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ON GENERALIZED THE CONFORMABLE FRACTIONAL CALCULUS

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Abstract. In this paper, we generalize the conformable fractional derivative and integral and obtain several results such as the product rule, quotient rule, chain rule.

1. Introduction

An important point is that the fractional derivative at a point x is a local property only when a is an integer; in non-integer cases we cannot say that the fractional derivative at x of a function f depends only on values of f very near x, in the way that integer-power derivatives certainly do. Therefore it is expected that the theory involves some sort of boundary conditions, involving information on the function further out. To use a metaphor, the fractional derivative requires some peripheral vision. As far as the existence of such a theory is concerned, the foundations of the subject were laid by Liouville in a paper from 1832. The fractional derivative of a function to order a is often now defined by means of the Fourier or Mellin integral transforms. Various types of fractional derivatives were introduced: Riemann-Liouville, Caputo, Hadamard, Erdelyi-Kober, Grunwald-Letnikov, Marchaud and Riesz are just a few to name [7]-[10]. Recently a new local, limit-based definition of a so-called conformable derivative has been formulated in [1], [6] as follows

$$D_\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon}$$

provied the limits exits. Note that if f is fully differentiable at t, then the derivative is $D_\alpha(f)(t)=t^{1-\alpha}f'(t)$.

The purpose of this paper is to establish some

2. Definitions and properties of a-conformable fractional derivative

In this section, we give a new definition and obtain several results such as the product rule, quotient rule, chain rule. We start with the following definition which is a generalization of the conformable fractional derivative.

Definition 1 (a-Conformable fractional derivative)**.** *Given a function $f:[a,b]\to\mathbb{R}$ with $0\le a<b$. Then the "a−conformable fractional derivative" of f of order α is defined by*

$$D_\alpha^a(f)(t)=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon t^{-\alpha}(t-a)\right)-f(t)}{\varepsilon\left(1-at^{-\alpha}\right)}, \tag{2.1}$$

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for all $t > a,\ \alpha \in (0,1)$ *. If* f *is* $(\alpha, a)-$*differentiable in some* (a,b) *,* $a > 0,\ \lim\limits_{t\to a^+} D_\alpha^a(f)(t)$ *exist, then define*

$$D_\alpha^a(f)(a) = \lim_{t\to a^+} D_\alpha^a(f)(t). \tag{2.2}$$

If the a-conformable fractional derivative of f *of order* α *exists, then we simply say* f *is* $(\alpha, a)-$*differentiable.*

Theorem 1. *Let* $0 \le a < b$ *and* $\alpha \in (0,1]$*. If a function* $f : [a,b] \to \mathbb{R}$ *with is* $(\alpha, a)-$*differentiable at* $t_0 > a$*, then* f *is continuous at* t_0*.*

Proof. Since $f\left(t_0 + \varepsilon t_0^{-\alpha}(t_0 - a)\right) - f(t_0) = \frac{f\left(t_0+\varepsilon t_0^{-\alpha}(t_0-a)\right)-f(t_0)}{\varepsilon\left(1-at_0^{-\alpha}\right)}\varepsilon\left(1 - at_0^{-\alpha}\right)$, we have

$$\lim_{\varepsilon\to 0} f\left(t_0 + \varepsilon t_0^{-\alpha}(t_0 - a)\right) - f(t_0) = \lim_{\varepsilon\to 0} \frac{f\left(t_0 + \varepsilon t_0^{-\alpha}(t_0 - a)\right) - f(t_0)}{\varepsilon\left(1 - at_0^{-\alpha}\right)} \lim_{\varepsilon\to 0} \varepsilon\left(1 - at_0^{-\alpha}\right).$$

Let $h = \varepsilon t_0^{-\alpha}(t_0 - a)$. Then we get

$$\lim_{\varepsilon\to 0} f(t_0 + h) - f(t_0) = D_\alpha^a(f)(t_0).0,$$

which implies that f is continuous at t_0. This completes the proof. $\square$

Having these definidions in hand we can present the following properties for (α, a)-differentiable functions:

Theorem 2. *Let* $\alpha \in (0,1]$ *and* f, g *be* $(\alpha, a)-$*differentiable at a point* $t > a$*. Then*

i. $D_\alpha^a(\lambda_1 f + \lambda_2 g) = \lambda_1 D_\alpha^a(f) + \lambda_2 D_\alpha^a(g)$, for all $\lambda_1, \lambda_2 \in \mathbb{R}$,

ii. $D_\alpha^a(t^n) = \frac{nt^{n-1}(t-a)}{(t^\alpha - a)}$ for all $n \in \mathbb{R}$

iii. $D_\alpha^a(c) = 0$, for all constant functions $f(t) = c$,

iv. $D_\alpha^a(fg) = gD_\alpha^a(f) + fD_\alpha^a(g)$,

v. $D_\alpha^a\left(\dfrac{f}{g}\right) = \dfrac{fD_\alpha^a(g) - gD_\alpha^a(f)}{g^2}$,

vi. $D_\alpha^a(f \circ g) = f'(g(t))\, D_\alpha^a(g)(t)$ for f differentiable at $g(t)$,

vii. If, in addition, f is a differantiable, then $D_\alpha^a(f)(t) = \frac{(t-a)}{(t^\alpha - a)} f'(t)$.

Proof. Parts *i* and *iii* follow directly from the definition. Now, we will prove *ii*, *iv*, *vi* and *vii*. For fixed and $t > a$, we have

$$\begin{aligned} D_\alpha^a(f)(t) &= \lim_{\varepsilon\to 0} \frac{\left(t + \varepsilon t^{-\alpha}(t-a)\right)^n - t^n}{\varepsilon\left(1 - at^{-\alpha}\right)} \\ &= \lim_{\varepsilon\to 0} \frac{t^n + n\varepsilon t^{-\alpha}(t-a)\, t^{n-1} + O(\varepsilon^2) - t^n}{\varepsilon\left(1 - at^{-\alpha}\right)} \\ &= \frac{nt^{n-1}(t-a)}{(t^\alpha - a)}. \end{aligned}$$

This completes the proof ii. Then, we wiil prove iv. For this purpose, since f, g are (α, a) $-$differentiable at a point $t > a$, we obtain

$$\begin{aligned} D_\alpha^a (fg)(t) &= \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{-\alpha}(t-a))\, g(t+\varepsilon t^{-\alpha}(t-a)) - f(t)g(t)}{\varepsilon(1-at^{-\alpha})} \\ &= \lim_{\varepsilon\to 0}\left[\frac{f(t+\varepsilon t^{-\alpha}(t-a)) - f(t)}{\varepsilon(1-at^{-\alpha})} g\left(t+\varepsilon t^{-\alpha}(t-a)\right)\right] + f(t)\lim_{\varepsilon\to 0}\frac{g(t+\varepsilon t^{-\alpha}(t-a)) - g(t)}{\varepsilon(1-at^{-\alpha})} \\ &= D_\alpha^a (f)(t) \lim_{\varepsilon\to 0} g\left(t+\varepsilon t^{-\alpha}(t-a)\right) + f(t) D_\alpha^a (g)(t). \end{aligned}$$

Since g is continuous at t, the $\lim_{\varepsilon\to 0} g(t+\varepsilon t^{-\alpha}(t-a)) = g(t)$. This completes the proof of iv. The proof of the v is similar to iv. Now, we prove the result vi. If the function g is a constant in a neighborhood t_0, then $D_\alpha^a (f\circ g)(t_0) = 0$. On the other hand, we assume that g is nonconstant function in the neighborhood of t_0. In this case, we can find an $\varepsilon_0 > 0$ such that $g(t_1) \neq g(t_2)$ for any $t_1, t_2 \in (t_0 - \varepsilon_0, t_0 + \varepsilon_0)$. Thus, since the function g is continuous at t_0, for $t_0 > a$ we get

