Received 14/07/16

SOME QUANTUM INTEGRAL INEQUALITIES FOR MIDPOINT FORMULA

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Abstract. In this paper we establish some new quantum integral inequalities for midpoint formula which the left hand of q-Hermite-Hadamard Inequalty for convex functions.

1. Introduction

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0,1]$. Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard's inequality [1], due to its rich geometrical significance and applications, which is stated as follows:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$, be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both the inequalities hold in reversed direction if f is concave. Since its discovery, Hermite-Hadamard's inequality has been considered the most useful inequality in mathematical analysis. This inequality has been extended in a number of ways and a number of papers have been written, we refer to [5]-[9].

The main aim of this paper is to establish some new quantum integral inequalities for midpoint formula on convex functions. Many consequences of Hermite-Hadamard type inequalities are obtained as special cases when $q \to 1$.

2. Preliminaries

Let $J := [a,b] \subset \mathbb{R}$, $J^{\circ} := (a,b)$ be interval and $0 < q < 1$ be a constant. We define q-derivative of a function $f : J \to \mathbb{R}$ at a point $x \in J$ on $[a,b]$ as follows.

Definition 1. *Assume $f : J \to \mathbb{R}$ is a continuous function and let $x \in J$. Then the expression*

$$ {}_aD_q\, f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a, \quad {}_aD_q\, f(a) = \lim_{x \to a} {}_aD_q\, f(x) \tag{2.1}$$

is called the q-derivative on J of function f at x.

Key words and phrases. Hermite-Hadamard's inequalities, convex functions, q-integrals, Hölder's inequality for q-integrals.

2010 Mathematics Subject Classification 26D15, 26A51, 26A42.

We say that f is q -differentiable on J provided ${}_aD_q\ f(x)$ exists for all $x \in J$. Note that if $a = 0$ in (2.1), then ${}_0D_q\ f = D_q f$, where D_q is the well-known q-derivative of the function $f(x)$ defined by

$$D_q\ f(x) = \frac{f(x) - f(qx)}{(1-q)x}. \tag{2.2}$$

For more details, see [2].

Lemma 1. [3] *Let $\alpha \in \mathbb{R}$, then we have*

$$ {}_aD_q\ (x-a)^{\alpha} = \left(\frac{1-q^{\alpha}}{1-q}\right)(x-a)^{\alpha-1}. \tag{2.3}$$

Definition 2. *Assume $f : J \to \mathbb{R}$ is a continuous function. Then the q-integral on J is defined by*

$$\int_a^x f(t)\ {}_ad_qt\ = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \tag{2.4}$$

for $x \in J$. Moreover, if $c \in (a, x)$ then the definite q-integral on J is defined by

$$\begin{aligned}\int_c^x f(t)\ {}_ad_qt\ &= \int_a^x f(t)\ {}_ad_qt\ - \int_a^c f(t)\ {}_ad_qt \\ &= (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \\ &\quad - (1-q)(c-a)\sum_{n=0}^{\infty} q^n f(q^n c + (1-q^n)a).\end{aligned}$$

Note that if $a = 0$, then (2.4) reduces to the classical q-integral of a function $f(x)$, defined by $\int_0^x f(t)\ {}_0d_qt\ = (1-q)\,x\sum_{n=0}^{\infty} q^n f(q^n x)$ for $x \in [0, \infty)$. For more details, see [2].

Theorem 1. [3] *Let $f : J \to \mathbb{R}$ be a continuous function. Then we have*

(i) ${}_aD_q \int_a^x f(t)\ {}_ad_qt\ = f(x)$;

(ii) $\int_c^x {}_aD_q\ f(t)\ {}_ad_qt\ = f(x) - f(c)$ for $c \in (a, x)$.

Theorem 2. [3] *Assume $f, g : J \to \mathbb{R}$ are continuous functions. $\alpha \in \mathbb{R}$. Then, for $x \in J$,*

(i) $\int_a^x [f(t) + g(t)]\ {}_ad_qt\ = \int_a^x f(t)\ {}_ad_qt\ + \int_a^x g(t)\ {}_ad_qt$;

(ii) $\int_a^x (\alpha f)(t)\ {}_ad_qt\ = \alpha\int_a^x f(t)\ {}_ad_qt$;

$$(iii) \int_c^x f(t) \ {}_aD_q \ g(t) \ {}_ad_qt \ = (fg)|_c^x - \int_c^x g(qt+(1-q)a) \ {}_aD_q \ f(t) \ {}_ad_qt \ ,$$

for $c \in (a,x)$.

Lemma 2. [4] *For $\alpha \in \mathbb{R}\backslash\{-1\}$, the following formula holds:*

$$\int_a^x (t-a)^{\alpha} \ {}_ad_qt \ = \left(\frac{1-q}{1-q^{\alpha+1}}\right)(x-a)^{\alpha+1}. \tag{2.5}$$

Theorem 3. [4] *(q-Hermite-Hadamard)) Let $f : J \to \mathbb{R}$ be a convex continuous function on J and $0 < q < 1$. Then we have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t) \ {}_ad_qt \ \leq \frac{qf(a)+f(b)}{1+q}. \tag{2.6}$$

The integral inequalities play a fundamental role in the theory of differential equations. The study of the fractional q-integral inequalities is also of great importance. Integral inequalities have been studied extensively by several researchers either in classical analysis or in the quantum one; see [2]-[4], [10] and references cited therein.

3. Auxiliary Results

In this section, we present some auxiliary results which are throughout this article.

