b-n)^{s+2}}{(s+1)(s+2)} - \frac{(b-n)(b-a)^{s+1}\left[2^{s+1}+1\right]}{2^{s+1}(s+1)} + \frac{(b-a)^{s+2}\left[2^{s+2}+1\right]}{2^{s+2}(s+2)},$$

$$\int_{\frac{a+b}{2}}^b |x-m|(b-x)^s = \frac{2(b-m)^{s+2}}{(s+1)(s+2)} + \frac{(b-a)^{s+2}}{2^{s+2}(s+2)} - \frac{(b-m)(b-a)^{s+1}}{2^{s+1}(s+1)}, \tag{2.2}$$

$$\int_a^{\frac{a+b}{2}} |x-n|(x-a)^s\,dx = \frac{2(n-a)^{s+2}}{(s+1)(s+2)} + \frac{(b-a)^{s+2}}{2^{s+2}(s+2)} - \frac{(n-a)(b-a)^{s+1}}{2^{s+1}(s+1)}, \tag{2.3}$$

$$\int_{\frac{a+b}{2}}^b |x-m|(x-a)^s\,dx \tag{2.4}$$
$$= \frac{2(m-a)^{s+2}}{(s+1)(s+2)} - \frac{(m-a)(b-a)^{s+1}\left[2^{s+1}+1\right]}{2^{s+1}(s+1)} + \frac{(b-a)^{s+2}\left[2^{s+2}+1\right]}{2^{s+2}(s+2)},$$

$$\int_a^{\frac{a+b}{2}} |x-n|(b-x)(x-a)\,dx \tag{2.5}$$
$$= \frac{(b-a)(n-a)^3}{3} - \frac{(n-a)^4}{6} + \frac{5(b-a)^4}{192} - \frac{(n-a)(b-a)^3}{12}$$

and

$$\int_{\frac{a+b}{2}}^{b} |x-m|\,(b-x)\,(x-a)\,dx = \frac{(b-a)(b-m)^3}{3} - \frac{(b-m)^4}{6} + \frac{5(b-a)^4}{192} - \frac{(b-m)(b-a)^3}{12}. \tag{2.6}$$

Theorem 2. *Let $f : I \to \mathbb{R}$ be a differantiable mapping on I^0 where $a, b \in I$ with $a < b$. If $|f'|$ is strongly s-convex on $[a, b]$, for some $s \in (0, 1]$ with modulus $c > 0$, then following inequality holds :*

$$\begin{aligned} &\left|(m-n)\,f\left(\frac{a+b}{2}\right) + (b-m)f(b) + (n-a)f(a) - \int_a^b f(x)\,dx\right| \\ &\leq \frac{|f'(a)|}{(b-a)^s}\left[\frac{2\left[(b-m)^{s+2}+(b-n)^{s+2}\right]}{(s+1)(s+2)}\right. \\ &\left.-\frac{(2b-n-m)(b-a)^{s+1}}{2^{s+1}(s+1)} - \frac{(b-n)(b-a)^{s+1}}{(s+1)} + \frac{(b-a)^{s+2}\left[2^{s+1}+1\right]}{2^{s+1}(s+2)}\right] \\ &+\frac{|f'(b)|}{(b-a)^s}\left[\frac{2\left[(b-m)^{s+2}+(b-n)^{s+2}\right]}{(s+1)(s+2)}\right. \\ &\left.-\frac{(m+n-2a)(b-a)^{s+1}}{2^{s+1}(s+1)} - \frac{(m-a)(b-a)^{s+1}}{(s+1)} + \frac{(b-a)^{s+2}\left[2^{s+1}+1\right]}{2^{s+1}(s+2)}\right] \\ &-c\left[(b-a)\frac{(n-a)^3+(b-m)^3}{3} - \frac{(n-a)^4+(b-m)^4}{6}\right. \\ &\left.+\frac{5(b-a)^4}{96} - \frac{((b-a)-(m-n))(b-a)^3}{12}\right] \end{aligned} \tag{2.7}$$

for $n \in \left[a, \frac{a+b}{2}\right]$ and $m \in \left[\frac{a+b}{2}, b\right]$.

