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BASIC INEQUALITIES FOR (m, M)-Ψ-CONVEX FUNCTIONS WHEN $\Psi = -\ln$

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Abstract. In this paper we establish some basic inequalities for (m, M)-Ψ-convex functions when $\Psi = -\ln$. Applications for power functions and weighted arithmetic mean and geometric mean are also provided.

1. Introduction

Assume that the function $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ (I is an interval) is convex on I and $m \in \mathbb{R}$. We shall say that the function $\Phi : I \to \mathbb{R}$ is m-Ψ-*lower convex* if $\Phi - m\Psi$ is a convex function on I. We may introduce the class of functions [1]

$$\mathcal{L}(I, m, \Psi) := \{\Phi : I \to \mathbb{R} | \Phi - m\Psi \text{ is convex on } I\}. \tag{1.1}$$

Similarly, for $M \in \mathbb{R}$ and Ψ as above, we can introduce the class of M-Ψ-*upper convex* functions by [1]

$$\mathcal{U}(I, M, \Psi) := \{\Phi : I \to \mathbb{R} | M\Psi - \Phi \text{ is convex on } I\}. \tag{1.2}$$

The intersection of these two classes will be called the class of (m, M)-Ψ-*convex functions* and will be denoted by [1]

$$\mathcal{B}(I, m, M, \Psi) := \mathcal{L}(I, m, \Psi) \cap \mathcal{U}(I, M, \Psi). \tag{1.3}$$

Remark 1. *If $\Phi \in \mathcal{B}(I, m, M, \Psi)$, then $\Phi - m\Psi$ and $M\Psi - \Phi$ are convex and then $(\Phi - m\Psi) + (M\Psi - \Phi)$ is also convex which shows that $(M - m)\Psi$ is convex, implying that $M \geq m$ (as Ψ is assumed not to be the trivial convex function $\Psi(t) = 0$, $t \in I$).*

The above concepts may be introduced in the general case of a convex subset in a real linear space, but we do not consider this extension here.

In [7], S. S. Dragomir and N. M. Ionescu introduced the concept of *g-convex dominated functions,* for a function $f : I \to \mathbb{R}$. We recall this, by saying, for a given convex function $g : I \to \mathbb{R}$, the function $f : I \to \mathbb{R}$ is *g-convex dominated* iff $g + f$ and $g - f$ are convex functions on I. In [7], the authors pointed out a number of inequalities for convex dominated functions related to Jensen's, Fuchs', Pečarić's, Barlow-Proschan and Vasić-Mijalković results, etc.

We observe that the concept of g-convex dominated functions can be obtained as a particular case from (m, M)-Ψ-convex functions by choosing $m = -1$, $M = 1$ and $\Psi = g$.

The following lemma holds [1].

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Lemma 1. *Let $\Psi, \Phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable functions on $\mathring{I}$, the interior of I and Ψ is a convex function on $\mathring{I}$.*

(i) *For $m \in \mathbb{R}$, the function $\Phi \in \mathcal{L}\left(\mathring{I}, m, \Psi\right)$ iff*

$$m\left[\Psi(t) - \Psi(s) - \Psi'(s)(t-s)\right] \leq \Phi(t) - \Phi(s) - \Phi'(s)(t-s) \tag{1.4}$$

for all $t, s \in \mathring{I}$.

(ii) *For $M \in \mathbb{R}$, the function $\Phi \in \mathcal{U}\left(\mathring{I}, M, \Psi\right)$ iff*

$$\Phi(t) - \Phi(s) - \Phi'(s)(t-s) \leq M\left[\Psi(t) - \Psi(s) - \Psi'(s)(t-s)\right] \tag{1.5}$$

for all $t, s \in \mathring{I}$.

(iii) *For $M, m \in \mathbb{R}$ with $M \geq m$, the function $\Phi \in \mathcal{B}\left(\mathring{I}, m, M, \Psi\right)$ iff both (1.4) and (1.5) hold.*

Another elementary fact for twice differentiable functions also holds [1].

