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APPLICATIONS OF YOUNG'S INEQUALITY FOR MATRICES

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Abstract. In this paper will be presented an application of some refinements of Young's inequality given by Dragomir S. S., using the same method as in the paper of Alzer, H., Fonseca, C. M. and Kovacec, A. for positive definite matrices.

1. **Introduction**

The famous Young's inequality, as a classical result, state that:

$$a^{\nu}b^{1-\nu} < \nu a + (1-\nu)b,$$

when a and b are positive numbers, $a \neq b$ and $\nu \in (0,1)$.

In these years, there are many interesting generalizations of this well-known inequality and its reverse, see for example [6],[7],[1],[3] many others and references therein. Related to these generalizations often appear the weighted arithmetic mean, geometric mean and harmonic mean defined by $A_{\nu}(a,b) = (1-\nu)a + \nu b$, $G_{\nu}(a,b) = a^{1-\nu}b^{\nu}$ and $H_{\nu}(a,b) = A_{\nu}^{-1}(a^{-1}, b^{-1}) = [(1-\nu)a^{-1} + \nu b^{-1}]^{-1}$.

It is necessary to recall, see [1], that for two positive definite matrices A, B, the ν-weighted arithmetic and geometric mean are defined as

$$A\nabla_{\mu}B = (1-\mu)A + \mu B$$

and

$$A\sharp_{\mu}B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\mu}A^{\frac{1}{2}}$$

when $\mu \in [0,1]$. If $\mu = \frac{1}{2}$ then we write only $A\nabla B$, $A\sharp B$.

It is known that for any two square matrices A, B, $A \leq B$ if $B - A$ is positive semidefinite. Also, $A < B$ if $B - A$ is positive definite, see [1] and [5].

The first matrix version of the Young inequality was proven for invertible matrices A in [8]. Recent improvement of the matrix Young inequality were given for example in [6], [7], [9], [1]. We use the following generalization of Young's inequality given in [2], see inequality (5.8), in order to obtains the matrix analogues in section 2. For any a, $b > 0$ and $\nu \in [0,1]$ we have:

$$2\frac{\nu(1-\nu)}{\nu^2-\nu+1}\left[A(a,b) - L(a,b)\right] \leq A_{\nu}(a,b) - G_{\nu}(a,b) \leq$$

$$\leq 2\left[A(a,b) - L(a,b)\right], \tag{1}$$

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where $A_\nu(a,b) = (1-\nu)a + \nu b,\ G_\nu(a,b) = a^{1-\nu}b^\nu,\ A(a,b) = \frac{a+b}{2},\ G(a,b) = \sqrt{ab}$ and $L(a,b) = \frac{b-a}{\log b - \log a}$.

We also have to mention the following inequalities used in [3] in the proof of Theorem 4:

For any $x > 0$ we have,

$$1 - \nu + \nu x - x^\nu \leq \begin{cases} \nu(x - 1 - \log x) \\ (1-\nu)(x\log x - x + 1) \end{cases} \tag{2}$$

and

$$\frac{x+1}{2} - \frac{x^\nu + x^{1-\nu}}{2} \leq \frac{1}{2}\min\{\nu.1-\nu\}(x-1)\log x \tag{3}$$

for any $\nu \in [o,1]$.

2. **A matrix analogue of a refinement of Young's inequality**

As in [1] we consider M_n the set of $n \times n$ square matrices. We denote by $\lambda_1(H) \leq \lambda_2(H) \leq ... \leq \lambda_n(H)$ the eigenvalues of a Hermitian matrix H of order n, in increasing order.

The scalar inequality from Lemma 1 and the Ostrowski's theorem, see [5] allows us to state the following result:

Theorem 1. *Let $\lambda \in (0,1)$ and $A,\ B$ be positive definite matrices. If $A \leq B$ then we have:*

$$A\nabla_\nu B - A\sharp_\nu B \leq 2\left[A\nabla B + \frac{B-A}{\log\frac{\lambda_1(A)}{\lambda_n(B)}}\right],$$

or

$$A\nabla_\nu B - A\sharp_\nu B \leq 2\left[A\nabla B - \frac{\frac{\lambda_1(A)}{\lambda_n(B)} - 1}{\log\frac{\lambda_1(A)}{\lambda_n(B)}}B\right].$$

Proof. We take into account the Hermitian matrix $C = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ which satisfy the inequality $0 < C \leq I$. As in the proof of Theorem 3.4, see [1], there is a unitary matrix U such that for some c_i we have $U^*CU = diag(c_1, ..., c_n) \leq I$ and thus by Ostrowski's theorem we get $\frac{\lambda_1(A)}{\lambda_n(B)} \leq c_i \leq 1$.