Proof. Taking madulus in Lemma 1 and using the strongly $s-$convexity of $|f'|$, we have

$$\begin{aligned} &\left|(m-n)\,f\left(\frac{a+b}{2}\right) + (b-m)f(b) + (n-a)f(a) - \int_a^b f(x)\,dx\right| \\ &\leq \int_a^{\frac{a+b}{2}} |x-n|\,|f'(x)|\,dx + \int_{\frac{a+b}{2}}^{b} |x-m|\,|f'(x)|\,dx \end{aligned} \tag{2.8}$$

$$
\begin{aligned}
&= \int_a^{\frac{a+b}{2}} |x-n|\, f\left(\frac{b-x}{b-a}a+\frac{x-a}{b-b}b\right)dx \\
&\quad + \int_{\frac{a+b}{2}}^{b} |x-m|\, f\left(\frac{b-x}{b-a}a+\frac{x-a}{b-b}b\right)dx \\
&\leq \int_a^{\frac{a+b}{2}} |x-n| \left[\left(\frac{b-x}{b-a}\right)^s |f'(a)| + \left(\frac{x-a}{b-a}\right)^s |f'(b)| - c\,(b-x)\,(x-a)\right] dx \\
&\quad + \int_{\frac{a+b}{2}}^{b} |x-m| \left[\left(\frac{b-x}{b-a}\right)^s |f'(a)| + \left(\frac{x-a}{b-a}\right)^s |f'(b)| - c\,(b-x)\,(x-a)\right] dx \\
&= \frac{|f'(a)|}{(b-a)^s}\left[\int_a^{\frac{a+b}{2}} |x-n|\,(b-x)^s\,dx + \int_{\frac{a+b}{2}}^{b} |x-m|\,(b-x)^s\,dx\right] \\
&\quad + \frac{|f'(b)|}{(b-a)^s}\left[\int_a^{\frac{a+b}{2}} |x-n|\,(x-a)^s\,dx + \int_{\frac{a+b}{2}}^{b} |x-m|\,(x-a)^s\,dx\right] \\
&\quad - c\left[\int_a^{\frac{a+b}{2}} |x-n|\,(b-x)\,(x-a)\,dx + \int_{\frac{a+b}{2}}^{b} |x-m|\,(b-x)\,(x-a)\,dx\right].
\end{aligned}
$$

If we subsitute the equalities (2.1)-(2.6) in (2.8), then we obtain required result (2.7). □

Remark 1. *If we choose* $m=b$, $n=a$ *in Theorem 2, then we have*

$$
\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq (b-a)\,\frac{2^{s+1}-1}{2^s\,(s+1)\,(s+2)}\left[\frac{|f'(a)|+|f'(b)|}{2}\right] - \frac{5c}{96}(b-a)^3
$$

Remark 2. *If we choose* $m=n=\frac{a+b}{2}$ *in Theorem 2, then we have*

$$
\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq (b-a)\,\frac{s2^s+1}{2^s\,(s+1)\,(s+2)}\left[\frac{|f'(a)|+|f'(b)|}{2}\right] - \frac{c}{32}(b-a)
$$

Remark 3. *If we choose* $m=\frac{a+5b}{6}$, $n=\frac{5a+b}{6}$ *in Theorem 2 , then we have*

$$
\begin{aligned}
&\left| \frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx \right| \\
&\leq (b-a)\,\frac{2\times 5^{s+2}-(4-s)6^{s+1}-2\times 3^{s+12}+2}{6^{s+2}\,(s+1)\,(s+2)}\left[|f'(a)|+|f'(b)|\right] - \frac{67}{2\times 6^4}c(b-a)^3
\end{aligned}
$$

For $c = 0$ it reduces to the Hermite–Hadamard-type inequalities for sconvex functions proved by Sarikaya et al. in [21].