Lemma 3. *Let $f : J \subset \mathbb{R} \to \mathbb{R}$ be a continuous function and $0 < q < 1$. If ${}_aD_q \ f$ is an integrable function on J°, then the following equality holds:*

$$\begin{aligned} & \frac{2}{1+q}f\left(\frac{a+b}{2}\right) - \frac{1-q}{1+q}f(b) - \frac{1}{q(b-a)}\int_a^{a+q(b-a)} f(s) \ {}_ad_qs \\ = \ & \frac{2(b-a)}{1+q}\left[\int_0^{1/2} \frac{1+q}{2}s \ {}_aD_q \ f(sb+(1-s)a) \ {}_0d_qs \right. \\ & \left. + \int_{1/2}^1 \left(\frac{1+q}{2}s-1\right) \ {}_aD_q \ f(sb+(1-s)a) \ {}_0d_qs \right]. \end{aligned} \tag{3.1}$$

Proof. Using integration by parts, we have

$$\begin{aligned}
I_1 &= \int_0^{1/2} \frac{1+q}{2} s \ {}_aD_q \ f\left(sb+(1-s)a\right) \ {}_0d_qs \\
&= \left[\frac{1+q}{2} s \frac{f\left(sb+(1-s)a\right)}{b-a}\right]_0^{1/2} \\
&\quad -\frac{1}{b-a}\int_0^{1/2} f\left(q\left(sb+(1-s)a\right)+(1-q)a\right) \ {}_aD_q \ \left(\frac{1+q}{2}s\right) \ {}_0d_qs \\
&= \frac{1}{b-a}\left(\frac{1+q}{4} f\left(\frac{a+b}{2}\right) - \frac{1+q}{2}\int_0^{1/2} f\left(qsb+(1-qs)a\right) \ {}_0d_qs \right)
\end{aligned}$$

and similarly

$$\begin{aligned}
I_2 &= \int_{1/2}^{1} \left(\frac{1+q}{2}s-1\right) \ {}_aD_q \ f\left(sb+(1-s)a\right) \ {}_0d_qs \\
&= \frac{1}{b-a}\left(\frac{q-1}{2} f\left(b\right) - \frac{q-3}{4} f\left(\frac{a+b}{2}\right) - \frac{1+q}{2}\int_{1/2}^{1} f\left(qsb+(1-qs)a\right) \ {}_0d_qs \right).
\end{aligned}$$

Now summing I_1 and I_2, then we have

$$I_1+I_2 = \frac{1}{b-a}\left[f\left(\frac{a+b}{2}\right) + \frac{q-1}{2} f\left(b\right) - \frac{1+q}{2}\frac{1}{q\left(b-a\right)}\int_a^{a+q(b-a)} f\left(s\right) \ {}_ad_qs \right].$$

Therefore, we obtain the desired result in (3,1) as required. The proof is completed. □

Remark 1. *If $q\to 1$, then (3,1) reduces to*

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f\left(s\right) ds \\
&= \left(b-a\right)\left[\int_0^{1/2} sf'\left(sb+(1-s)a\right) ds + \int_{1/2}^{1} \left(s-1\right) f'\left(sb+(1-s)a\right) ds\right]
\end{aligned}$$

which is proved by Kirmaci in [*8, Lemma 2.1 page 138*].

Lemma 4. *Let $f : J \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on J°, $a, b \in J^\circ$ with $a < b$. If ${}_aD_q^2\ f$ integrable function on J°, then*

$$\begin{aligned}
&\frac{(b-a)^2}{2}\left(\int_0^1 m(s)\ {}_aD_q^2\ f(sb+(1-s)a)\ {}_0d_qs\right) \qquad (3.2)\\
&= \frac{(1-q)^2(b-a)}{8}\ {}_aD_q\ f(b) + \frac{(1+q)(1-q)(3+q)}{8}f(b)\\
&\quad -\frac{(1-q)(b-a)}{4}\ {}_aD_q\ f\left(\frac{a+b}{2}\right) - \frac{(1+q)}{2}f\left(\frac{a+b}{2}\right)\\
&\quad +\frac{(1+q)^3}{8(b-a)}\int_a^b f(x)\ {}_ad_qx
\end{aligned}$$

where

$$m(s) = \begin{cases} \left(\frac{1+q}{2}s\right)^2 & , s \in \left[0, \frac{1}{2}\right) \\ \left(1 - \frac{1+q}{2}s\right)^2 & , s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Proof. Clearly to write that

$$\begin{aligned}
&\int_0^1 m(s)\ {}_aD_q^2\ f(sb+(1-s)a)\ {}_0d_qs\\
&= \int_0^{1/2}\left(\frac{1+q}{2}s\right)^2\ {}_aD_q^2\ f(sb+(1-s)a)\ {}_0d_qs\\
&\quad +\int_{1/2}^1\left(1-\frac{1+q}{2}s\right)^2\ {}_aD_q^2\ f(sb+(1-s)a)\ {}_0d_qs\\
&= J_1 + J_2.
\end{aligned}$$

Using twice integration by parts, it follows that

$$\begin{aligned}
J_1 &= \int_0^{1/2}\left(\frac{1+q}{2}s\right)^2\ {}_aD_q^2\ f(sb+(1-s)a)\ {}_0d_qs\\
&= \left[\left(\frac{1+q}{2}s\right)^2\frac{{}_aD_q\ f(sb+(1-s)a)}{b-a}\right]_0^{1/2} - \frac{(1+q)^3}{4(b-a)}\int_0^{1/2} s\ {}_aD_q\ f(sb+(1-s)a)\ {}_0d_qs\\
&= \frac{(1+q)^2}{16(b-a)}\ {}_aD_q\ f\left(\frac{a+b}{2}\right) - \frac{(1+q)^3}{8(b-a)^2}f\left(\frac{a+b}{2}\right) + \frac{(1+q)^3}{4(b-a)^2}\int_0^{1/2} f(sb+(1-s)a)\ {}_0d_qs.
\end{aligned}$$

Using tha change of the variable $x = sb + (1-s)a$ for $s \in [0,1]$, which gives

$$J_1 = \frac{(1+q)^2}{16(b-a)} \; {}_aD_q \; f\left(\frac{a+b}{2}\right) - \frac{(1+q)^3}{8(b-a)^2} f\left(\frac{a+b}{2}\right) + \frac{(1+q)^3}{4(b-a)^3} \int_a^{(a+b)/2} f(x) \; {}_ad_qx \; . \tag{3.3}$$

