Received 12/07/16

ON SOME NEW INEQUALITIES VIA GG-CONVEXITY AND $GA-$CONVEXITY FUNCTIONS

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Abstract. In this paper, we established some integral inequalities for functions whose derivatives of absolute values are $GG-$convex and $GA-$convex.

1. Introduction

We will start with the definiton of convexity:

Definition 1. *The function $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on I, if the inequality*

$$f\left(tx + (1-t)y\right) \leq tf(x) + (1-t) f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$. We say that f is concave if $-f$ is convex.

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function where $a, b \in I$ with $a < b$. Then the following double inequality hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

This inequality is well-known in the literature as Hermite-Hadamard inequality that gives us upper and lower bounds for the mean-value of a convex function. If f is concave function both of the inequalities in above hold in reversed direction.

Anderson *et. al.* mentioned mean function in [2] as following*:*

Definition 2. *A function $M : (0,\infty) \times (0,\infty) \to (0,\infty)$ is called a Mean function if*

(1) $M(x,y) = M(y,x)$,
(2) $M(x,x) = x$,
(3) $x < M(x,y) < y$, whenever $x < y$,
(4) $M(ax,ay) = aM(x,y)$ for all $a > 0$.

Based on the definition of mean function, let us recall special means (See [2])

1. Arithmetic Mean: $M(x,y) = A(x,y) = \frac{x+y}{2}$.
2. Geometric Mean: $M(x,y) = G(x,y) = \sqrt{xy}$.
3. Harmonic Mean: $M(x,y) = H(x,y) = 1/A\left(\frac{1}{x}, \frac{1}{y}\right)$.
4. Logarithmic Mean: $M(x,y) = L(x,y) = (x-y)/(\log x - \log y)$ for $x \neq y$ nd $L(x,x) = x$.

1991 *Mathematics Subject Classification.* 26D15, 26A51, 26E60, 41A55.

Key words and phrases. $GG-$convex and $GA-$convex functions, Hölder inequality, Power-mean integral inequality.

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5. Identric Mean: $M(x,y) = I(x,y) = (1/e)\left(x^x/y^y\right)^{1/(x-y)}$ for $x \neq y$ nd $I(x,x) = x$.

In [2], Anderson *et. al.* also gave a definition that include several different classes of convex functions as the following:

Definition 3. *Let $f : I \to (0,\infty)$ be continuous, where I is subinterval of $(0,\infty)$. Let M and N be any two Mean functions. We say f is MN-convex (concave) if*

$$f(M(x,y)) \leq (\geq) N(f(x), f(y))$$

for all $x, y \in I$.

In [1], Niculescu mentioned the following considerable definition:

Definition 4. *The $GG-$convex functions are those functions $f : I \to J$ (acting on subintervals of $(0,\infty)$) such that*

$$x, y \in I \text{ and } \lambda \in [0,1] \Longrightarrow f\left(x^{1-\lambda}y^{\lambda}\right) \leq f(x)^{1-\lambda} f(y)^{\lambda}. \tag{1.1}$$

In [3], authors proved the following lemma and establish new inequalities.

Lemma 1. *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I° and $a, b \in I^\circ$ with $a < b$.if $f' \in L[a,b]$ then following identity holds:*

$$\begin{aligned} & bf(b) - af(a) - \int_a^b f(u)du \\ = \; & (\ln x - \ln a)\int_0^1 \left(x^{2t}a^{2(1-t)}\right) f'\left(x^t a^{1-t}\right)dt - (\ln x - \ln b)\int_0^1 \left(x^{2t}b^{2(1-t)}\right) f'\left(x^t b^{1-t}\right)dt \end{aligned}$$

for all $x \in [a,b]$.

For recent results, generalizations, improvements see the papers [1] - [10].

The main aim of this paper is to prove some new integral inequalities for $GG-$convex and $GA-$convex functions by using a new integral identity.

2. NEW INEQUALITIES FOR GG–CONVEX FUNCTIONS

We need the following integral identity to get our new results.