Theorem 3. *Let $f : I \to \mathbb{R}$ be a differantiable mapping on I^0 where $a, b \in I$ with $a < b$. If $|f'|^q$ is strongly $s-$convex on $[a, b]$ for some $s \in (0, 1]$ with modulus $c > 0$, then following inequality holds :*

$$
\begin{aligned}
(2.9)\quad & \left|(m-n) f\left(\frac{a+b}{2}\right)+(b-m)f(b)+(n-a)f(a)-\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{(b-a)^{\frac{1}{q}}}{(p+1)^{\frac{1}{p}}}\left\{\left[(n-a)^{p+1}+\left(\frac{a+b}{2}-n\right)^{p+1}\right]^{\frac{1}{p}}\right. \\
& \times\left(\frac{1}{2^{s+1}(s+1)}\left[\left[2^{s+1}-1\right]|f'(a)|^q+|f'(b)|^q\right]-c\frac{(b-a)^2}{12}\right)^{\frac{1}{q}} \\
& +\left[(b-m)^{p+1}+\left(m-\frac{a+b}{2}\right)^{p+1}\right]^{\frac{1}{p}} \\
& \left.\times\left(\frac{1}{2^{s+1}(s+1)}\left[|f'(a)|^q+\left[2^{s+1}-1\right]|f'(b)|^q\right]-c\frac{(b-a)^2}{12}\right)^{\frac{1}{q}}\right\}
\end{aligned}
$$

for $n \in \left[a, \frac{a+b}{2}\right]$ and $m \in \left[\frac{a+b}{2}, b\right]$ where $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 1 and by using the Hölder inequality , then we have

$$
\begin{aligned}
(2.10)\quad & \left|(m-n) f\left(\frac{a+b}{2}\right)+(b-m)f(b)+(n-a)f(a)-\int_a^b f(x)\,dx\right| \\
=\ & \int_a^{\frac{a+b}{2}} |x-n|\,|f'(x)|\,dx+\int_{\frac{a+b}{2}}^b |x-m|\,|f'(x)|\,dx \\
\leq\ & \left(\int_a^{\frac{a+b}{2}} |x-n|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}} |f'(x)|^q\,dx\right)^{\frac{1}{q}} \\
& +\left(\int_{\frac{a+b}{2}}^b |x-m|^p\,dx\right)^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^b |f'(x)|^q\,dx\right)^{\frac{1}{q}}
\end{aligned}
$$

$$= \frac{1}{(p+1)^{\frac{1}{p}}}\left[(n-a)^{p+1}+\left(\frac{a+b}{2}-n\right)^{p+1}\right]^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}}\left|f'(x)\right|^q dx\right)^{\frac{1}{q}}$$

$$+\frac{1}{(p+1)^{\frac{1}{p}}}\left[(b-m)^{p+1}+\left(m-\frac{a+b}{2}\right)^{p+1}\right]^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^{b}\left|f'(x)\right|^q dx\right)^{\frac{1}{q}}$$

Using the strongly $s-$convexity of $|f'|$, we have

$$\left|(m-n)\,f\left(\frac{a+b}{2}\right)+(b-m)f(b)+(n-a)f(a)-\int_a^b f(x)\,dx\right| \tag{2.11}$$

$$\leq \frac{1}{(p+1)^{\frac{1}{p}}}\left\{\left[(n-a)^{p+1}+\left(\frac{a+b}{2}-n\right)^{p+1}\right]^{\frac{1}{p}}\right.$$

$$\times\left(\int_a^{\frac{a+b}{2}}\left[\left(\frac{b-x}{b-a}\right)^s\left|f'(a)\right|^q+\left(\frac{x-a}{b-a}\right)^s\left|f'(b)\right|^q-c\,(b-x)\,(x-a)\right]dx\right)^{\frac{1}{q}}$$

$$+\left[(b-m)^{p+1}+\left(m-\frac{a+b}{2}\right)^{p+1}\right]^{\frac{1}{p}}$$

$$\left.\times\left(\int_{\frac{a+b}{2}}^{b}\left[\left(\frac{b-x}{b-a}\right)^s\left|f'(a)\right|^q+\left(\frac{x-a}{b-a}\right)^s\left|f'(b)\right|^q-c\,(b-x)\,(x-a)\right]dx\right)^{\frac{1}{q}}\right\}$$