Lemma 2. *Let $\Psi, \Phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable on $\mathring{I}$ and Ψ is convex on $\mathring{I}$.*

(i) *For $m \in \mathbb{R}$, the function $\Phi \in \mathcal{L}\left(\mathring{I}, m, \Psi\right)$ iff*

$$m\Psi''(t) \leq \Phi''(t) \quad \text{for all } t \in \mathring{I}. \tag{1.6}$$

(ii) *For $M \in \mathbb{R}$, the function $\Phi \in \mathcal{U}\left(\mathring{I}, M, \Psi\right)$ iff*

$$\Phi''(t) \leq M\Psi''(t) \quad \text{for all } t \in \mathring{I}. \tag{1.7}$$

(iii) *For M, $m \in \mathbb{R}$ with $M \geq m$, the function $\Phi \in \mathcal{B}\left(\mathring{I}, m, M, \Psi\right)$ iff both (1.6) and (1.7) hold.*

For various inequalities concerning these classes of function, see the survey paper [3].

In what follows, we consider the class of functions $\mathcal{B}(I, m, M, -\ln)$ for $M, m \in \mathbb{R}$ with $M \geq m$ that is obtained for $\Psi : I \subseteq (0, \infty) \to \mathbb{R}$, $\Psi(t) = -\ln t$.

If $\Phi : I \subseteq (0, \infty) \to \mathbb{R}$ is a differentiable function on $\mathring{I}$ then by Lemma 1 we have $\Phi \in \mathcal{B}(I, m, M, -\ln)$ iff

$$\begin{aligned} m\left[\ln s - \ln t - \frac{1}{s}(s-t)\right] &\leq \Phi(t) - \Phi(s) - \Phi'(s)(t-s) \\ &\leq M\left[\ln s - \ln t - \frac{1}{s}(s-t)\right] \end{aligned} \tag{1.8}$$

for any $s, t \in \mathring{I}$.

If $\Phi : I \subseteq (0, \infty) \to \mathbb{R}$ is a twice differentiable function on $\mathring{I}$ then by Lemma 2 we have $\Phi \in \mathcal{B}(I, m, M, -\ln)$ iff

$$m \leq t^2\Phi''(t) \leq M, \tag{1.9}$$

which is a convenient condition to verify in applications.

In this paper we establish some basic inequalities for (m, M)-Ψ-convex functions when $\Psi = -\ln$. Applications for power functions and weighted arithmetic mean and geometric mean are also provided.

For recent results concerning inequalities for weighted arithmetic mean and geometric mean, see [4], [5] and [8]-[15].

2. Some Inequalities From Definition of Convexity

We define the weighted arithmetic and geometric means

$$A_{\nu}(a,b):=(1-\nu)a+\nu b \text{ and } G_{\nu}(a,b):=a^{1-\nu}b^{\nu}$$

where $\nu\in[0,1]$ and $a,\ b>0$. If $\nu=\frac{1}{2}$, then we write for brevity $A(a,b)$ and $G(a,b)$, respectively.

The following double inequality holds, see also [6]:

Theorem 1. *Let $M,\ m\in\mathbb{R}$ with $M>m$ and $\Phi\in\mathcal{B}((0,\infty),m,M,-\ln)$. Then for any $a,\ b>0$ and $\nu\in[0,1]$ we have*

$$\ln\left(\frac{A_{\nu}(a,b)}{G_{\nu}(a,b)}\right)^{m}\leq(1-\nu)\Phi(a)+\nu\Phi(b)-\Phi((1-\nu)a+\nu b)\leq\ln\left(\frac{A_{\nu}(a,b)}{G_{\nu}(a,b)}\right)^{M}. \tag{2.1}$$

Proof. Since $\Phi\in\mathcal{B}((0,\infty),m,M,-\ln)$, then $\Phi_m:=\Phi+m\ln$ is convex and by the definition of convexity, we have

$$\begin{aligned}&\Phi((1-\nu)a+\nu b)+m\ln A_{\nu}(a,b)\\&\leq(1-\nu)[\Phi(a)+m\ln a]+\nu[\Phi(b)+m\ln b]\\&=(1-\nu)\Phi(a)+\nu\Phi(b)+(1-\nu)m\ln a+\nu m\ln b\\&=(1-\nu)\Phi(a)+\nu\Phi(b)+m\ln G_{\nu}(a,b)\end{aligned}$$

that is equivalent to

$$m\ln\frac{A_{\nu}(a,b)}{G_{\nu}(a,b)}\leq(1-\nu)\Phi(a)+\nu\Phi(b)-\Phi((1-\nu)a+\nu b)$$

and the first inequality in (2.1) is proved.