Similarly, we can show that

$$\begin{aligned} J_2 &= \int_{1/2}^{1} \left(1 - \frac{1+q}{2}s\right)^2 \; {}_aD_q^2 \; f(sb + (1-s)a) \; {}_0d_qs \\ &= \frac{(1-q)^2}{4(b-a)} \; {}_aD_q \; f(b) - \frac{(3-q)^2}{16(b-a)} \; {}_aD_q \; f\left(\frac{a+b}{2}\right) - \frac{(1+q)(q-1)(3+q)}{4(b-a)^2} f(b) \\ &\quad + \frac{(1+q)\left(q^2+2q-7\right)}{8(b-a)^2} f\left(\frac{a+b}{2}\right) + \frac{(1+q)^3}{4(b-a)^2} \int_{1/2}^{1} f(sb + (1-s)a) \; {}_0d_qs. \end{aligned}$$

Using tha change of the variable $x = sb + (1-s)a$ for $s \in [0,1]$, which gives

$$\begin{aligned} J_2 &= \frac{(1-q)^2}{4(b-a)} \; {}_aD_q \; f(b) - \frac{(3-q)^2}{16(b-a)} \; {}_aD_q \; f\left(\frac{a+b}{2}\right) \\ &\quad - \frac{(1+q)(q-1)(3+q)}{4(b-a)^2} f(b) \qquad (3.4) \\ &\quad + \frac{(1+q)\left(q^2+2q-7\right)}{8(b-a)^2} f\left(\frac{a+b}{2}\right) + \frac{(1+q)^3}{4(b-a)^3} \int_{(a+b)/2}^{b} f(x) \; {}_ad_qx \; . \end{aligned}$$

Now, finally summing (3.3)-(3.4) and multiplying the both sides by $\frac{(b-a)^2}{2}$, we obtain

$$\begin{aligned} \frac{(b-a)^2}{2}(J_1 + J_2) &= \frac{(1-q)^2(b-a)}{8} \; {}_aD_q \; f(b) + \frac{(1+q)(1-q)(3+q)}{8} f(b) \\ &\quad - \frac{(1-q)(b-a)}{4} \; {}_aD_q \; f\left(\frac{a+b}{2}\right) - \frac{(1+q)}{2} f\left(\frac{a+b}{2}\right) \\ &\quad + \frac{(1+q)^3}{8(b-a)} \int_a^b f(x) \; {}_ad_qx \end{aligned}$$

which is required. □

Remark 2. *If $q \to 1$, then (3.2) reduces to*

$$\begin{aligned} &\frac{(b-a)^2}{2}\left(\int_0^{1/2} s^2 f''(sb + (1-s)a)\,ds + \int_{1/2}^{1} (1-s)^2 f''(sb + (1-s)a)\,ds\right) \\ &= \frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \end{aligned}$$

which is proved by Sarikaya et. al in [9, Lemma 2].

Lemma 5. *For $0<q<1$ and $p>1$*

$$\int_{1/2}^{1}\left|\frac{2}{1+q}-s\right|^{p}\ {}_0d_qs=\frac{1-q}{1-q^{p+1}}\left[\frac{(3-q)^{p+1}-(2-2q)^{p+1}}{2^{p+1}(1+q)^{p+1}}\right]. \tag{3.5}$$

Proof. Since $\frac{2}{1+q}\geq 1$ for $0<q<1$, then we have

$$\begin{aligned}\int_{1/2}^{1}\left|\frac{2}{1+q}-s\right|^{p}\ {}_0d_qs &= \int_{1/2}^{1}\left(\frac{2}{1+q}-s\right)^{p}\ {}_0d_qs\\ &= (-1)^p\int_{1/2}^{1}\left(s-\frac{2}{1+q}\right)^{p}\ {}_0d_qs\\ &= (-1)^p\left[\int_{0}^{1}\left(s-\frac{2}{1+q}\right)^{p}\ {}_0d_qs-\int_{0}^{1/2}\left(s-\frac{2}{1+q}\right)^{p}\ {}_0d_qs\right]\\ &= (-1)^p\frac{1-q}{1-q^{p+1}}\left[\left(s-\frac{2}{1+q}\right)^{p+1}\Bigg|_0^1-\left(s-\frac{2}{1+q}\right)^{p+1}\Bigg|_0^{1/2}\right]\\ &= \frac{1-q}{1-q^{p+1}}\left[\left(\frac{3-q}{2(1+q)}\right)^{p+1}-\left(\frac{1-q}{1+q}\right)^{p+1}\right]\\ &= \frac{1-q}{1-q^{p+1}}\left[\frac{(3-q)^{p+1}-(2-2q)^{p+1}}{2^{p+1}(1+q)^{p+1}}\right].\end{aligned}$$

The proof is completed. □

4. Main Results

In this section, we present some q-integral inequalities for midpoint type formula which the left hand side of q-Hermite- Hadamard type inequality for convex functions.