By a simple calculation , we have

$$\int_a^{\frac{a+b}{2}}(b-x)^s\,dx=\int_{\frac{a+b}{2}}^{b}(x-a)^s\,dx=\frac{(b-a)^{s+1}\left[2^{s+1}-1\right]}{2^{s+1}\,(s+1)} \tag{2.12}$$

$$\int_a^{\frac{a+b}{2}}(x-a)^s\,dx=\int_{\frac{a+b}{2}}^{b}(b-x)^s\,dx=\frac{(b-a)^{s+1}}{2^{s+1}\,(s+1)} \tag{2.13}$$

$$\int_a^{\frac{a+b}{2}}(b-x)\,(x-a)\,dx=\int_{\frac{a+b}{2}}^{b}(b-x)\,(x-a)\,dx=\frac{(b-a)^3}{12} \tag{2.14}$$

If we subsitute the equalities 2.12-2.14 in 2.11, then we obtain result 2.9. □

Corollary 2. *If we choose $m = b$ and $n = a$ in Theorem 3, then we have*

$$\left| f\left(\frac{a+b}{2}\right) - \int_a^b f(x)\,dx \right| \tag{2.15}$$
$$\leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left\{ \left(\frac{\left[2^{s+1}-1\right]|f'(a)|^q + |f'(b)|^q}{2^s(s+1)} - c\frac{(b-a)^2}{6} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + \left[2^{s+1}-1\right]|f'(b)|^q}{2^s(s+1)} - c\frac{(b-a)^2}{6} \right)^{\frac{1}{q}} \right\}.$$

Remark 4. *Choosing $s = 1$ in Corollary 2, we obtain the inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \int_a^b f(x)\,dx \right|$$
$$\leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left\{ \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} - c\frac{(b-a)^2}{12} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} - c\frac{(b-a)^2}{12} \right)^{\frac{1}{q}} \right\}.$$

Corollary 3. *If we choose $m = n = \frac{a+b}{2}$ in Theorem 3, then we have*

$$\left| \frac{f(a)+f(b)}{2} - \int_a^b f(x)\,dx \right| \tag{2.16}$$
$$\leq \frac{b-a}{2\times 2^{\frac{1}{p}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left\{ \left(\frac{\left[2^{s+1}-1\right]|f'(a)|^q + |f'(b)|^q}{2^{s+1}(s+1)} - c\frac{(b-a)^2}{12} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + \left[2^{s+1}-1\right]|f'(b)|^q}{2^{s+1}(s+1)} - c\frac{(b-a)^2}{12} \right)^{\frac{1}{q}} \right\}.$$

Remark 5. *Choosing $s = 1$ in Corollary 3, we obtain the inequality*

$$\left| \frac{f(a)+f(b)}{2} - \int_a^b f(x)\,dx \right|$$
$$\leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left\{ \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} - c\frac{(b-a)^2}{12} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} - c\frac{(b-a)^2}{12} \right)^{\frac{1}{q}} \right\}.$$

Corollary 4. *If we choose* $m=\frac{a+5b}{6}$ *and* $n=\frac{5a+b}{6}$ *in Theorem 3 , then we have*

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\int\limits_{a}^{b}f\left(x\right)dx\right| \\ \leq\ & \frac{b-a}{6}\left(\frac{2^{p+1}+1}{6(p+1)}\right)^{\frac{1}{p}} \\ & \times\left\{\left(\frac{\left[2^{s+1}-1\right]\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{2^{s+1}\left(s+1\right)}-c\frac{(b-a)^{2}}{12}\right)^{\frac{1}{q}}\right. \\ & \left.+\left(\frac{\left|f'(a)\right|^{q}+\left[2^{s+1}-1\right]\left|f'(b)\right|^{q}}{2^{s+1}\left(s+1\right)}-c\frac{(b-a)^{2}}{12}\right)^{\frac{1}{q}}\right\}. \end{aligned} \tag{2.17}$$

Remark 6. *Choosing* $s=1$ *in Corollary 3, we obtain the inequality*

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\int\limits_{a}^{b}f\left(x\right)dx\right| \\ \leq\ & \frac{b-a}{12}\left(\frac{2^{p+1}+1}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{3\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{4}-c\frac{(b-a)^{2}}{12}\right)^{\frac{1}{q}}\right. \\ & \left.+\left(\frac{\left|f'(a)\right|^{q}+3\left|f'(b)\right|^{q}}{4}-c\frac{(b-a)^{2}}{12}\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Theorem 4. *Let* $f:I\rightarrow\mathbb{R}$ *be a differantiable mapping on* I^{0} *where* $a,b\in I$ *with* $a<b$. *If* $|f'|$ *is strongly* $s-$*convex on* $[a,b]$ *for some* $s\in(0,1]$ *with modulus* $c>0$ *, then following inequality holds :*