Similarly, by the convexity of $\Phi_M:=-M\ln-\Phi$ we get the second part of (2.1). □

For $m,\ M$ with $M>m>0$ we define

$$M_p:=\begin{cases}M^p \text{ if } p>1\\ \\ m^p \text{ if } p<0\end{cases} \text{ and } m_p:=\begin{cases}m^p \text{ if } p>1\\ \\ M^p \text{ if } p<0\end{cases}. \tag{2.2}$$

Consider the function $\Phi(t)=t^p,\ p\in(-\infty,0)\cup(1,\infty)$. This is a convex function and $\Phi''(t)=p(p-1)t^{p-2},\ t>0$. Consider $\kappa(t):=t^2\Phi''(t)=p(p-1)t^p$. We observe that

$$\max_{t\in[m,M]}\kappa(t)=p(p-1)M_p \text{ and } \min_{t\in[m,M]}\kappa(t)=p(p-1)m_p.$$

Corollary 1. *Let $m,\ M$ with $M>m>0$ and $p\in(-\infty,0)\cup(1,\infty)$. Then for any $a,\ b\in[m,M]$ and $\nu\in[0,1]$ we have*

$$\ln\left(\frac{A_{\nu}(a,b)}{G_{\nu}(a,b)}\right)^{p(p-1)m_p}\leq(1-\nu)a^p+\nu b^p-((1-\nu)a+\nu b)^p\leq\ln\left(\frac{A_{\nu}(a,b)}{G_{\nu}(a,b)}\right)^{p(p-1)M_p}, \tag{2.3}$$

where M_p and m_p are defined by (2.2).

By taking the exponential in (2.3) we get the equivalent inequality

$$\begin{aligned}\exp\left[\frac{(1-\nu)a^p+\nu b^p-((1-\nu)a+\nu b)^p}{p(p-1)M_p}\right] &\\ \leq \frac{A_\nu(a,b)}{G_\nu(a,b)} &\\ \leq \exp\left[\frac{(1-\nu)a^p+\nu b^p-((1-\nu)a+\nu b)^p}{p(p-1)m_p}\right]\end{aligned} \tag{2.4}$$

for any $p\in(-\infty,0)\cup(1,\infty)$, $\nu\in[0,1]$ and any $a,\ b\in[m,M]$.

If we take $p=2$ in (2.4) and perform the calculations, then we get

$$\exp\left[\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{M^2}\right]\leq\frac{A_\nu(a,b)}{G_\nu(a,b)}\leq\exp\left[\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{m^2}\right] \tag{2.5}$$

for any $a,b\in[m,M]$.

If $a,\ b>0$ then by taking $M=\max\{a,b\}$ and $m=\min\{a,b\}$ in (2.5) we have

$$\exp\left[\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\max^2\{a,b\}}\right]\leq\frac{A_\nu(a,b)}{G_\nu(a,b)}\leq\exp\left[\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\min^2\{a,b\}}\right]. \tag{2.6}$$

Since

$$\frac{(b-a)^2}{\max^2\{a,b\}}=\left(\frac{b-a}{\max\{a,b\}}\right)^2=\left(\frac{\min\{a,b\}}{\max\{a,b\}}-1\right)^2$$

and

$$\frac{(b-a)^2}{\min^2\{a,b\}}=\left(\frac{b-a}{\min\{a,b\}}\right)^2=\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2$$

for any $a,\ b>0$, then (2.6) can be written as

$$\begin{aligned}\exp\left[\frac{1}{2}(1-\nu)\nu\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] &\\ \leq\frac{A_\nu(a,b)}{G_\nu(a,b)} &\\ \leq\exp\left[\frac{1}{2}(1-\nu)\nu\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right].\end{aligned} \tag{2.7}$$

This inequality was obtained in a different way in [5].