Theorem 4. *Let $f:J\to\mathbb{R}$ be a continuous function and $0<q<1$. If $\left|{}_aD_q\ f\right|$ is convex and integrable function on J°, then the following inequality holds.*

$$\begin{aligned}&\left|\frac{2}{1+q}f\left(\frac{a+b}{2}\right)-\frac{1-q}{1+q}f(b)-\frac{1}{q(b-a)}\int_a^{a+q(b-a)}f(s)\ {}_ad_qs\right| \\ &\leq \frac{(b-a)}{4(1+q)^2(1+q+q^2)}\\ &\quad\times\left[\left(2q^3+q^2+4q-1\right)\left|{}_aD_q\ f(a)\right|+\left(3q^2+3\right)\left|{}_aD_q\ f(b)\right|\right].\end{aligned} \tag{4.1}$$

Proof. Using (3.1) and the convexity of ${}_aD_q f$ on J°, we have

$$\begin{aligned}
&\left|\frac{2}{1+q}f\left(\frac{a+b}{2}\right)-\frac{1-q}{1+q}f(b)-\frac{1}{q(b-a)}\int_a^{a+q(b-a)} f(s)\ {}_ad_qs\right| \\
\leq\ & (b-a)\left[\int_0^{1/2} s\left|{}_aD_q f(sb+(1-s)a)\right|\ {}_0d_qs\right. \\
&\left.+\int_{1/2}^{1}\left|s-\frac{2}{1+q}\right|\left|{}_aD_q f(sb+(1-s)a)\right|\ {}_0d_qs\right] \\
\leq\ & (b-a)\int_0^{1/2}\left[s^2\left|{}_aD_q f(b)\right|+s(1-s)\left|{}_aD_q f(a)\right|\right]\ {}_0d_qs \\
&+(b-a)\int_{1/2}^{1}\left[\left|s-\frac{2}{1+q}\right|s\left|{}_aD_q f(b)\right|+\left|s-\frac{2}{1+q}\right|(1-s)\left|{}_aD_q f(a)\right|\right]\ {}_0d_qs \\
=\ & \frac{(b-a)}{4(1+q)^2(1+q+q^2)}\left[\left(2q^3+q^2+4q-1\right)\left|{}_aD_q f(a)\right|+\left(3q^2+3\right)\left|{}_aD_q f(b)\right|\right].
\end{aligned}$$

The proof is completed. ∎

Remark 3. *If $q\to 1$, then (4.1) reduces to*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(s)\ ds\right|\leq\frac{(b-a)}{4}\left(\frac{|f'(a)|+|f'(b)|}{2}\right)$$

which is proved by Kirmaci in [8, Teorem 2.2, page 138].

Theorem 5. *Let $f:J\to\mathbb{R}$ be a continuous function and $0<q<1$. If $\left|{}_aD_q f\right|^p$ is convex and integrable function on J° and $p>1$ then the following inequality holds:*

$$\left|\frac{2}{1+q}f\left(\frac{a+b}{2}\right)-\frac{1-q}{1+q}f(b)-\frac{1}{q(b-a)}\int_a^{a+q(b-a)} f(s)\ {}_ad_qs\right| \tag{4.2}$$

$$\begin{aligned}
\leq\ & \frac{(b-a)}{\left(\sum_{n=0}^{p}q^n\right)^{1/p}}\frac{1}{2^{\frac{3p-1}{p}}(1+q)^{\frac{p-1}{p}}} \\
&\times\left[\left(\left|{}_aD_q f(b)\right|^{\frac{p}{p-1}}+(2q+1)\left|{}_aD_q f(a)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right. \\
&\left.+\left\{\frac{(3-q)^{p+1}-(2-2q)^{p+1}}{(1+q)^{p+1}}\right\}^{1/p}\left(3\left|{}_aD_q f(b)\right|^{\frac{p}{p-1}}+(2q-1)\left|{}_aD_q f(a)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right].
\end{aligned}$$

Proof. From (3.1) and by Hölder inequality for $p>1$ we have

$$
\begin{aligned}
& \left| \frac{2}{1+q} f\left(\frac{a+b}{2}\right) - \frac{1-q}{1+q} f(b) - \frac{1}{q(b-a)} \int_a^{a+q(b-a)} f(s) \ {}_a d_q s \right| \\
\leq \ & (b-a) \left[\int_0^{1/2} s \left| \ {}_a D_q \ f(sb+(1-s)a) \right| \ {}_0 d_q s \right. \\
& \left. + \int_{1/2}^{1} \left| \frac{2}{1+q} - s \right| \left| \ {}_a D_q \ f(sb+(1-s)a) \right| \ {}_0 d_q s \right] \\
= \ & (b-a)(L_1 + L_2).
\end{aligned}
$$

By simple computation, it follows that

$$
\begin{aligned}
L_1 &= \int_0^{1/2} s \left| \ {}_a D_q \ f(sb+(1-s)a) \right| \ {}_0 d_q s \\
&\leq \left(\int_0^{1/2} s^p \ {}_0 d_q s \right)^{1/p} \left(\int_0^{1/2} \left| \ {}_a D_q \ f(sb+(1-s)a) \right|^{\frac{p}{p-1}} \ {}_0 d_q s \right)^{\frac{p-1}{p}} \\
&\leq \left(\frac{1-q}{1-q^{p+1}} \right)^{1/p} \frac{1}{2^{\frac{p+1}{p}}} \left(\frac{\left| \ {}_a D_q \ f(b) \right|^{\frac{p}{p-1}}}{4(1+q)} + \frac{(2q+1) \left| \ {}_a D_q \ f(a) \right|^{\frac{p}{p-1}}}{4(1+q)} \right)^{\frac{p-1}{p}} \\
&= \frac{1}{\left(\sum\limits_{n=0}^{p} q^n \right)^{1/p}} \frac{1}{2^{\frac{3p-1}{p}} (1+q)^{\frac{p-1}{p}}} \left(\left| \ {}_a D_q \ f(b) \right|^{\frac{p}{p-1}} + (2q+1) \left| \ {}_a D_q \ f(a) \right|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}}
\end{aligned}
$$