Lemma 2. *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I° where $\eta, \mu \in I^\circ$ with $\eta < \mu$. If $f' \in L[\eta,\mu]$, then the following equality holds:*

$$\begin{aligned} & \mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_a^b \psi f(\psi)d\psi \\ = \; & (\ln\mu - \ln\xi)\int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right) f'\left(\mu^\tau \xi^{1-\tau}\right)d\tau + (\ln\xi - \ln\eta)\int_0^1 \left(\xi^{3\tau}\eta^{3(1-\tau)}\right) f'\left(\xi^\tau \eta^{1-\tau}\right)d\tau \end{aligned}$$

Proof. Let

$$I_1 = \int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right) f'\left(\mu^\tau \xi^{1-\tau}\right)d\tau$$

and

$$I_2 = \int_0^1 \left(\xi^{3\tau}\eta^{3(1-\tau)}\right) f'\left(\xi^\tau \eta^{1-\tau}\right)d\tau$$

We notice that

$$\begin{aligned} I_1 &= \int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right) f'\left(\mu^{\tau}\xi^{1-\tau}\right) d\tau \\ &= \frac{1}{\ln\mu - \ln\xi}\int_0^1 \left(\mu^{2\tau}\xi^{2(1-\tau)}\right) f'\left(\mu^{\tau}\xi^{1-\tau}\right) d\left(\mu^{\tau}\xi^{1-\tau}\right). \end{aligned}$$

By the change of the variable $\psi = \mu^{\tau}\xi^{1-\tau}$ and integrating by parts , we have

$$I_1 = \frac{1}{\ln\mu - \ln\xi}\left[\mu^2 f(\mu) - \xi^2 f(\xi) - 2\int_\xi^\mu \psi f(\psi)d\psi\right].$$

Conformably, we have

$$I_2 = \frac{1}{\ln\xi - \ln\eta}\left[\xi^2 f(\xi) - \eta^2 f(\eta) - 2\int_\eta^\xi \psi f(\psi)d\psi\right].$$

Multiplying I_1 by $(\ln\mu - \ln\xi)$, I_2 by $(\ln\xi - \ln\eta)$ and adding the results we get the desired identity. □

Our first result is given in the following Theorem.

Theorem 1. *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I° where $\eta, \mu \in I^\circ$ with $\eta < \mu$,and $f' \in L[\eta,\mu]$. If $\left|f'\right|$ is $GG-$ convex on $[\eta,\mu]$, then the following inequality holds:*

$$\begin{aligned} &\left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_a^b \psi f(\psi)d\psi\right| \\ \leq\ & (\ln\mu - \ln\xi)\, L\left(\mu^3\left|f'(\mu)\right|, \xi^3\left|f'(\xi)\right|\right) + (\ln\xi - \ln\eta)\, L\left(\xi^3\left|f'(\xi)\right|, \eta^3\left|f'(\eta)\right|\right). \end{aligned}$$

Proof. From Lemma 2, using the property of the modulus and $GG-$ convexity of $\left|f'\right|$ we can write

$$\begin{aligned} &\left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_a^b \psi f(\psi)d\psi\right| \\ \leq\ & (\ln\mu - \ln\xi)\int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f'\left(\mu^{\tau}\xi^{1-\tau}\right)\right| d\tau + (\ln\xi - \ln\eta)\int_0^1 \left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f'\left(\xi^{\tau}\eta^{1-\tau}\right)\right| d\tau \\ \leq\ & (\ln\mu - \ln\xi)\int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f'(\mu)\right|^{\tau}\left|f'(\xi)\right|^{1-\tau} d\tau \\ & + (\ln\xi - \ln\eta)\int_0^1 \left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f'(\eta)\right|^{\tau}\left|f'(\mu)\right|^{1-\tau} d\tau \\ =\ & (\ln\mu - \ln\xi)\,\xi^3\left|f'(\xi)\right|\int_0^1 \left(\frac{\mu^3\left|f'(\mu)\right|}{\xi^3\left|f'(\xi)\right|}\right)^{\tau} d\tau + (\ln\xi - \ln\eta)\,\eta^3\left|f'(\eta)\right|\int_0^1 \left(\frac{\xi^3\left|f'(\xi)\right|}{\eta^3\left|f'(\eta)\right|}\right)^{\tau} d\tau. \end{aligned}$$

If we calculate the integrals above, we get the desired result. □

Theorem 2. *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I° where $\eta,\mu \in I^\circ$ with $\eta < \mu$,and $f' \in L[\eta,\mu]$. If $\left|f'\right|^q$ is GG− convex on $[\eta,\mu]$ for all $x \in [\eta,\mu]$, the following inequality*

$$\begin{aligned}
&\left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_a^b \psi f(\psi)d\psi\right| \\
\leq\ & (\ln\mu - \ln\xi)\left(L\left(\mu^{3p},\xi^{3p}\right)\right)^{\frac{1}{p}}\left(L\left(\left|f'(\mu)\right|^q,\left|f'(\xi)\right|^q\right)\right)^{\frac{1}{q}} \\
&+ (\ln\xi - \ln\eta)\left(L\left(\xi^{3p},\eta^{3p}\right)\right)^{\frac{1}{p}}\left(L\left(\left|f'(\xi)\right|^q,\eta^3\left|f'(\eta)\right|^q\right)\right)^{\frac{1}{q}}
\end{aligned}$$