If we take $p=-1$ in (2.4) and perform the calculations, then we get

$$\exp\left[\frac{1}{2}(1-\nu)\nu\frac{m(b-a)^2}{abA_\nu(a,b)}\right]\leq\frac{A_\nu(a,b)}{G_\nu(a,b)}\leq\exp\left[\frac{1}{2}(1-\nu)\nu\frac{M(b-a)^2}{abA_\nu(a,b)}\right] \tag{2.8}$$

for any $a,\ b\in[m,M]$ and $\nu\in[0,1]$.

If $a, b > 0$ then by taking $M = \max\{a, b\}$ and $m = \min\{a, b\}$ in (2.8) and since $ab = \max\{a, b\}\min\{a, b\}$ we have

$$\begin{aligned}\exp\left[\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\max\{a,b\}A_\nu(a,b)}\right] \\ \leq \frac{A_\nu(a,b)}{G_\nu(a,b)} \\ \leq \exp\left[\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\min\{a,b\}A_\nu(a,b)}\right]\end{aligned} \tag{2.9}$$

for any $\nu \in [0, 1]$.

Since

$$\frac{1}{\max\{a,b\}} \leq \frac{1}{A_\nu(a,b)} \leq \frac{1}{\min\{a,b\}}$$

hence

$$\exp\left[\frac{1}{2}(1-\nu)\nu\left(\frac{\min\{a,b\}}{\max\{a,b\}}-1\right)^2\right] \leq \exp\left[\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\max\{a,b\}A_\nu(a,b)}\right]$$

and

$$\exp\left[\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\min\{a,b\}A_\nu(a,b)}\right] \leq \exp\left[\frac{1}{2}(1-\nu)\nu\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right],$$

showing that the double inequality (2.9) is better than (2.7).

3. Some Perturbed Inequalities

Recall the following result obtained by Dragomir in 2006 [2] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned}n\min_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^n f(x_j) - f\left(\frac{1}{n}\sum_{j=1}^n x_j\right)\right] \\ \leq \frac{1}{P_n}\sum_{j=1}^n p_j f(x_j) - f\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right) \\ \leq n\max_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^n f(x_j) - f\left(\frac{1}{n}\sum_{j=1}^n x_j\right)\right],\end{aligned} \tag{3.1}$$

where $f : C \to \mathbb{R}$ is a convex function defined on convex subset C of the linear space X, $\{x_j\}_{j\in\{1,2,\dots,n\}}$ are vectors in C and $\{p_j\}_{j\in\{1,2,\dots,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^n p_j > 0$.

For $n = 2$, we deduce from (3.1) that

$$\begin{aligned}2r\left[\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right] \\ \leq \nu f(x) + (1-\nu)f(y) - f(\nu x + (1-\nu)y) \\ \leq 2R\left[\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right]\end{aligned} \tag{3.2}$$

for any $x, y \in C$ and $\nu \in [0,1]$ where $r := \min\{\nu, 1-\nu\}$ and $R := \max\{\nu, 1-\nu\}$.

Theorem 2. *Let $M,\ m \in \mathbb{R}$ with $M > m$ and $\Phi \in \mathcal{B}((0,\infty), m, M, -\ln)$. Then for any $a,\ b > 0$ and $\nu \in [0,1]$ we have*

$$\begin{aligned}
&\ln\left[\frac{A_\nu(a,b)}{G_\nu(a,b)}\left(\frac{G(a,b)}{A(a,b)}\right)^{2r}\right]^m \\
&\le (1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a + \nu b) \\
&- 2r\left[\frac{\Phi(a)+\Phi(b)}{2} - \Phi\left(\frac{a+b}{2}\right)\right] \\
&\le \ln\left[\left(\frac{G(a,b)}{A(a,b)}\right)^{2r}\frac{A_\nu(a,b)}{G_\nu(a,b)}\right]^M
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\ln\left[\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\frac{G_\nu(a,b)}{A_\nu(a,b)}\right]^m \\
&\le 2R\left[\frac{\Phi(a)+\Phi(b)}{2} - \Phi\left(\frac{a+b}{2}\right)\right] \\
&- [\nu\Phi(a) + (1-\nu)\Phi(b) - \Phi(\nu a + (1-\nu)b)] \\
&\le \ln\left[\frac{G_\nu(a,b)}{A_\nu(a,b)}\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\right]^M,
\end{aligned} \tag{3.4}$$

where $r := \min\{\nu, 1-\nu\}$ and $R := \max\{\nu, 1-\nu\}$.