$$\begin{aligned} D_\alpha^a (f\circ g)(t_0) &= \lim_{\varepsilon\to 0} \frac{f\left(g\left(t_0+\varepsilon t_0^{-\alpha}(t_0-a)\right)\right) - f(g(t_0))}{\varepsilon\left(1-at_0^{-\alpha}\right)} \\ &= \lim_{\varepsilon\to 0} \frac{f\left(g\left(t_0+\varepsilon t_0^{-\alpha}(t_0-a)\right)\right) - f(g(t_0))}{g\left(t_0+\varepsilon t_0^{-\alpha}(t_0-a)\right) - g(t_0)} . \frac{g\left(t_0+\varepsilon t_0^{-\alpha}(t_0-a)\right) - g(t_0)}{\varepsilon\left(1-at_0^{-\alpha}\right)} \\ &= \lim_{\varepsilon_1\to 0} \frac{f(g(t_0)+\varepsilon_1) - f(g(t_0))}{\varepsilon_1} . \lim_{\varepsilon\to 0} \frac{g\left(t_0+\varepsilon t_0^{-\alpha}(t_0-a)\right) - g(t_0)}{\varepsilon\left(1-at_0^{-\alpha}\right)} \\ &= f'(g(t_0))\, D_\alpha^a (g)(t_0). \end{aligned}$$

This shows the Chain rule is proved.

To prove part vii, let $h = \varepsilon t^{-\alpha}(t-a)$ in Definition 1 and taking $\varepsilon = \frac{ht^\alpha}{(t-a)}$. Therefore, we get

$$\begin{aligned} D_\alpha^a (f)(t) &= \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{-\alpha}(t-a)) - f(t)}{\varepsilon(1-at^{-\alpha})} \\ &= \frac{(t-a)}{(t^\alpha - a)} \lim_{\varepsilon\to 0} \frac{f(t+h) - f(t)}{h} \\ &= \frac{(t-a)}{(t^\alpha - a)} f'(t) \end{aligned}$$

since, by assumptaion f is differentiable at $t > 0$. This completes the proof of the theorem. □

The following theorem lists (α, a)-fractional derivative of several familiar functions.

Theorem 3. *Let $\alpha \in (0,1]$, $t > a$ and $c, n \in \mathbb{R}$. Then we have the following results*

i. $D_\alpha^a (t^n) = \frac{nt^{n-1}(t-a)}{(t^\alpha - a)}$
ii. $D_\alpha^a (1) = 0$
iii. $D_\alpha^a (e^{ct}) = c\frac{(t-a)}{(t^\alpha - a)} e^{ct}$
iv. $D_\alpha^a (\sin ct) = c\frac{(t-a)}{(t^\alpha - a)} \cos ct$

v. $D_{\alpha}^{a}(\cos ct) = -c\frac{(t-a)}{(t^{\alpha}-a)}\sin ct$

vi. $D_{\alpha}^{a}\left(\frac{t^{\alpha}}{\alpha}\right) = \frac{t^{\alpha-1}(t-a)}{(t^{\alpha}-a)}$.

It is easy to see from part *vi* of Theorem 2 that we have rather unusual results given in the following theorem.

Theorem 4. *Let $\alpha \in (0,1]$ and $t > a$. Then we have the following results*

i. $D_{\alpha}^{a}\left(\sin\frac{t^{\alpha}}{\alpha}\right) = \frac{t^{\alpha-1}(t-a)}{(t^{\alpha}-a)}\cos\frac{t^{\alpha}}{\alpha}$

v. $D_{\alpha}^{a}\left(\cos\frac{t^{\alpha}}{\alpha}\right) = -\frac{t^{\alpha-1}(t-a)}{(t^{\alpha}-a)}\sin\frac{t^{\alpha}}{\alpha}$

vi. $D_{\alpha}^{a}\left(e^{\frac{t^{\alpha}}{\alpha}}\right) = \frac{t^{\alpha-1}(t-a)}{(t^{\alpha}-a)}e^{\frac{t^{\alpha}}{\alpha}}$.

We begin by proving the Rolle's theorem, the mean value theorem, and the extended mean value theorem for a-conformable fractional differentiable functions.

Theorem 5 (Rolle's theorem)**.** *Let $\alpha \in (0,1]$ and $0 \le a < b$. If $f : [a,b] \to \mathbb{R}$ be a given function that satisfies*

i. f is continuous on $[a,b]$

ii. f is (α,a)-differentiable for some $\alpha \in (0,1)$

iii. $f(a) = f(b)$.