holds where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2, using the property of the modulus, $GG-$ convexity of $\left|f'\right|^q$ and Hölder integral inequality, we can write

$$\begin{aligned}
&\left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_a^b \psi f(\psi)d\psi\right| \\
=\ & (\ln\mu - \ln\xi)\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f'\left(\mu^\tau\xi^{1-\tau}\right)\right|d\tau + (\ln\xi - \ln\eta)\int_0^1\left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f'\left(\xi^\tau\eta^{1-\tau}\right)\right|d\tau \\
\leq\ & (\ln\mu - \ln\xi)\left(\int_0^1 \mu^{3\tau p}\xi^{3(1-\tau)p}d\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\mu^\tau\xi^{1-\tau}\right)\right|^q d\tau\right)^{\frac{1}{q}} \\
&+ (\ln\xi - \ln\eta)\left(\int_0^1 \xi^{3\tau p}\eta^{3(1-\tau)p}d\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\xi^\tau\eta^{1-\tau}\right)\right|^q d\tau\right)^{\frac{1}{q}} \\
\leq\ & (\ln\mu - \ln\xi)\left(\xi^{3p}\int_0^1\left(\frac{\mu^{3p}}{\xi^{3p}}\right)^\tau d\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(\mu)\right|^{q\tau}\left|f'(\xi)\right|^{(1-\xi)q}d\tau\right)^{\frac{1}{q}} \\
&+ (\ln\xi - \ln\eta)\left(\eta^{3p}\int_0^1\left(\frac{\xi^{3p}}{\eta^{3p}}\right)^\tau d\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(\xi)\right|^{q\tau}\left|f'(\eta)\right|^{(1-\xi)q}d\tau\right)^{\frac{1}{q}}
\end{aligned}$$

If we calculate the integrals above, we get the desired result. □

Theorem 3. *Under the assumptions of Theorem 2, the following inequality holds:*

$$\begin{aligned}
&\left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_a^b \psi f(\psi)d\psi\right| \\
\leq\ & \ln\mu - \ln\xi\left(\frac{\mu^{3p}-1}{\ln\mu^{3p}}\right)^{\frac{1}{p}}\left(L\left(\left|f'(\mu)\right|^q,\xi^{3q}\left|f'(\xi)\right|^q\right)\right)^{\frac{1}{q}} \\
& \ln\xi - \ln\eta\left(\frac{\xi^{3p}-1}{\ln\xi^{3p}}\right)^{\frac{1}{p}}\left(L\left(\left|f'(\xi)\right|^q,\eta^{3q}\left|f'(\eta)\right|^q\right)\right)^{\frac{1}{q}}
\end{aligned}$$

holds where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2, using the property of the modulus, $GG-$ convexity of $\left|f^{'}\right|^{q}$ and Hölder integral inequality, we can write

$$\begin{aligned}
&\left|\mu^{2}f(\mu)-\eta^{2}f(\eta)-2\int_{a}^{b}\psi f(\psi)d\psi\right| \\
=\ & (\ln\mu-\ln\xi)\int_{0}^{1}\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f^{'}\left(\mu^{\tau}\xi^{1-\tau}\right)\right|d\tau+(\ln\xi-\ln\eta)\int_{0}^{1}\left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f^{'}\left(\xi^{\tau}\eta^{1-\tau}\right)\right|d\tau \\
\leq\ & (\ln\mu-\ln\xi)\left(\int_{0}^{1}\mu^{3\tau p}d\tau\right)^{\frac{1}{p}}\left(\int_{0}^{1}\xi^{3(1-\tau)q}\left|f^{'}\left(\mu^{\tau}\xi^{1-\tau}\right)\right|^{q}d\tau\right)^{\frac{1}{q}} \\
& +(\ln\xi-\ln\eta)\left(\int_{0}^{1}\xi^{3\tau p}d\tau\right)^{\frac{1}{p}}\left(\int_{0}^{1}\eta^{3(1-\tau)q}\left|f^{'}\left(\xi^{\tau}\eta^{1-\tau}\right)\right|^{q}d\tau\right)^{\frac{1}{q}} \\
\leq\ & (\ln\mu-\ln\xi)\left(\int_{0}^{1}\mu^{3\tau p}d\tau\right)^{\frac{1}{p}}\left(\int_{0}^{1}\xi^{3(1-\tau)q}\left|f^{'}(\mu)\right|^{q\tau}\left|f^{'}(\xi)\right|^{(1-\tau)q}dt\right)^{\frac{1}{q}} \\
& +(\ln\xi-\ln\eta)\left(\int_{0}^{1}\xi^{3\tau p}d\tau\right)^{\frac{1}{p}}\left(\int_{0}^{1}\eta^{3(1-\tau)q}\left|f^{'}(\xi)\right|^{q\tau}\left|f^{'}(\eta)\right|^{(1-\tau)q}dt\right)^{\frac{1}{q}}
\end{aligned}$$