Proof. Since $\Phi \in \mathcal{B}((0,\infty), m, M, -\ln)$, then $f_m := \Phi + m\ln$ is convex and by (3.2) we have

$$\begin{aligned}
&2r\left[\frac{\Phi(a)+\Phi(b)}{2} - \Phi\left(\frac{a+b}{2}\right)\right] - 2rm\ln\frac{A(a,b)}{G(a,b)} \\
&\le \nu\Phi(a) + (1-\nu)\Phi(b) - \Phi(\nu a + (1-\nu)b) - m\ln\frac{A_\nu(a,b)}{G_\nu(a,b)} \\
&\le 2R\left[\frac{\Phi(a)+\Phi(b)}{2} - \Phi\left(\frac{a+b}{2}\right)\right] - 2Rm\ln\frac{A(a,b)}{G(a,b)},
\end{aligned} \tag{3.5}$$

for any $a, b > 0$ and $\nu \in [0,1]$.

Since $\Phi \in \mathcal{B}((0,\infty), m, M, -\ln)$, then also $f_M := -\Phi - M\ln$ is convex and by (3.2) we have

$$\begin{aligned}
&2r\left[\Phi\left(\frac{a+b}{2}\right) - \frac{\Phi(a)+\Phi(b)}{2}\right] + 2rM\ln\frac{A(a,b)}{G(a,b)} \\
&\le \Phi(\nu a + (1-\nu)b) - \nu\Phi(a) - (1-\nu)\Phi(b) + M\ln\frac{A_\nu(a,b)}{G_\nu(a,b)} \\
&\le 2R\left[\Phi\left(\frac{a+b}{2}\right) - \frac{\Phi(a)+\Phi(b)}{2}\right] + 2RM\ln\frac{A(a,b)}{G(a,b)},
\end{aligned} \tag{3.6}$$

for any $a, b > 0$ and $\nu \in [0,1]$.

From the first inequality in (3.5) we have

$$\begin{aligned}&\ln\left[\frac{A_\nu(a,b)}{G_\nu(a,b)}\left(\frac{G(a,b)}{A(a,b)}\right)^{2r}\right]^m\\&\leq\nu\Phi(a)+(1-\nu)\Phi(b)-\Phi(\nu a+(1-\nu)b)\\&-2r\left[\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right]\end{aligned}$$

while from the first inequality in (3.6) we also have

$$\begin{aligned}&\nu\Phi(a)+(1-\nu)\Phi(b)-\Phi(\nu a+(1-\nu)b)\\&-2r\left[\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right]\\&\leq\ln\left[\left(\frac{G(a,b)}{A(a,b)}\right)^{2r}\frac{A_\nu(a,b)}{G_\nu(a,b)}\right]^M\end{aligned}$$

for any $a,b>0$ and $\nu\in[0,1]$.

These prove the desired result (3.3).

From the second inequality in (3.5) we have

$$\begin{aligned}&\ln\left[\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\frac{G_\nu(a,b)}{A_\nu(a,b)}\right]^m\\&\leq 2R\left[\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right]\\&-[\nu\Phi(a)+(1-\nu)\Phi(b)-\Phi(\nu a+(1-\nu)b)]\end{aligned}$$

while from the second inequality in (3.6) we also have

$$\begin{aligned}&2R\left[\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right]\\&-[\nu\Phi(a)+(1-\nu)\Phi(b)-\Phi(\nu a+(1-\nu)b)]\\&\leq\ln\left[\frac{G_\nu(a,b)}{A_\nu(a,b)}\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\right]^M,\end{aligned}$$

for any $a,b>0$ and $\nu\in[0,1]$.