$$\begin{aligned} &\left|\frac{n}{b-a}\int_{a}^{\frac{a+b}{2}}f\left(x\right)dx+\frac{m}{b-a}\int_{\frac{a+b}{2}}^{b}f\left(x\right)dx-\frac{n+m}{2}f\left(\frac{a+b}{2}\right)\right| \\ \leq\ & (b-a)\left\{\left[\frac{m}{2^{s+2}\left(s+2\right)}+n\frac{2^{s+2}-s-3}{2^{s+2}\left(s+1\right)\left(s+2\right)}\right]\left|f'\left(a\right)\right|\right. \\ & \left.+\left[\frac{m}{2^{s+2}\left(s+2\right)}+n\frac{2^{s+2}-s-3}{2^{s+2}\left(s+1\right)\left(s+2\right)}\right]\left|f'\left(b\right)\right|-\frac{5c\left(m+n\right)}{192}\left(b-a\right)^{2}\right\} \end{aligned} \tag{2.18}$$

where $n\in\left[a,\frac{a+b}{2}\right],m\in\left[\frac{a+b}{2},b\right]$.

Proof. From Lemma 2 and using strongly s-convex of $|f'|$, we have

$$
\begin{aligned}
&\left|\frac{n}{b-a}\int_a^{\frac{a+b}{2}} f(x)\,dx+\frac{m}{b-a}\int_{\frac{a+b}{2}}^{b} f(x)\,dx-\frac{n+m}{2}f\left(\frac{a+b}{2}\right)\right| \qquad (2.19)\\
\leq\;& (b-a)\left\{\int_0^{\frac{1}{2}} mt\left|f'\left((ta)+(1-t)b\right)\right|dt\right.\\
&\left.+\int_{\frac{1}{2}}^{1} n(1-t)\left|f'\left((ta)+(1-t)b\right)\right|dt\right\}\\
\leq\;& (b-a)\left\{\int_0^{\frac{1}{2}} mt\left[t^s\left|f'(a)\right|+(1-t)^s\left|f'(b)\right|-ct(1-t)(b-a)^2\right]\right.\\
&\left.+\int_{\frac{1}{2}}^{1} n(1-t)\left[t^s\left|f'(a)\right|+(1-t)^s\left|f'(b)\right|-ct(1-t)(b-a)^2\right]\right\}\\
=\;& (b-a)\left\{m\left|f'(a)\right|\int_0^{\frac{1}{2}} t^{s+1}dt+m\left|f'(b)\right|\int_0^{\frac{1}{2}} t(1-t)^s\,dt-mc(b-a)^2\int_0^{\frac{1}{2}} t^2(1-t)\,dt\right.\\
&\left.+n\left|f'(a)\right|\int_{\frac{1}{2}}^{1}(1-t)t^s dt+n\left|f'(b)\right|\int_{\frac{1}{2}}^{1}(1-t)^{s+1}dt-nc(b-a)^2\int_{\frac{1}{2}}^{1} t(1-t)^2\,dt\right\}
\end{aligned}
$$

Using the facts that

$$\int_0^{\frac{1}{2}} t^{s+1}dt=\int_{\frac{1}{2}}^{1}(1-t)^{s+1}dt=\frac{1}{2^{s+2}(s+2)},$$

$$\int_0^{\frac{1}{2}} t(1-t)^s\,dt=\int_{\frac{1}{2}}^{1}(1-t)t^s dt=\frac{2^{s+2}-s-3}{2^{s+2}(s+1)(s+2)},$$

and

$$\int_0^{\frac{1}{2}} t^2(1-t)\,dt=\int_{\frac{1}{2}}^{1} t(1-t)^2\,dt=\frac{5}{192},$$

one can obtain required result. □

Remark 7. *If we choose $m=n$ in Theorem 4, then Theorem 4 reduces Remark 1.*

Corollary 5. *Under assumption of Theorem 4 with $s=1$, we have*

$$
\begin{aligned}
&\left|\frac{n}{b-a}\int_a^{\frac{a+b}{2}} f(x)\,dx+\frac{m}{b-a}\int_{\frac{a+b}{2}}^{b} f(x)\,dx-\frac{n+m}{2}f\left(\frac{a+b}{2}\right)\right| \qquad (2.20)\\
\leq\;& \frac{b-a}{24}\left[(m+2n)\left|f'(a)\right|+(2m+n)\left|f'(b)\right|\right]-\frac{5c(m+n)}{192}(b-a)^3.
\end{aligned}
$$