and by (3.5) with Hölder inequality

$$\begin{aligned}
L_2 &= \int_0^{1/2} \left|\frac{2}{1+q} - s\right| \left| \ {}_aD_q \ f(sb+(1-s)a)\right| \ {}_0d_qs \\
&\le \left(\int_{1/2}^{1} \left|\frac{2}{1+q} - s\right|^p \ {}_0d_qs\right)^{1/p} \left(\int_{1/2}^{1} \left| \ {}_aD_q \ f(sb+(1-s)a)\right|^{\frac{p}{p-1}} \ {}_0d_qs\right)^{\frac{p-1}{p}} \\
&\le \left(\frac{1-q}{1-q^{p+1}}\right)^{1/p} \left\{\frac{(3-q)^{p+1}-(2-2q)^{p+1}}{2^{p+1}(1+q)^{p+1}}\right\}^{1/p} \\
&\quad \times \left(\frac{3\left| \ {}_aD_q \ f(b)\right|^{\frac{p}{p-1}}}{4(1+q)} + \frac{(2q-1)\left| \ {}_aD_q \ f(a)\right|^{\frac{p}{p-1}}}{4(1+q)}\right)^{\frac{p-1}{p}} \\
&= \frac{1}{\left(\sum\limits_{n=0}^{p} q^n\right)^{1/p}} \frac{1}{2^{\frac{3p-1}{p}}(1+q)^{\frac{p-1}{p}}} \left\{\frac{(3-q)^{p+1}-(2-2q)^{p+1}}{(1+q)^{p+1}}\right\}^{1/p} \\
&\quad \times \left(3\left| \ {}_aD_q \ f(b)\right|^{\frac{p}{p-1}} + (2q-1)\left| \ {}_aD_q \ f(a)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}.
\end{aligned}$$

Summing $L_1 + L_2$, then we have

$$\begin{aligned}
&\left|\frac{2}{1+q} f\left(\frac{a+b}{2}\right) - \frac{1-q}{1+q} f(b) - \frac{1}{q(b-a)} \int_a^{a+q(b-a)} f(s) \ {}_ad_qs\right| \\
&= \frac{(b-a)}{\left(\sum\limits_{n=0}^{p} q^n\right)^{1/p}} \frac{1}{2^{\frac{3p-1}{p}}(1+q)^{\frac{p-1}{p}}} \left[\left(\left| \ {}_aD_q \ f(b)\right|^{\frac{p}{p-1}} + (2q+1)\left| \ {}_aD_q \ f(a)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right. \\
&\left. + \left\{\frac{(3-q)^{p+1}-(2-2q)^{p+1}}{(1+q)^{p+1}}\right\}^{1/p} \left(3\left| \ {}_aD_q \ f(b)\right|^{\frac{p}{p-1}} + (2q-1)\left| \ {}_aD_q \ f(a)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right].
\end{aligned}$$

The proof is completed. □

Remark 4. *If $q \to 1$, then (4.2) reduces to*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| \\
&\le \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p}\left[\left(|f'(a)|^{\frac{p}{p-1}} + 3|f'(b)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right. \\
&\left. + \left(3|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right]
\end{aligned}$$

which is proved by Kirmaci in [8, Teorem 2.3, page 139].

Theorem 6. *Let $f : J \to \mathbb{R}$ be a continuous function and $0 < q < 1$. If $\left| {}_aD_q\ f\right|^p$ is convex and integrable function on J° and $p > 1$ then the following inequality holds:*

$$\left| \frac{2}{1+q} f\left(\frac{a+b}{2}\right) - \frac{1-q}{1+q} f(b) - \frac{1}{q(b-a)} \int_a^{a+q(b-a)} f(s)\ {}_ad_q s \right| \tag{4.3}$$
$$\leq \frac{(b-a)}{\left(\sum_{n=0}^{p} q^n\right)^{1/p}} \frac{1}{2^{\frac{3p-1}{p}}(1+q)^{\frac{p-1}{p}}} \left[\left| {}_aD_q\ f(b)\right| + (2q+1)\left| {}_aD_q\ f(a)\right| \right.$$
$$\left. + \left\{ \frac{(3-q)^{p+1} - (2q-2)^{p+1}}{(1+q)^{p+1}} \right\}^{1/p} \left(3\left| {}_aD_q\ f(b)\right| + (2q-1)\left| {}_aD_q\ f(a)\right|\right) \right].$$

Proof. We consider inequality (4.2)

$$\left| \frac{2}{1+q} f\left(\frac{a+b}{2}\right) - \frac{1-q}{1+q} f(b) - \frac{1}{q(b-a)} \int_a^{a+q(b-a)} f(s)\ {}_ad_q s \right|$$
$$\leq \frac{(b-a)}{\left(\sum_{n=0}^{p} q^n\right)^{1/p}} \frac{1}{2^{\frac{3p-1}{p}}(1+q)^{\frac{p-1}{p}}}$$
$$\times \left[\left(\left| {}_aD_q\ f(b)\right|^{\frac{p}{p-1}} + (2q+1)\left| {}_aD_q\ f(a)\right|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \right.$$
$$\left. + \left\{ \frac{(3-q)^{p+1} - (2q-2)^{p+1}}{(1+q)^{p+1}} \right\}^{1/p} \left(3\left| {}_aD_q\ f(b)\right|^{\frac{p}{p-1}} + (2q-1)\left| {}_aD_q\ f(a)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \right]$$

Let $a_1 = (2q+1)\left| {}_aD_q\ f(a)\right|^{\frac{p}{p-1}}$, $b_1 = \left| {}_aD_q\ f(b)\right|^{\frac{p}{p-1}}$, $a_2 = (2q-1)\left| {}_aD_q\ f(a)\right|^{\frac{p}{p-1}}$, $b_2 = 3\left| {}_aD_q\ f(b)\right|^{\frac{p}{p-1}}$.