These prove the desired result (3.4). □

Corollary 2. *Let m,M with $M>m>0$ and $p\in(-\infty,0)\cup(1,\infty)$. Then for any $a,b\in[m,M]$ and $\nu\in[0,1]$ we have*

$$\begin{aligned}&\ln\left[\frac{A_\nu(a,b)}{G_\nu(a,b)}\left(\frac{G(a,b)}{A(a,b)}\right)^{2r}\right]^{p(p-1)m_p}\\&\leq(1-\nu)a^p+\nu b^p-((1-\nu)a+\nu b)^p\\&-2r\left[\frac{a^p+b^p}{2}-\left(\frac{a+b}{2}\right)^p\right]\\&\leq\ln\left[\left(\frac{G(a,b)}{A(a,b)}\right)^{2r}\frac{A_\nu(a,b)}{G_\nu(a,b)}\right]^{p(p-1)M_p}\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}(3.8)\qquad & \ln\left[\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\frac{G_\nu(a,b)}{A_\nu(a,b)}\right]^{p(p-1)m_p} \\ & \leq 2R\left[\frac{a^p+b^p}{2}-\left(\frac{a+b}{2}\right)^p\right] \\ & -\left[(1-\nu)a^p+\nu b^p-((1-\nu)a+\nu b)^p\right] \\ & \leq \ln\left[\frac{G_\nu(a,b)}{A_\nu(a,b)}\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\right]^{p(p-1)M_p},\end{aligned}$$

where $r := \min\{\nu, 1-\nu\}$ *and* $R := \max\{\nu, 1-\nu\}$ *and* M_p *and* m_p *are defined by (2.2).*

Observe, by simple calculation, we have that

$$\begin{aligned}(3.9)\qquad & (1-\nu)a^2+\nu b^2-((1-\nu)a+\nu b)^2-2r\left[\frac{a^2+b^2}{2}-\left(\frac{a+b}{2}\right)^2\right] \\ & =(1-\nu)\nu(b-a)^2-\frac{r}{2}(b-a)^2=r\left(R-\frac{1}{2}\right)(b-a)^2\end{aligned}$$

and

$$\begin{aligned}(3.10)\qquad & 2R\left[\frac{a^2+b^2}{2}-\left(\frac{a+b}{2}\right)^2\right]-\left[(1-\nu)a^2+\nu b^2-((1-\nu)a+\nu b)^2\right] \\ & =\frac{R}{2}(b-a)^2-(1-\nu)\nu(b-a)^2=R\left(\frac{1}{2}-r\right)(b-a)^2\end{aligned}$$

for any $a, b \in [m, M]$ and $\nu \in [0,1]$.

If we write the inequalities (3.7) and (3.8) for $p = 2$, then we get

$$\begin{aligned}(3.11)\qquad \ln\left[\frac{A_\nu(a,b)}{G_\nu(a,b)}\left(\frac{G(a,b)}{A(a,b)}\right)^{2r}\right]^{2m^2} & \leq r\left(R-\frac{1}{2}\right)(b-a)^2 \\ & \leq \ln\left[\left(\frac{G(a,b)}{A(a,b)}\right)^{2r}\frac{A_\nu(a,b)}{G_\nu(a,b)}\right]^{2M^2}\end{aligned}$$

and

$$\begin{aligned}(3.12)\qquad \ln\left[\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\frac{G_\nu(a,b)}{A_\nu(a,b)}\right]^{2m^2} & \leq R\left(\frac{1}{2}-r\right)(b-a)^2 \\ & \leq \ln\left[\frac{G_\nu(a,b)}{A_\nu(a,b)}\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\right]^{2M^2},\end{aligned}$$

for any $a, b \in [m, M]$ and $\nu \in [0,1]$.