By a simple computation we get the desired result. □

Theorem 4. *Under the assumptions of Theorem 2, the following inequality holds:*

$$\begin{aligned}
&\left|\mu^{2}f(\mu)-\eta^{2}f(\eta)-2\int_{a}^{b}\psi f(\psi)d\psi\right| \\
\leq\ & (\ln\mu-\ln\xi)\left(L\left(\mu^{3q}\left|f^{'}(\mu)\right|^{q},\xi^{3q}\left|f^{'}(\xi)\right|^{q}\right)\right)^{\frac{1}{q}} \\
& +(\ln\xi-\ln\eta)\left(L\left(\xi^{3q}\left|f^{'}(\xi)\right|^{q},\eta^{3q}\left|f^{'}(\eta)\right|^{q}\right)\right)^{\frac{1}{q}}
\end{aligned}$$

holds where $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2, using the property of the modulus, $GG-$ convexity of $\left|f^{'}\right|^{q}$ and Hölder integral inequality, we can write

$$\begin{aligned}
&\left|\mu^2 f(\mu)-\eta^2 f(\eta)-2\int_a^b \psi f(\psi)d\psi\right| \\
=\ & (\ln\mu-\ln\xi)\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f'\left(\mu^{\tau}\xi^{1-\tau}\right)\right|d\tau+(\ln\xi-\ln\eta)\int_0^1\left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f'\left(\xi^{\tau}\eta^{1-\tau}\right)\right|d\tau \\
\leq\ & (\ln\mu-\ln\xi)\left(\int_0^1 d\tau\right)^{\frac{1}{p}}\left(\int_0^1 \mu^{3\tau q}\xi^{3(1-\tau)q}\left|f'\left(\mu^{\tau}\xi^{1-\tau}\right)\right|^q d\tau\right)^{\frac{1}{q}} \\
&+(\ln\xi-\ln\eta)\left(\int_0^1 d\tau\right)^{\frac{1}{p}}\left(\int_0^1 \xi^{3\tau p}\eta^{3(1-\tau)q}\left|f'\left(\xi^{\tau}\eta^{1-\tau}\right)\right|^q d\tau\right)^{\frac{1}{q}} \\
\leq\ & (\ln\mu-\ln\xi)\left(\int_0^1 d\tau\right)^{\frac{1}{p}}\left(\int_0^1 \mu^{3\tau q}\xi^{3(1-\tau)q}\left|f'(\mu)\right|^{q\tau}\left|f'(\xi)\right|^{(1-\tau)q}dt\right)^{\frac{1}{q}} \\
&+(\ln\xi-\ln\eta)\left(\int_0^1 d\tau\right)^{\frac{1}{p}}\left(\int_0^1 \xi^{3\tau p}\eta^{3(1-\tau)q}\left|f'(\xi)\right|^{q\tau}\left|f'(\eta)\right|^{(1-\tau)q}dt\right)^{\frac{1}{q}}
\end{aligned}$$

If we calculate the integrals above, we get the desired result. □

Theorem 5. *Under the assumptions of Theorem 2, the following inequality holds:*

$$\begin{aligned}
&\left|\mu^2 f(\mu)-\eta^2 f(\eta)-2\int_a^b \psi f(\psi)d\psi\right| \\
\leq\ & (\ln\mu-\ln\xi)\left(\frac{\mu^3-1}{3\ln\mu}\right)^{\frac{1}{p}} L^{\frac{1}{q}}\left(\mu^3\left|f'(\mu)\right|^q,\xi^{3q}\left|f'(\xi)\right|^q\right) \\
&+(\ln\xi-\ln\eta)\left(\frac{\xi^3-1}{3\ln\xi}\right)^{\frac{1}{p}} L^{\frac{1}{q}}\left(\xi^3\left|f'(\xi)\right|^q,\eta^{3q}\left|f'(\eta)\right|^q\right).
\end{aligned}$$

holds where $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2, using the property of the modulus, $GG-$ convexity of $\left|f'\right|^q$ and Hölder integral inequality, we can write