If we write the inequality (1) when $0 < b \leq 1$ for these positive real numbers $c_i,\ i = \overline{1,n}$ and replace b by c_i and a by 1 then we have:

$$1 - \nu + \nu c_i - c_i^\nu \leq 2\left[\frac{c_i+1}{2} - \frac{c_i - 1}{\log c_i}\right],\quad i = \overline{1,n},$$

or

$$1 - \nu + \nu c_i - c_i^\nu \leq 2\left[\frac{c_i+1}{2} + \frac{1 - c_i}{\log c_i}\right],\ i = \overline{1,n},$$

when $c_i \leq 1$. We consider the function $f(t) = \frac{1}{\log(t)}$, $t \in (0,1)$ defined on a compact subinterval of $(0,1)$, function which attains its maximum at its left endpoint and we get:

$$1 - \nu + \nu c_i - c_i^{\nu} \leq 2 \left[\frac{c_i + 1}{2} + \frac{1 - c_i}{\log \frac{\lambda_1(A)}{\lambda_n(B)}} \right], \ i = \overline{1, n},$$

when $c_i \leq 1$, or by calculus, the diagonal matrix inequality,

$$I\nabla_{\nu} diag(c_1, ..., c_n) - I\sharp_{\nu} diag(c_1, ..., c_n) \leq$$

$$\leq 2 \left[I\nabla diag(c_1, ..., c_n) + \frac{1}{\log \frac{\lambda_1(A)}{\lambda_n(B)}} (I - diag(c_1, ..., c_n)) \right].$$

Then applying the conjugation $\bullet \to B^{\frac{1}{2}} U \bullet U^* B^{\frac{1}{2}}$ we get the desired inequality.

For the second inequality we proceed like before, but in inequality

$$1 - \nu + \nu c_i - c_i^{\nu} \leq 2 \left[\frac{c_i + 1}{2} - \frac{c_i - 1}{\log c_i} \right], \ \ i = \overline{1, n},$$

we replace $-\frac{c_i - 1}{\log c_i}$ by $-\frac{\frac{\lambda_1(A)}{\lambda_n(B)} - 1}{\log \frac{\lambda_1(A)}{\lambda_n(B)}}$, because the function $f(t) = -\frac{t-1}{\log t}$, $t \in (0,1)$ attains its maximum at its left endpoint on a compact subinterval of $(0,1)$. Then by calculus, the diagonal matrix inequality,

$$I\nabla_{\nu} diag(c_1, ..., c_n) - I\sharp_{\nu} diag(c_1, ..., c_n) \leq$$

$$\leq 2 \left[I\nabla diag(c_1, ..., c_n) - \frac{\frac{\lambda_1(A)}{\lambda_n(B)} - 1}{\log \frac{\lambda_1(A)}{\lambda_n(B)}} I \right].$$

Then applying the conjugation $\bullet \to B^{\frac{1}{2}} U \bullet U^* B^{\frac{1}{2}}$ we get the desired inequality.
∎

The following result takes place if we put instead of $\lambda = 1$ in inequality (2.1), Theorem 2.1 from [1], $\lambda = n \in \mathbf{N}^*$.

Remark 1. *Let ν and τ be real numbers with $0 < \nu \leq \tau < 1$. If A, B are positive definite matrices, then*

$$\frac{\nu^n}{\tau^n} \left[\sum_{k=0}^{n} \binom{n}{k} (1-\tau)^{n-k} \tau^k B\sharp_k A - B\sharp_{\tau n} A \right] <$$

$$< \sum_{k=0}^{n} \binom{n}{k} (1-\nu)^{n-k} \nu^k B\sharp_k A - B\sharp_{\nu n} A <$$

$$< \frac{(1-\nu)^n}{(1-\tau)^n} \left[\sum_{k=0}^{n} \binom{n}{k} (1-\tau)^{n-k} \tau^k B\sharp_k A - B\sharp_{\tau n} A \right]$$

for any real numbers ν and τ with $0 < \nu < \tau < 1$.