From the first inequality in (3.11) we have

$$(3.13)\qquad \frac{A_\nu(a,b)}{G_\nu(a,b)} \leq \left(\frac{A(a,b)}{G(a,b)}\right)^{2r}\exp\left(\frac{1}{2m^2}r\left(R-\frac{1}{2}\right)(b-a)^2\right),$$

while from the second inequality in (3.11) we have

$$\left(\frac{A(a,b)}{G(a,b)}\right)^{2r}\exp\left[\frac{1}{2M^2}r\left(R-\frac{1}{2}\right)(b-a)^2\right]\le\frac{A_\nu(a,b)}{G_\nu(a,b)}. \tag{3.14}$$

From the first inequality in (3.12) we have

$$\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\exp\left[-\frac{1}{2m^2}R\left(\frac{1}{2}-r\right)(b-a)^2\right]\le\frac{A_\nu(a,b)}{G_\nu(a,b)} \tag{3.15}$$

while from the second inequality in (3.12) we have

$$\frac{A_\nu(a,b)}{G_\nu(a,b)}\le\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\exp\left[-\frac{1}{2M^2}R\left(\frac{1}{2}-r\right)(b-a)^2\right]. \tag{3.16}$$

In conclusion, from (3.13)-(3.16) we have the following result:

$$\begin{aligned}&\max\left\{\left(\frac{A(a,b)}{G(a,b)}\right)^{2r}\exp\left[\frac{1}{2M^2}r\left(R-\frac{1}{2}\right)(b-a)^2\right],\right.\\&\left.\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\exp\left[-\frac{1}{2m^2}R\left(\frac{1}{2}-r\right)(b-a)^2\right]\right\}\\&\le\frac{A_\nu(a,b)}{G_\nu(a,b)}\\&\le\min\left\{\left(\frac{A(a,b)}{G(a,b)}\right)^{2r}\exp\left(\frac{1}{2m^2}r\left(R-\frac{1}{2}\right)(b-a)^2\right),\right.\\&\left.\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\exp\left[-\frac{1}{2M^2}R\left(\frac{1}{2}-r\right)(b-a)^2\right]\right\}\end{aligned} \tag{3.17}$$

for any $a,b\in[m,M]$ and $\nu\in[0,1]$.

We need the following lemma [4]:

Lemma 3. *If the function $f:I\subset\mathbb{R}\to\mathbb{R}$ is a differentiable convex function on $\mathring{I}$, then for any a, $b\in\mathring{I}$ and $\nu\in[0,1]$ we have*

$$\begin{aligned}0&\le(1-\nu)f(a)+\nu f(b)-f((1-\nu)a+\nu b)\\&\le\nu(1-\nu)(b-a)\left[f'(b)-f'(a)\right].\end{aligned} \tag{3.18}$$

We have:

Theorem 3. *Let M, $m\in\mathbb{R}$ with $M>m$ and $\Phi\in\mathcal{B}((0,\infty),m,M,-\ln)$. Then for any $a,b>0$ and $\nu\in[0,1]$ we have*

$$\begin{aligned}&m\left[\nu(1-\nu)\frac{(b-a)^2}{ab}-\ln\frac{A_\nu(a,b)}{G_\nu(a,b)}\right]\\&\le\nu(1-\nu)(b-a)\left(\Phi'(b)-\Phi'(a)\right)\\&-\left[(1-\nu)\Phi(a)+\nu\Phi(b)-\Phi((1-\nu)a+\nu b)\right]\\&\le M\left[\nu(1-\nu)\frac{(b-a)^2}{ab}-\ln\frac{A_\nu(a,b)}{G_\nu(a,b)}\right].\end{aligned} \tag{3.19}$$

Proof. Since $\Phi \in \mathcal{B}((0,\infty), m, M, -\ln)$, then $f_m := \Phi + m\ln$ is convex and by (3.18) we have

$$\begin{aligned}
0 &\leq (1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a + \nu b) \\
&\quad - m\ln\frac{A_\nu(a,b)}{G_\nu(a,b)} \\
&\leq \nu(1-\nu)(b-a)\left[\Phi'(b) - \Phi'(a) + \frac{m}{b} - \frac{m}{a}\right] \\
&= \nu(1-\nu)(b-a)(\Phi'(b) - \Phi'(a)) - \frac{m}{ab}\nu(1-\nu)(b-a)^2
\end{aligned}$$

that is equivalent to

$$\begin{aligned}
& m\left[\nu(1-\nu)\frac{(b-a)^2}{ab} - \ln\frac{A_\nu(a,b)}{G_\nu(a,b)}\right] \\
&\leq \nu(1-\nu)(b-a)(\Phi'(b) - \Phi'(a)) \\
&\quad - [(1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a + \nu b)]
\end{aligned}$$

for any a, $b \in [m, M]$ and $\nu \in [0,1]$ and the first inequality in (3.19) is proved.