Here $0 < (p-1)/p < 1$, for $p > 1$. Using the fact that,

$$\sum_{k=0}^{n} (a_k + b_k)^s \leq \sum_{k=0}^{n} a_k^s + \sum_{k=0}^{n} b_k^s$$

for $0 < s \leq 1$, $a_1, a_2, ..., a_n \geq 0$, $b_1, b_2, ..., b_n \geq 0$ we obtain

$$\left| \frac{2}{1+q} f\left(\frac{a+b}{2}\right) - \frac{1-q}{1+q} f(b) - \frac{1}{q(b-a)} \int_a^{a+q(b-a)} f(s)\ {}_ad_q s \right|$$
$$\leq \frac{(b-a)}{\left(\sum_{n=0}^{p} q^n\right)^{1/p}} \frac{1}{2^{\frac{3p-1}{p}}(1+q)^{\frac{p-1}{p}}} \times \left[\left| {}_aD_q\ f(b)\right| + (2q+1)\left| {}_aD_q\ f(a)\right| \right.$$
$$\left. + \left\{ \frac{(3-q)^{p+1} - (2q-2)^{p+1}}{(1+q)^{p+1}} \right\}^{1/p} \left(3\left| {}_aD_q\ f(b)\right| + (2q-1)\left| {}_aD_q\ f(a)\right|\right) \right].$$

The proof is completed. □

Remark 5. *If $q \to 1$, then (4.3) reduces to*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| \leq \frac{b-a}{2}\left(\frac{4}{p+1}\right)^{1/p}\left[\frac{|f'(a)|+|f'(b)|}{2}\right].$$

which is proved by Kirmaci in [8, Teorem 2.4, page 140].

Theorem 7. *Let $f : J \to \mathbb{R}$ be a continuous function and $0 < q < 1$. If $\left| {}_aD_q \; f\right|^p$ is convex and integrable function on J° and $p > 1$, then the following inequality holds:*

$$\begin{aligned}
(4.4) \quad & \left| \frac{2}{1+q} f\left(\frac{a+b}{2}\right) - \frac{1-q}{1+q} f(b) - \frac{1}{q(b-a)} \int_a^{a+q(b-a)} f(s) \; {}_ad_qs \right| \\
\leq \; & \frac{(b-a)}{4(1+q)\left[2\left(1+q+q^2\right)(1+q)\right]^{1/p}} \\
& \times \left[\left(\left(1+q+2q^2\right)(1+q) \left| {}_aD_q \; f(a)\right|^p + (1+q)^2 \left| {}_aD_q \; f(b)\right|^p \right)^{1/p} \right. \\
& \left. + \left((2q-1)\left(3+q^2\right) \left| {}_aD_q \; f(a)\right|^p + \left(5-2q+5q^2\right) \left| {}_aD_q \; f(b)\right|^p \right)^{1/p} \right].
\end{aligned}$$

Proof. From (3.1) and $\frac{1}{p} + \frac{1}{r} = 1$ $(p > 1)$, we have

$$\begin{aligned}
& \left| \frac{2}{1+q} f\left(\frac{a+b}{2}\right) - \frac{1-q}{1+q} f(b) - \frac{1}{q(b-a)} \int_a^{a+q(b-a)} f(s) \; {}_ad_qs \right| \\
\leq \; & \frac{2(b-a)}{1+q} \left[\int_0^{1/2} \frac{1+q}{2} s \left| {}_aD_q \; f(sb+(1-s)a)\right| \; {}_0d_qs \right. \\
& \left. + \int_{1/2}^{1} \left| \frac{1+q}{2} s - 1 \right| \left| {}_aD_q \; f(sb+(1-s)a)\right| \; {}_0d_qs \right]
\end{aligned}$$

and by Hölder inequality we have

$$\begin{aligned}
&\left|\frac{2}{1+q}f\left(\frac{a+b}{2}\right)-\frac{1-q}{1+q}f(b)-\frac{1}{q(b-a)}\int_a^{a+q(b-a)} f(s)\ {}_ad_qs\right| \\
\leq\ & (b-a)\left[\left(\int_0^{1/2} s\ {}_0d_qs\right)^{1/r}\left(\int_0^{1/2} s\left|\ {}_aD_q\ f(sb+(1-s)a)\right|^p\ {}_0d_qs\right)^{1/p}\right. \\
&\left.+\left(\int_{1/2}^{1}\left|s-\frac{2}{1+q}\right|\ {}_0d_qs\right)^{1/r}\left(\int_{1/2}^{1}\left|s-\frac{2}{1+q}\right|\left|\ {}_aD_q\ f(sb+(1-s)a)\right|^p\ {}_0d_qs\right)^{1/p}\right] \\
\leq\ & \frac{(b-a)}{4^{1/r}(1+q)^{1/r}}\left[\left(\int_0^{1/2}\left(s^2\left|\ {}_aD_q\ f(b)\right|^p+\left(s-s^2\right)\left|\ {}_aD_q\ f(a)\right|^p\right)\ {}_0d_qs\right)^{1/p}\right. \\
&\left.+\left(\int_{1/2}^{1}\left\{\left(\frac{2}{1+q}s-s^2\right)\left|\ {}_aD_q\ f(b)\right|^p+\left(\frac{2}{1+q}-\frac{3+q}{1+q}s+s^2\right)\left|\ {}_aD_q\ f(a)\right|^p\right\}\ {}_0d_qs\right)^{1/p}\right] \\
=\ & \frac{(b-a)}{4^{1/r}(1+q)^{1/r}}\left[\left(\frac{\left|\ {}_aD_q\ f(b)\right|^p}{8(1+q+q^2)}+\frac{\left(1+q+2q^2\right)\left|\ {}_aD_q\ f(a)\right|^p}{8(1+q+q^2)(1+q)}\right)^{1/p}\right. \\
&\left.+\left(\frac{\left(5-2q+5q^2\right)\left|\ {}_aD_q\ f(b)\right|^p}{8(1+q+q^2)(1+q)^2}+\frac{(2q-1)\left(3+q^2\right)\left|\ {}_aD_q\ f(a)\right|^p}{8(1+q+q^2)(1+q)^2}\right)^{1/p}\right] \\
=\ & \frac{(b-a)}{4(1+q)}\frac{1}{\left[2(1+q+q^2)(1+q)\right]^{1/p}} \\
&\times\left[\left((1+q)^2\left|\ {}_aD_q\ f(b)\right|^p+\left(1+q+2q^2\right)(1+q)\left|\ {}_aD_q\ f(a)\right|^p\right)^{1/p}\right. \\
&\left.+\left(\left(5-2q+5q^2\right)\left|\ {}_aD_q\ f(b)\right|^p+(2q-1)\left(3+q^2\right)\left|\ {}_aD_q\ f(a)\right|^p\right)^{1/p}\right].
\end{aligned}$$