$$\begin{aligned}
&\left|\mu^2 f(\mu)-\eta^2 f(\eta)-2\int_a^b \psi f(\psi)d\psi\right| \\
=\ & (\ln\mu-\ln\xi)\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f'\left(\mu^{\tau}\xi^{1-\tau}\right)\right|d\tau+(\ln\xi-\ln\eta)\int_0^1\left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f'\left(\xi^{\tau}\eta^{1-\tau}\right)\right|d\tau \\
\leq\ & (\ln\mu-\ln\xi)\left(\int_0^1 \mu^{3\tau}d\tau\right)^{\frac{1}{p}}\left(\int_0^1 \mu^{3\tau}\xi^{3(1-\tau)q}\left|f'(\mu)\right|^{q\tau}\left|f'(\xi)\right|^{(1-\tau)q}d\tau\right)^{\frac{1}{q}} \\
&+(\ln\xi-\ln\eta)\left(\int_0^1 \xi^{3\tau}d\tau\right)^{\frac{1}{p}}\left(\int_0^1 \xi^{3\tau}\eta^{3(1-\tau)q}\left|f'(\xi)\right|^{q\tau}\left|f'(\eta)\right|^{(1-\tau)q}d\tau\right)^{\frac{1}{q}}
\end{aligned}$$

If we calculate the integrals above, we get the desired result. □

Theorem 6. *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I° where $\eta, \mu \in I^\circ$ with $\eta < \mu$,and $f^{'} \in L[\eta,\mu]$. If $\left|f^{'}\right|^q$ is GG− convex on $[\eta,\mu]$ for all $x \in [\eta,\mu]$, the following inequality*

$$\begin{aligned}
&\left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_a^b \psi f(\psi)d\psi\right| \\
\leq\ & (\ln\mu - \ln\xi)\left(L(\mu^3,\xi^3)\right)^{1-\frac{1}{q}}\left(L\left(\mu^3\left|f^{'}(\mu)\right|^q, \xi^3\left|f^{'}(\xi)\right|^q\right)\right)^{\frac{1}{q}} \\
&+ (\ln\xi - \ln\eta)\left(L(\xi^3,\eta^3)\right)^{1-\frac{1}{q}}\left(L\left(\xi^3\left|f^{'}(\xi)\right|^q, \eta^3\left|f^{'}(\eta)\right|^q\right)\right)^{\frac{1}{q}}
\end{aligned}$$

holds for $q \geq 1$.

Proof. From Lemma 2, using the property of the modulus, $GG-$ convexity of $\left|f^{'}\right|^q$ and power-mean integral inequality, we can write

$$\begin{aligned}
&\left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_a^b \psi f(\psi)d\psi\right| \\
=\ & (\ln\mu - \ln\xi)\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f^{'}\left(\mu^\tau\xi^{1-\tau}\right)\right|d\tau + (\ln\xi - \ln\eta)\int_0^1\left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f^{'}\left(\xi^\tau\eta^{1-\tau}\right)\right|d\tau \\
\leq\ & (\ln\mu - \ln\xi)\left(\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f^{'}\left(\mu^\tau a^{1-\tau}\right)\right|^q d\tau\right)^{\frac{1}{q}} \\
&+ (\ln\xi - \ln\eta)\left(\int_0^1\left(\xi^{3\tau}\eta^{3(1-\tau)}\right)d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f^{'}\left(\xi^\tau\eta^{1-\tau}\right)\right|^q d\tau\right)^{\frac{1}{q}} \\
\leq\ & (\ln\mu - \ln\xi)\left(\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f^{'}(\mu)\right|^{q\tau}\left|f^{'}(\xi)\right|^{(1-\tau)q}dt\right)^{\frac{1}{q}} \\
&+ (\ln\xi - \ln\eta)\left(\int_0^1\left(\xi^{3\tau}\eta^{3(1-\tau)}\right)d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\left|f^{'}(\xi)\right|^{q\tau}\left|f^{'}(\eta)\right|^{(1-\tau)q}d\tau\right)^{\frac{1}{q}}.
\end{aligned}$$

We get the desired result by simple calculation. □

3. NEW INEQUALITIES FOR $GA-$CONVEX FUNCTIONS

In this section, we obtain some inequalities for $GA-$ convex functions.