Proof. We use the same method as in [1], starting from inequality

$$\left(\frac{\nu}{\tau}\right)^n \left[\sum_{k=o}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (1-\tau)^{n-k}\tau^k a^{n-k}b^k - a^{n(1-\tau)}b^{n\tau}\right] <$$

$$< \sum_{k=o}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (1-\nu)^{n-k}\nu^k a^{n-k}b^k - a^{n(1-\nu)}b^{n\nu} <$$

$$< \left(\frac{1-\nu}{1-\tau}\right)^n \left[\sum_{k=o}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (1-\tau)^{n-k}\tau^k a^{n-k}b^k - a^{n(1-\tau)}b^{n\tau}\right],$$

where we put $a = 1$ and $b = l_i > 0$, see the proof of Lemma 3.1. Now, using the spectral theorem for Hermitian matrices, see [5] Theorem 2.5.6, there is a unitary matrix and a real diagonal matrix $\Lambda = diag(l_1, ..., l_n)$ so that $Q = U^*\Lambda U$. Then we have the following matrix inequality for diagonal matrices

$$\left(\frac{\nu}{\tau}\right)^n \left[\sum_{k=o}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (1-\tau)^{n-k}\tau^k \Lambda^k - \Lambda^{n\tau}\right] < \sum_{k=o}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (1-\nu)^{n-k}\nu^k \Lambda^k - \Lambda^{n\nu} <$$

$$< \left(\frac{1-\nu}{1-\tau}\right)^n \left[\sum_{k=o}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (1-\tau)^{n-k}\Lambda^k - \Lambda^{n\tau}\right]$$

which can be read, as in [1], either as entrywise companion or in the positive semidefinite ordering. Applying the $*$-conjugation $\bullet \to U^* \bullet U$ we get

$$\left(\frac{\nu}{\tau}\right)^n \left[\sum_{k=o}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (1-\tau)^{n-k}\tau^k Q^k - Q^{n\tau}\right] < \sum_{k=o}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (1-\nu)^{n-k}\nu^k Q^k - Q^{n\nu} <$$

$$< \left(\frac{1-\nu}{1-\tau}\right)^n \left[\sum_{k=o}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (1-\tau)^{n-k}Q^k - Q^{n\tau}\right].$$

But $A > 0$ impliies $A^{-\frac{1}{2}}$ and $A^{\frac{1}{2}}$ are Hermitian positive definite and then by [5], page 494, $Q = A^{-\frac{1}{2}}BA^{\frac{1}{2}}$ is a positive definite $*$- conjugation of B. Applying here the $*$-conjugation $\bullet \to A^{\frac{1}{2}} \bullet A^{\frac{1}{2}}$ to last inequality we get the desired inequality.
∎

Theorem 2. *Let $\lambda \in (0,1)$ and A, B be positive definite matrices. If $A \leq B$ then we have:*

$$A\nabla_\nu B - A\sharp_\nu B \leq \nu \left[A - B - B\log\frac{\lambda_1(A)}{\lambda_n(B)}\right],$$

or

$$A\nabla_\nu B - A\sharp_\nu B \leq \nu \left[\frac{\lambda_1(A)}{\lambda_n(B)} - 1 - \log\frac{\lambda_1(A)}{\lambda_n(B)}\right] B,$$

or

$$A\nabla_\nu B - A\sharp_\nu B \leq (1-\nu) \left[\frac{\lambda_1(A)}{\lambda_n(B)}\log\frac{\lambda_1(A)}{\lambda_n(B)} - \frac{\lambda_1(A)}{\lambda_n(B)} + 1\right] B.$$

Moreover, the following inequality takes place:

$$A\nabla_\nu B - \frac{1}{2}(A\sharp_\nu B + A\sharp_{1-\nu} B) \leq \frac{1}{2}\min\{\nu.1-\nu\} \left(\frac{\lambda_1(A)}{\lambda_n(B)} - 1\right)\left(\log\frac{\lambda_1(A)}{\lambda_n(B)}\right) B.$$

Proof. We take into account the Hermitian matrix $C = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ which satisfy the inequality $0 < C \le I$. As in the proof of Theorem 3.4, see [1], there is a unitary matrix U such that for some c_i we have $U^*CU = diag(c_1, ..., c_n) \le I$ and thus by Ostrowski's theorem we get $\frac{\lambda_1(A)}{\lambda_n(B)} \le c_i \le 1$.

If we write the inequalities (2) and (3) when $0 < x \le 1$ for these positive real numbers $c_i,\ i = \overline{1,n}$ then we have:

$$1 - \nu + \nu c_i - c_i^\nu \le \nu(c_i - 1 - \log c_i),\ \ i = \overline{1,n},$$

or

$$1 - \nu + \nu c_i - c_i^\nu \le (1-\nu)[c_i \log c_i - c_i + 1],\ i = \overline{1,n}.$$

Moreover, we also have:

$$\frac{c_i+1}{2} - \frac{c_i^\nu + c_i^{1-\nu}}{2} \le \frac{1}{2}\min\{\nu, 1-\nu\}(c_i - 1)\log c_i,\ \ i = \overline{1,n}.$$