The proof is completed $\square$

Corollary 1. *If $q\to 1$, then (4.4) reduces to*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)}\int_a^b f(s)\,ds\right| \\
\leq\ & \frac{(b-a)}{8}\left[\left(\frac{|f'(b)|^p+2|f'(a)|^p}{3}\right)^{1/p}+\left(\frac{2|f'(b)|^p+|f'(a)|^p}{3}\right)^{1/p}\right].
\end{aligned}$$

Theorem 8. *Let $f:J\to\mathbb{R}$ be a continuous function and $0<q<1$. If $\left|\ {}_aD_q\ f\right|$ is convex and integrable function on J° and $p>1$ then the following inequality* *holds:*

$$\left|\frac{2}{1+q}f\left(\frac{a+b}{2}\right)-\frac{1-q}{1+q}f(b)-\frac{1}{q(b-a)}\int\limits_{a}^{a+q(b-a)}f(s)\ {}_{a}d_{q}s\right| \tag{4.5}$$
$$\leq \frac{(b-a)}{4(1+q)}\frac{\left[\left(-2+8q+2q^{2}+4q^{3}\right)\left|\ {}_{a}D_{q}\ f(a)\right|+6\left(1+q^{2}\right)\left|\ {}_{a}D_{q}\ f(b)\right|\right]}{\left[2\left(1+q+q^{2}\right)(1+q)\right]^{1/p}}$$

Proof. We consider inequality (4.4)

$$\left|\frac{2}{1+q}f\left(\frac{a+b}{2}\right)-\frac{1-q}{1+q}f(b)-\frac{1}{q(b-a)}\int\limits_{a}^{a+q(b-a)}f(s)\ {}_{a}d_{q}s\right|$$
$$\leq \frac{(b-a)}{4(1+q)}\frac{1}{\left[2\left(1+q+q^{2}\right)(1+q)\right]^{1/p}}$$
$$\times\left[\left(\left(1+q+2q^{2}\right)(1+q)\left|\ {}_{a}D_{q}\ f(a)\right|^{p}+(1+q)^{2}\left|\ {}_{a}D_{q}\ f(b)\right|^{p}\right)^{1/p}\right.$$
$$\left.+\left((2q-1)\left(3+q^{2}\right)\left|\ {}_{a}D_{q}\ f(a)\right|^{p}+\left(5-2q+5q^{2}\right)\left|\ {}_{a}D_{q}\ f(b)\right|^{p}\right)^{1/p}\right]$$

Let

$$a_{1}=\left(1+q+2q^{2}\right)(1+q)\left|\ {}_{a}D_{q}\ f(a)\right|^{p},b_{1}=(1+q)^{2}\left|\ {}_{a}D_{q}\ f(b)\right|^{p},$$

$$a_{2}=(2q-1)\left(3+q^{2}\right)\left|\ {}_{a}D_{q}\ f(a)\right|^{p},b_{2}=\left(5-2q+5q^{2}\right)\left|\ {}_{a}D_{q}\ f(b)\right|^{p}.$$

Here $0<(p-1)/p<1$, for $p>1$. Using the fact that,

$$\sum_{k=0}^{n}(a_{k}+b_{k})^{s}\leq\sum_{k=0}^{n}a_{k}^{s}+\sum_{k=0}^{n}b_{k}^{s}$$

for $0<s\leq 1$, $a_{1},a_{2},...,a_{n}\geq 0$, $b_{1},b_{2},...,b_{n}\geq 0$, we obtain

$$\left|\frac{2}{1+q}f\left(\frac{a+b}{2}\right)-\frac{1-q}{1+q}f(b)-\frac{1}{q(b-a)}\int\limits_{a}^{a+q(b-a)}f(s)\ {}_{a}d_{q}s\right| \tag{4.6}$$
$$\leq \frac{(b-a)}{4(1+q)}\frac{\left[\left(-2+8q+2q^{2}+4q^{3}\right)\left|\ {}_{a}D_{q}\ f(a)\right|+6\left(1+q^{2}\right)\left|\ {}_{a}D_{q}\ f(b)\right|\right]}{\left[2\left(1+q+q^{2}\right)(1+q)\right]^{1/p}}$$

which is required. □

Corollary 2. *If $q\to 1$, then (4.5) reduces to*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int\limits_{a}^{b}f(s)\,ds\right|\leq\frac{(b-a)}{12^{\frac{1-p}{p}}}\left(\frac{|f'(a)|+|f'(b)|}{8}\right).$$

Proof.

Theorem 9. *Let $f : J \to \mathbb{R}$ be a continuous function and $0 < q < 1$. If $\left| {}_aD_q^2\ f \right|$ is convex function. Then the following inequality holds:*

$$\begin{aligned}
&\left| \frac{(1-q)^2 (b-a)}{8}\ {}_aD_q\ f(b) + \frac{(1+q)(1-q)(3+q)}{8} f(b) \right. \\
&-\frac{(1-q)(b-a)}{4}\ {}_aD_q\ f\left(\frac{a+b}{2}\right) - \frac{(1+q)}{2} f\left(\frac{a+b}{2}\right) \\
&\left. +\frac{(1+q)^3}{8(b-a)} \int_a^b f(x)\ {}_ad_qx \right| \\
\leq\ & \left| {}_aD_q^2\ f(b) \right| \left(\frac{3}{4(1+q)} - \frac{7(1+q)}{8(1+q+q^2)} + \frac{(1+q)^2}{4(1+q+q^2+q^3)} \right) \\
&+\left| {}_aD_q^2\ f(a) \right| \left(\frac{1}{2} - \frac{3(2+q)}{4(1+q)} + \frac{(1+q)(9+2q)}{8(1+q+q^2)} - \frac{(1+q)^2}{4(1+q+q^2+q^3)} \right).
\end{aligned} \tag{4.7}$$