Theorem 7. *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I° where $\eta, \mu \in I^\circ$ with $\eta < \mu$,and $f^{'} \in L[\eta,\mu]$. If $\left|f^{'}\right|$ is GA− convex on $[\eta,\mu]$ for all $x \in [\eta,\mu]$, the following inequality*

$$\begin{aligned}
&\left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_a^b \psi f(\psi)d\psi\right| \\
\leq\ & \frac{\left|f^{'}(\mu)\right|}{3}\left[\mu^3 - L(\xi^3,\mu^3)\right] + \frac{\left|f^{'}(\xi)\right|}{3}\left[L(\xi^3,\mu^3) - L(\eta^3,\xi^3)\right] + \frac{\left|f^{'}(\eta)\right|}{3}\left[L(\eta^3,\xi^3) - \xi^3\right]
\end{aligned}$$

holds.

Proof. From Lemma 2, using the property of the modulus, $GA-$ convexity of $\left|f^{'}\right|$, we can write

$$\begin{aligned}
&\left|\mu^2 f(\mu)-\eta^2 f(\eta)-2\int_a^b \psi f(\psi)d\psi\right| \\
&= (\ln\mu-\ln\xi)\int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f^{'}\left(\mu^{\tau}\xi^{1-\tau}\right)\right|d\tau+(\ln\xi-\ln\eta)\int_0^1 \left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f^{'}\left(\xi^{\tau}\eta^{1-\tau}\right)\right|d\tau \\
&\leq (\ln\mu-\ln\xi)\int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left[\tau\left|f^{'}(\mu)\right|+(1-\tau)\left|f^{'}(\xi)\right|\right]d\tau \\
&\quad+(\ln\xi-\ln\eta)\int_0^1 \left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left[\tau\left|f^{'}(\xi)\right|+(1-\tau)\left|f^{'}(\eta)\right|\right]d\tau
\end{aligned}$$

We get the desired result by simple calculation. □

Theorem 8. *Let $f: I\subseteq\mathbb{R}_+=(0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function on I° where $\eta,\mu\in I^\circ$ with $\eta<\mu$,and $f^{'}\in L[\eta,\mu]$. If $\left|f^{'}\right|^q$ is $GA-$ convex on $[\eta,\mu]$ for all $x\in[\eta,\mu]$, the following inequality*

$$\begin{aligned}
&\left|\mu^2 f(\mu)-\eta^2 f(\eta)-2\int_a^b \psi f(\psi)d\psi\right| \\
&\leq (\ln\mu-\ln\xi)^{1-\frac{1}{q}}L^{1-\frac{1}{q}}\left(\xi^3,\mu^3\right)\left(\frac{\left|f^{'}(\mu)\right|^q\left[\mu^3-L(\xi^3,\mu^3)\right]+\left|f^{'}(\xi)\right|^q\left[L(\xi^3,\mu^3)-\xi^3\right]}{3}\right)^{\frac{1}{q}} \\
&\quad+(\ln\xi-\ln\eta)^{1-\frac{1}{q}}L^{1-\frac{1}{q}}\left(\eta^3,\xi^3\right)\left(\frac{\left|f^{'}(\xi)\right|^q\left[\xi^3-L(\eta^3,\xi^3)\right]+\left|f^{'}(\eta)\right|^q\left[L(\eta^3,\xi^3)-\eta^3\right]}{3}\right)^{\frac{1}{q}}
\end{aligned}$$

holds for $q\geq 1$.

Proof. From Lemma 2, using the property of the modulus, $GA-$ convexity of $\left|f^{'}\right|^q$ and power-mean integral inequality, we can write

$$\begin{aligned}
&\left|\mu^2 f(\mu)-\eta^2 f(\eta)-2\int_a^b \psi f(\psi)d\psi\right| \\
&= (\ln\mu-\ln\xi)\int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f^{'}\left(\mu^{\tau}\xi^{1-\tau}\right)\right|d\tau+(\ln\xi-\ln\eta)\int_0^1 \left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f^{'}\left(\xi^{\tau}\eta^{1-\tau}\right)\right|d\tau \\
&\leq (\ln\mu-\ln\xi)\left(\int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right)d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left[\tau\left|f^{'}(\mu)\right|^q+(1-\tau)\left|f^{'}(\xi)\right|^q\right]d\tau\right)^{\frac{1}{q}} \\
&\quad+(\ln\xi-\ln\eta)\left(\int_0^1 \left(\xi^{3\tau}\eta^{3(1-\tau)}\right)d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left[\tau\left|f^{'}(\xi)\right|^q+(1-\tau)\left|f^{'}(\eta)\right|^q\right]d\tau\right)^{\frac{1}{q}}
\end{aligned}$$