We consider the functions $f(x) = -\log x$, $g(x) = x - 1 - \log x$, $h(x) = x\log x - x + 1$, $t(x) = (x-1)\log x$, $x \in (0,1)$ defined on a compact subinterval of $(0,1)$, function which attains its maximum at its left endpoint and we get:

$$1 - \nu + \nu c_i - c_i^\nu \le \nu\left(c_i - 1 - \log\frac{\lambda_1(A)}{\lambda_n(B)}\right),\ i = \overline{1,n},$$

$$1 - \nu + \nu c_i - c_i^\nu \le \nu\left(\frac{\lambda_1(A)}{\lambda_n(B)} - 1 - \log\frac{\lambda_1(A)}{\lambda_n(B)}\right),\ i = \overline{1,n},$$

$$1 - \nu + \nu c_i - c_i^\nu \le (1-\nu)\left(\frac{\lambda_1(A)}{\lambda_n(B)}\log\frac{\lambda_1(A)}{\lambda_n(B)} - \frac{\lambda_1(A)}{\lambda_n(B)} + 1\right),\ i = \overline{1,n},$$

and

$$\frac{c_i+1}{2} - \frac{c_i^\nu + c_i^{1-\nu}}{2} \le \frac{1}{2}\min\{\nu, 1-\nu\}\left(\frac{\lambda_1(A)}{\lambda_n(B)} - 1\right)\log\frac{\lambda_1(A)}{\lambda_n(B)},\ i = \overline{1,n}$$

respectively, when $c_i \le 1$,

By calculus, the diagonal matrix inequalities become,

$$I\nabla_\nu diag(c_1, ..., c_n) - I\sharp_\nu diag(c_1, ..., c_n) \le \nu\left[diag(c_1, ..., c_n) - I - \log\frac{\lambda_1(A)}{\lambda_n(B)}I\right],$$

$$I\nabla_\nu diag(c_1, ..., c_n) - I\sharp_\nu diag(c_1, ..., c_n) \le \nu\left[diag(c_1, ..., c_n) - 1 - \log\frac{\lambda_1(A)}{\lambda_n(B)}\right]I,$$

$$I\nabla_\nu diag(c_1, ..., c_n) - I\sharp_\nu diag(c_1, ..., c_n) \le (1-\nu)\left(\frac{\lambda_1(A)}{\lambda_n(B)}\log\frac{\lambda_1(A)}{\lambda_n(B)} - \frac{\lambda_1(A)}{\lambda_n(B)} + 1\right)I$$

and

$$\frac{diag(c_1, ..., c_n) + I}{2} - \frac{[diag(c_1, ..., c_n)]^\nu + [diag(c_1, ..., c_n)]^{1-\nu}}{2} \le$$
$$\le \frac{1}{2}\min\{\nu, 1-\nu\}\left(\frac{\lambda_1(A)}{\lambda_n(B)} - 1\right)\left(\log\frac{\lambda_1(A)}{\lambda_n(B)}\right)I$$

respectively.

Then applying the conjugation $\bullet \to B^{\frac{1}{2}}U\bullet U^*B^{\frac{1}{2}}$ we get the desired inequalities.

∎

As in [4], let H be a Hilbert space and $\mathcal{B}_1(H)$ the trace class operators in $\mathcal{B}(H)$. We define the trace of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$tr(A) = \sum_{i \in I} < Ae_i, e_i >,$$

where $\{e_i\}_{i \in I}$ is an orthonormal basis of H.

Theorem 3. *Let A, B be two positive operators and P, $Q \in \mathcal{B}_1(H)$ with P, $Q > 0$. Then for any $\nu \in [0,1]$ we have:*

$$\left(\frac{\nu}{\tau}\right)^n \left[\sum_{k=0}^{n} \binom{n}{k} (1-\tau)^{n-k} \tau^k tr(QB^k) tr(PA^{n-k}) - tr(QB^{n\tau}) tr(PA^{n(1-\tau)})\right] <$$

$$< \sum_{k=0}^{n} \binom{n}{k} (1-\nu)^{n-k} \nu^k tr(QB^k) tr(PA^{n-k}) - tr(QB^{n\nu}) tr(PA^{n(1-\nu)}) <$$

$$< \left(\frac{1-\nu}{1-\tau}\right)^n \left[\sum_{k=0}^{n} \binom{n}{k} (1-\tau)^{n-k} \tau^k tr(QB^k) tr(PA^{n-k}) - tr(QB^{n\tau}) tr(PA^{n(1-\tau)})\right].$$

Proof. The proof will be as in [4], but we use the inequality (2.1) from [1]. ∎

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