Since $\Phi \in \mathcal{B}((0,\infty), m, M, -\ln)$, then also $f_M := -\Phi - M\ln$ is convex and by (3.18) we have

$$\begin{aligned}
0 &\leq -(1-\nu)\Phi(a) - \nu\Phi(b) + f((1-\nu)a + \nu b) \\
&\quad + M\ln\frac{A_\nu(a,b)}{G_\nu(a,b)} \\
&\leq -\nu(1-\nu)(b-a)[\Phi'(b) - \Phi'(a)] + M\nu(1-\nu)\frac{(b-a)^2}{ab}
\end{aligned}$$

that is equivalent to

$$\begin{aligned}
& \nu(1-\nu)(b-a)[\Phi'(b) - \Phi'(a)] \\
&\quad - (1-\nu)\Phi(a) - \nu\Phi(b) + f((1-\nu)a + \nu b) \\
&\leq M\left[\nu(1-\nu)\frac{(b-a)^2}{ab} - \ln\frac{A_\nu(a,b)}{G_\nu(a,b)}\right]
\end{aligned}$$

for any $a, b \in [m, M]$ and $\nu \in [0,1]$ and the second inequality in (3.19) is proved. □

Corollary 3. *Let m, M with $M > m > 0$ and $p \in (-\infty, 0) \cup (1, \infty)$. Then for any $a, b \in [m, M]$ and $\nu \in [0,1]$ we have*

$$\begin{aligned}
& p(p-1)m_p\left[\nu(1-\nu)\frac{(b-a)^2}{ab} - \ln\frac{A_\nu(a,b)}{G_\nu(a,b)}\right] \\
&\leq p\nu(1-\nu)(b-a)\left(b^{p-1} - a^{p-1}\right) \\
&\quad - [(1-\nu)a^p + \nu b^p - ((1-\nu)a + \nu b)^p] \\
&\leq p(p-1)M_p\left[\nu(1-\nu)\frac{(b-a)^2}{ab} - \ln\frac{A_\nu(a,b)}{G_\nu(a,b)}\right],
\end{aligned} \tag{3.20}$$

where M_p and m_p are defined by (2.2).

The case $p=2$ is of interest. Observe that

$$2\nu(1-\nu)(b-a)^2 - \left[(1-\nu)a^2 + \nu b^2 - ((1-\nu)a + \nu b)^2\right]$$
$$= 2\nu(1-\nu)(b-a)^2 - \nu(1-\nu)(b-a)^2 = \nu(1-\nu)(b-a)^2$$

and by (3.20) we have

$$\begin{aligned} &2m^2\left[\nu(1-\nu)\frac{(b-a)^2}{ab} - \ln\frac{A_\nu(a,b)}{G_\nu(a,b)}\right] \\ &\le \nu(1-\nu)(b-a)^2 \\ &\le 2M^2\left[\nu(1-\nu)\frac{(b-a)^2}{ab} - \ln\frac{A_\nu(a,b)}{G_\nu(a,b)}\right], \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\exp\left(\nu(1-\nu)(b-a)^2\left(\frac{1}{ab} - \frac{1}{2m^2}\right)\right) \\ &\le \frac{A_\nu(a,b)}{G_\nu(a,b)} \\ &\le \exp\left(\nu(1-\nu)(b-a)^2\left(\frac{1}{ab} - \frac{1}{2M^2}\right)\right) \end{aligned} \tag{3.21}$$

for any a, $b \in [m,M]$ and $\nu \in [0,1]$.

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