We get the desired result by simple calculation. □

Remark 1. *In Theorem 8, if we choose $q=1$, Theorem 8 reduces to Theorem 7.*

Theorem 9. *Under the assumptions of Theorem 2, the following inequality holds:*

$$\begin{aligned}
&\left|\mu^2 f(\mu)-\eta^2 f(\eta)-2\int_a^b \psi f(\psi)d\psi\right| \\
\leq\ & (\ln\mu-\ln\xi)^{\frac{1}{q}}\left(\frac{q-1}{3q}\right)^{1-\frac{1}{q}}\left(\mu^{\frac{3q}{q-1}}-\xi^{\frac{3q}{q-1}}\right)^{1-\frac{1}{q}}\left(A\left(\left|f'(\mu)\right|^q+\left|f'(\xi)\right|^q\right)\right)^{\frac{1}{q}} \\
&+(\ln\xi-\ln\eta)^{\frac{1}{q}}\left(\frac{q-1}{3q}\right)^{1-\frac{1}{q}}\left(\xi^{\frac{3q}{q-1}}-\eta^{\frac{3q}{q-1}}\right)^{1-\frac{1}{q}}\left(A\left(\left|f'(\xi)\right|^q+\left|f'(\eta)\right|^q\right)\right)^{\frac{1}{q}}
\end{aligned}$$

holds where $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2, using the property of the modulus, $GA-$ convexity of $\left|f'\right|^q$ and Hölder integral inequality, we can write

$$\begin{aligned}
&\left|\mu^2 f(\mu)-\eta^2 f(\eta)-2\int_a^b \psi f(\psi)d\psi\right| \\
=\ & (\ln\mu-\ln\xi)\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f'\left(\mu^\tau\xi^{1-\tau}\right)\right|d\tau+(\ln\xi-\ln\eta)\int_0^1\left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f'\left(\xi^\tau\eta^{1-\tau}\right)\right|d\tau \\
\leq\ & (\ln\mu-\ln\xi)\left(\int_0^1\mu^{3\tau}\xi^{3(1-\tau)}d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\tau\left|f'(\mu)\right|^q+(1-\tau)\left|f'(\xi)\right|^q d\tau\right)^{\frac{1}{q}} \\
&+(\ln\xi-\ln\eta)\left(\int_0^1\xi^{3\tau}\eta^{3(1-\tau)}d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\tau\left|f'(\xi)\right|^q+(1-\tau)\left|f'(\eta)\right|^q d\tau\right)^{\frac{1}{q}}
\end{aligned}$$

We get the desired result by simple calculation. □

Theorem 10. *Let $f: I\subseteq\mathbb{R}_+=(0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function on I° where $\eta,\mu\in I^\circ$ with $\eta<\mu$,and $f'\in L[\eta,\mu]$. If $\left|f'\right|^q$ is $GA-$ convex on $[\eta,\mu]$ for all $x\in[\eta,\mu]$, the following inequality*

$$\begin{aligned}
&\left|\mu^2 f(\mu)-\eta^2 f(\eta)-2\int_a^b \psi f(\psi)d\psi\right| \\
\leq\ & \frac{(\ln\mu-\ln\xi)^{1-\frac{1}{q}}}{q^{\frac{1}{q}}}\left(K_q(\mu,\xi)\right)^{\frac{1}{q}}+\frac{(\ln\xi-\ln\eta)^{1-\frac{1}{q}}}{q^{\frac{1}{q}}}\left(K_q(\xi,\eta)\right)^{\frac{1}{q}}
\end{aligned}$$

where

$$\begin{aligned}
K_\alpha(\mu,\xi) &= \frac{\left|f'(\mu)\right|^q\left[\mu^{3q}-L(\xi^{3q},\mu^{3q})\right]+\left|f'(\xi)\right|^q\left[L(\xi^{3q},\mu^{3q})-\xi^3\right]}{3} \\
K_\alpha(\xi,\eta) &= \frac{\left|f'(\xi)\right|^q\left[\xi^{3q}-L(\eta^{3q},\xi^{3q})\right]+\left|f'(\eta)\right|^q\left[L(\eta^{3q},\xi^{3q})-\eta^{3q}\right]}{3}
\end{aligned}$$

for all $x\in[\eta,\mu]$ and $q\geq 1$.