Theorem 5. *Let $f:I\to\mathbb{R}$ be a differantiable mapping on I^0 where $a,b\in I$ with $a<b$. If $|f'|^q$ is strongly $s-$convex on $[a,b]$ for some $s\in(0,1]$ with modulus $c>0$*

, then following inequality holds :

$$
\begin{aligned}
(2.21)\qquad & \left|\frac{n}{b-a}\int_a^{\frac{a+b}{2}} f(x)\,dx+\frac{m}{b-a}\int_{\frac{a+b}{2}}^b f(x)\,dx-\frac{n+m}{2}f\left(\frac{a+b}{2}\right)\right| \\
\leq\quad & \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{m\left(\frac{|f'(a)|^q+|f'(b)|^q\left[2^{s+1}-1\right]}{2^s(s+1)}-\frac{c(b-a)^2}{6}\right)^{\frac{1}{q}}\right. \\
& \left.+n\left(\frac{|f'(a)|^q\left[2^{s+1}-1\right]+|f'(b)|^q}{2^s(s+1)}-\frac{c(b-a)^2}{6}\right)^{\frac{1}{q}}\right\}
\end{aligned}
$$

where $n\in\left[a,\frac{a+b}{2}\right],m\in\left[\frac{a+b}{2},b\right]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2 and using Hölder Inequality, then we have

$$
\begin{aligned}
& \left|\frac{n}{b-a}\int_a^{\frac{a+b}{2}} f(x)\,dx+\frac{m}{b-a}\int_{\frac{a+b}{2}}^b f(x)\,dx-\frac{n+m}{2}f\left(\frac{a+b}{2}\right)\right| \\
\leq\quad & (b-a)\left\{\int_0^{\frac{1}{2}} mt\,|f'(ta+(1-t)b)|\,dt+\int_{\frac{1}{2}}^1 n(1-t)\,|f'(ta+(1-t)b)|\,dt\right\} \\
\leq\quad & (b-a)\left\{\left(\int_0^{\frac{1}{2}}(mt)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\right. \\
& \left.+\left(\int_{\frac{1}{2}}^1 n^p(1-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1|f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\right\}
\end{aligned}
$$

Using srtongly s-convex of $|f'|^q$;

$$
\begin{aligned}
& \left|\frac{n}{b-a}\int_a^{\frac{a+b}{2}} f(x)\,dx+\frac{m}{b-a}\int_{\frac{a+b}{2}}^b f(x)\,dx-\frac{n+m}{2}f\left(\frac{a+b}{2}\right)\right| \\
\leq\quad & (b-a)\left\{\left(\int_0^{\frac{1}{2}}(mt)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left[t^s|f'(a)|^q+(1-t)^s|f'(b)|^q-ct(1-t)(b-a)^2\right]dt\right)^{\frac{1}{q}}\right. \\
& \left.+\left(\int_{\frac{1}{2}}^1\left[n(1-t)^p\,dt\right]\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left[t^s|f'(a)|^q+(1-t)^s|f'(b)|^q-ct(1-t)(b-a)^2\right]dt\right)^{\frac{1}{q}}\right\}
\end{aligned}
$$

By simple computation, the required result (2.21) can be easily established. □

Remark 8. *If we choose $m=n$ in Theorem 5, then Theorem 5 reduces Remark 2.*

Corollary 6. *Under assumption of Theorem 4 with $s=1$, we have*

$$\left|\frac{n}{b-a}\int_{a}^{\frac{a+b}{2}} f(x)\,dx+\frac{m}{b-a}\int_{\frac{a+b}{2}}^{b} f(x)\,dx-\frac{n+m}{2}f\left(\frac{a+b}{2}\right)\right| \tag{2.22}$$
$$\leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{m\left(\frac{|f'(a)|^{q}+3|f'(b)|^{q}}{4}-\frac{c(b-a)^{2}}{6}\right)^{\frac{1}{q}}\right.$$
$$\left.+n\left(\frac{3|f'(a)|^{q}+|f'(b)|^{q}}{4}-\frac{c(b-a)^{2}}{6}\right)^{\frac{1}{q}}\right\}.$$