□

Proof. From (3.2) it follows that

$$\begin{aligned}
&\left| \frac{(1-q)^2 (b-a)}{8}\ {}_aD_q\ f(b) + \frac{(1+q)(1-q)(3+q)}{8} f(b) \right. \\
&-\frac{(1-q)(b-a)}{4}\ {}_aD_q\ f\left(\frac{a+b}{2}\right) - \frac{(1+q)}{2} f\left(\frac{a+b}{2}\right) \\
&\left. +\frac{(1+q)^3}{8(b-a)} \int_a^b f(x)\ {}_ad_qx \right| \\
\leq\ & \frac{(b-a)^2}{2} \left(\int_0^1 |m(s)| \left| {}_aD_q^2\ f(sb+(1-s)a) \right|\ {}_0d_qs \right).
\end{aligned}$$

Clearly write that

$$\begin{aligned}
&\int_0^1 |m(s)| \left| {}_aD_q^2\ f(sb+(1-s)a) \right|\ {}_0d_qs \\
=\ & \int_0^{1/2} \left(\frac{1+q}{2}s\right) |^2\ {}_aD_q^2\ f(sb+(1-s)a) \Big|\ {}_0d_qs \\
&+\int_{1/2}^1 \left(1-\frac{1+q}{2}s\right)^2 \left| {}_aD_q^2\ f(sb+(1-s)a) \right|\ {}_0d_qs \\
=\ & l_1 + l_2.
\end{aligned} \tag{4.8}$$

From (4.8) and using convexity of $\left| {}_aD_q^2 \ f\right|$, it follows that

$$l_1 = \int_0^{1/2} \left(\frac{1+q}{2}s\right)^2 \left| {}_aD_q^2 \ f\left(sb+(1-s)a\right)\right| \ {}_0d_qs \tag{4.9}$$

$$\begin{aligned}
&\leq \frac{(1+q)^2}{4}\left[\left| {}_aD_q^2 \ f(b)\right| \int_0^{1/2} s^3 \ {}_0d_qs + \left| {}_aD_q^2 \ f(a)\right| \int_0^{1/2}\left(s^2-s^3\right) \ {}_0d_qs\right] \\
&\leq \frac{(1+q)^2}{4}\left[\left| {}_aD_q^2 \ f(b)\right| \left(\frac{1-q}{1-q^4}\right)\frac{1}{16} + \left| {}_aD_q^2 \ f(a)\right| \left(\frac{1-q}{1-q^3}\frac{1}{8} - \frac{1-q}{1-q^4}\frac{1}{16}\right)\right] \\
&\leq \frac{(1+q)^2}{4}\left[\frac{\left| {}_aD_q^2 \ f(b)\right|}{16\left(1+q+q^2+q^3\right)} + \frac{1+q+q^2+2q^3}{1+q+q^2}\frac{\left| {}_aD_q^2 \ f(a)\right|}{16\left(1+q+q^2+q^3\right)}\right]
\end{aligned}$$

and similarly

$$l_2 = \int_{1/2}^{1} \left(1-\frac{1+q}{2}s\right)^2 \left| {}_aD_q^2 \ f\left(sb+(1-s)a\right)\right| \ {}_0d_qs \tag{4.10}$$

$$\begin{aligned}
&= \int_{1/2}^{1} \left(1-(1+q)s+\frac{(1+q)^2}{4}s^2\right) \left| {}_aD_q^2 \ f\left(sb+(1-s)a\right)\right| \ {}_0d_qs \\
&\leq \left| {}_aD_q^2 \ f(b)\right| \int_{1/2}^{1} \left(s-(1+q)s^2+\frac{(1+q)^2}{4}s^3\right) \ {}_0d_qs \\
&\quad + \left| {}_aD_q^2 \ f(a)\right| \int_{1/2}^{1} \left(1-(2+q)s+\frac{(1+q)(5+q)}{4}s^2-\frac{(1+q)^2}{4}s^3\right) \ {}_0d_qs \\
&= \left| {}_aD_q^2 \ f(b)\right| \left(\frac{3}{4(1+q)}-\frac{7(1+q)}{8\left(1+q+q^2\right)}+\frac{15(1+q)^2}{64\left(1+q+q^2+q^3\right)}\right) \\
&\quad + \left| {}_aD_q^2 \ f(a)\right| \left(\frac{1}{2}-\frac{3(2+q)}{4(1+q)}+\frac{7(1+q)(5+q)}{32\left(1+q+q^2\right)}-\frac{15(1+q)^2}{64\left(1+q+q^2+q^3\right)}\right).
\end{aligned}$$

Finally summing (4.9) and (4.10)

$$\begin{aligned}
l_1+l_2 &= \left| {}_aD_q^2 \ f(b)\right| \left(\frac{3}{4(1+q)}-\frac{7(1+q)}{8\left(1+q+q^2\right)}+\frac{(1+q)^2}{4\left(1+q+q^2+q^3\right)}\right) \\
&\quad + \left| {}_aD_q^2 \ f(a)\right| \left(\frac{1}{2}-\frac{3(2+q)}{4(1+q)}+\frac{(1+q)(9+2q)}{8\left(1+q+q^2\right)}-\frac{(1+q)^2}{4\left(1+q+q^2+q^3\right)}\right).
\end{aligned}$$

The proof is completed. □

Remark 6. *If $q \to 1$, then (4,7) reduces to*

$$\left| \frac{1}{b-a} \int_a^b f(s)\, ds - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{24} \frac{|f''(b)| + |f''(a)|}{2}.$$

which is proved by Sarikaya et. al in [9, Teorem 5].

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