Proof. From Lemma 2, using the property of the modulus, $GA-$ convexity of $\left|f^{'}\right|^{q}$ and power-mean integral inequality, we can write

$$\begin{aligned} & \left|\mu^2 f(\mu)-\eta^2 f(\eta)-2\int_a^b \psi f(\psi)d\psi\right| \\ = & (\ln\mu-\ln\xi)\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f^{'}\left(\mu^{\tau}\xi^{1-\tau}\right)\right|d\tau+(\ln\xi-\ln\eta)\int_0^1\left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f^{'}\left(\xi^{\tau}\eta^{1-\tau}\right)\right|d\tau \\ \leq & (\ln\mu-\ln\xi)\left(\int_0^1 d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\left[\mu^{3\tau}\xi^{3(1-\tau)}\right]^q\left[\tau\left|f^{'}(\mu)\right|^q+(1-\tau)\left|f^{'}(\xi)\right|^q\right]d\tau\right)^{\frac{1}{q}} \\ & +(\ln\xi-\ln\eta)\left(\int_0^1 d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\left[\xi^{3\tau}\eta^{3(1-\tau)}\right]^q\left[\tau\left|f^{'}(\xi)\right|^q+(1-\tau)\left|f^{'}(\eta)\right|^q\right]d\tau\right)^{\frac{1}{q}} \end{aligned}$$

We get the desired result by simple calculation. □

Theorem 11. *Let* $f: I\subseteq\mathbb{R}_+=(0,\infty)\longrightarrow\mathbb{R}$ *be a differentiable function on* I° *where* $\eta,\mu\in I^{\circ}$ *with* $\eta<\mu$ *,and* $f^{'}\in L[\eta,\mu]$*. If* $\left|f^{'}\right|^{q}$ *is* $GA-$ *convex on* $[\eta,\mu]$ *for all* $x\in[\eta,\mu]$*, the following inequality*

$$\begin{aligned} & \left|\mu^2 f(\mu)-\eta^2 f(\eta)-2\int_a^b \psi f(\psi)d\psi\right| \\ \leq & (\ln\mu-\ln\xi)^{1-\frac{1}{q}}\left[\mu^{\frac{3q-3p}{q-1}}-\xi^{\frac{3q-3p}{q-1}}\right]^{\frac{q-1}{q}}\left(\frac{q-1}{3q-3p}\right)^{1-\frac{1}{q}}\left(K_q(\mu,\xi)\right)^{\frac{1}{q}} \\ & +(\ln\xi-\ln\eta)^{1-\frac{1}{q}}\left[\xi^{\frac{3q-3p}{q-1}}-\eta^{\frac{3q-3p}{q-1}}\right]^{\frac{q-1}{q}}\left(\frac{q-1}{3q-3p}\right)^{1-\frac{1}{q}}\left(K_q(\xi,\eta)\right)^{\frac{1}{q}} \end{aligned}$$

where

$$\begin{aligned} K_{\alpha}(\mu,\xi) &= \frac{\left|f^{'}(\mu)\right|^q\left[\mu^{3q}-L(\xi^{3q},\mu^{3q})\right]+\left|f^{'}(\xi)\right|^q\left[L(\xi^{3q},\mu^{3q})-\xi^{3}\right]}{3} \\ K_{\alpha}(\xi,\eta) &= \frac{\left|f^{'}(\xi)\right|^q\left[\xi^{3q}-L(\eta^{3q},\xi^{3q})\right]+\left|f^{'}(\eta)\right|^q\left[L(\eta^{3q},\xi^{3q})-\eta^{3q}\right]}{3} \end{aligned}$$

for all $x\in[\eta,\mu]$ and $q\geq 1$.

Proof. From Lemma 2, using the property of the modulus, $GA-$ convexity of $\left|f^{'}\right|^{q}$ and power-mean integral inequality, we can write

$$\begin{aligned}
&\left|\mu^2 f(\mu)-\eta^2 f(\eta)-2\int_a^b \psi f(\psi)d\psi\right| \\
=\ & (\ln\mu-\ln\xi)\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f^{'}\left(\mu^{\tau}\xi^{1-\tau}\right)\right|d\tau+(\ln\xi-\ln\eta)\int_0^1\left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f^{'}\left(\xi^{\tau}\eta^{1-\tau}\right)\right|d\tau \\
\leq\ & (\ln\mu-\ln\xi)\left(\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)^{\frac{q-p}{q-1}}d\tau\right)^{\frac{q-1}{q}}\left(\int_0^1\left[\mu^{3\tau}\xi^{3(1-\tau)}\right]^p\left[\tau\left|f^{'}(\mu)\right|^q+(1-\tau)\left|f^{'}(\xi)\right|^q\right]d\tau\right)^{\frac{1}{q}} \\
&+(\ln\xi-\ln\eta)\left(\int_0^1 d\tau\right)^{\frac{q-1}{q}}\left(\int_0^1\left[\xi^{3\tau}\eta^{3(1-\tau)}\right]^p\left[\tau\left|f^{'}(\xi)\right|^q+(1-\tau)\left|f^{'}(\eta)\right|^q\right]d\tau\right)^{\frac{1}{q}}
\end{aligned}$$

We get the desired result by simple calculation. □

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