References

[1] H. Angulo, J. Gimenez, A. M. Moros, and K. Nikodem, *On strongly $h-$convex function*, Ann. Funct. Anal. 2(2), 2011, 85–91

[2] M. K. Bakula and J. Pečarić, *Note on some Hadamard-type inequalities,* Journal of Inequalities in Pure and Applied Mathematics, vol. 5, no. 3, article 74, 2004.

[3] M. V. Cortez, Relative strongly $h-$convex functions and integral inequalities, Appl. Math. Inf. Sci. Lett. 4, No. 2, 1-7 (2016).

[4] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[5] S. S. Dragomir and R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. lett., 11(5) (1998), 91-95.

[6] H. Hudzik, L. Maligranda, *Some remarks on $s-$convex functions.* Aequ. Math. 48, 100-111 (1994).

[7] M. E. Kiris and M. Z. Sarikaya, *Some generalized inequalities for Hermite-Hadamard's integral inequalities and applications*, submited (2015).

[8] J. Makó and A. Házy, *On strongly convex functions.* Carpathian Journal of Mathematics, 32 (1). 87-95.

[9] N. Merentes, K. Nikodem, *Remarks on strongly convex functions*, Aequationes Math. 80 (2010) 193–199.

[10] K. Nikodem, Z. Pales, *Characterizations of inner product spaces be strongly convex functions*, Banach J. Math. Anal. 5 (2011) 83–87.

[11] M. E. Özdemir, Ç. Yıldız, A. O. Akdemir and E. Set, *On some inequalities for $s-$convex functions and applications,* Journal of Inequalities and Applications 2013, 2013:333.

[12] J.E. Pečarić, F. Proschan and Y.L. Tong, *Convex functions, partial orderings and statistical applications*, Academic Press, Boston, 1992.

[13] B.T. Polyak, *Existence theorems and convergence of minimizing sequences in extremum problems with restictions*, Soviet Math. Dokl. 7 (1966), 72–75.

[14] M. Z. Sarikaya, T. Tunc and M. K. Yildiz, *Some generalized integral inequalities for convex functions and applications*, AIP Conference Proceedings, 1726, 020047 (2016); doi: 10.1063/1.4945873.

[15] M. Z. Sarikaya, *On strongly φ_h-convex functions in inner product spaces*, Arabian Journal of Mathematics, (2013) 2:295–302.

[16] M. Z. Sarikaya, E. Set, M. E. Ozdemir and S. S. Dragomir, *New some Hadamard's type inequalities for co-ordinated convex functions*, Tamsui Oxford Journal of Information and Mathematical Sciences, 28(2) (2012) 137-152.

[17] M. Z. Sarikaya and H. Yaldiz, *On Hermite Hadamard-type inequalities for strongly log-convex functions*, International Journal of Modern Mathematical Sciences, 2013, 5(3): 92-98.

[18] M. Z. Sarikaya and H. Yaldiz, *On the Hadamard's type inequalities for L-Lipschitzian mapping*, Konuralp Journal of Mathematics, 1(2), 33-40 (2013)

[19] M. Z. Sarikaya and K. Ozcelik, *On Hermite-Hadamard type integral inequalities for strongly φ_h-convex functions*, International Journal of Advanced Research in Engineering and Applied Sciences (IJAREAS), 1(1), pp:34-52, 2014.

[20] M. Z. Sarikaya, O*n Hermite Hadamard-type inequalities for strongly* φ*-convex functions*, Southeast Asian Bull. Math., 39(1) (2015) , pp: 123-132.
[21] M. Z. Sarikaya, E. Set and M. E. Ozdemir, *On new inequalities of Simpson's type for* $s-$*convex functions,* Computers and Mathematics with Applications 60 (2010) 2191–2199.

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