Then , there exist $c \in (a,b)$, such that $D_{\alpha}^{a}(f)(c) = 0$.

Proof. The proof is done in a similar way in [6]. □

Theorem 6 (Mean Value theorem)**.** *Let $\alpha \in (0,1]$ and $0 \le a < b$. If $f : [a,b] \to \mathbb{R}$ be a given function that satisfies*

i. f is continuous on $[a,b]$

ii. f is (α,a)-differentiable for some $\alpha \in (0,1)$.

Then , there exist $c \in (a,b)$, such that

$$D_{\alpha}^{a}(f)(c) = \frac{f(b)-f(a)}{\frac{b^{\alpha}}{\alpha}-\frac{a^{\alpha}}{\alpha}}\frac{c^{\alpha-1}(c-a)}{(c^{\alpha}-a)}.$$

Proof. Let's now define a new function, as follow

$$g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{\frac{b^{\alpha}}{\alpha}-\frac{a^{\alpha}}{\alpha}}\left(\frac{x^{\alpha}}{\alpha}-\frac{a^{\alpha}}{\alpha}\right).$$

Since g is continuous on $[a,b]$, (α,a)-differentiable, and $g(a) = 0 = g(b)$, then by Rolle's theorem, there exist a $c \in (a,b)$ such that $D_{\alpha}^{a}(g)(c) = 0$ for some $\alpha \in (0,1)$. Using the fact that $D_{\alpha}^{a}\left(\frac{t^{\alpha}}{\alpha}\right) = \frac{t^{\alpha-1}(t-a)}{(t^{\alpha}-a)}$, reach to the desired result. □

Theorem 7 (Extended Mean Value theorem)**.** *Let $\alpha \in (0,1]$ and $0 \le a < b$. If $f : [a,b] \to \mathbb{R}$ be a given function that satisfies*

i. f,g is continuous on $[a,b]$

ii. f,g is (α,a)-differentiable for some $\alpha \in (0,1)$

iii. $D_{\alpha}^{a}(g)(t) \neq 0$ for all $t \in (a,b)$.

Then , there exist $c \in (a,b)$ such that

$$\frac{D_{\alpha}^{a}(f)(c)}{D_{\alpha}^{a}(g)(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Remark 1. *If $g(t) = \frac{t^{\alpha}}{\alpha}$, then this is just the statement of the Mean Value Theorem for a-conformable fractional differentiable functions.*

Proof. Let's now define a new function, as follow

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} \left(g(x) - g(a)\right).$$

Then the function F satisfies the conditions of Rolle's theorem. Thus, there exist a $c \in (a, b)$ such that $D_\alpha^a (F)(c) = 0$ for some $\alpha \in (0, 1)$. Using the linearity of D_α^a, we have

$$0 = D_\alpha^a (F)(c) = D_\alpha^a (f)(c) - \frac{f(b) - f(a)}{g(b) - g(a)} D_\alpha^a (g)(c).$$

Therefore, we get desired result. □

3. Definitions and properties of (α, a)-conformable fractional integral

Now we introduce the (α, a)-conformable fractional integral (or (α, a)-fractional integral) as follows:

Definition 2 ((α, a)-Conformable fractional integral)**.** *Let $\alpha \in (0, 1]$ and $0 \le a < b$. A function $f : [a, b] \to \mathbb{R}$ is (α, a)-fractional integrable on $[a, b]$ if the integral*

$$\int_a^b f(x)\, d_\alpha^a x := \int_a^b f(x) \frac{(x^\alpha - a)}{x - a} dx \tag{3.1}$$

exists and is finite. All α-fractional integrable on $[a, b]$ is indicated by $L^1_{(\alpha,a)}([a, b])$.

Remark 2.

$$I_{(\alpha,a)}^a (f)(t) = \int_a^t f(x) \frac{(x^\alpha - a)}{x - a} dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1]$.

Theorem 8 (Inverse property)**.** *Let $\alpha \in (0, 1]$ and $0 \le a < b$. Also, let $f : (a, b) \to \mathbb{R}$ be a continuous function such that $I_{(\alpha,a)}^a (f)$ exists. Then, for all $t > a$ we have*

$$D_\alpha^a \left(I_{(\alpha,a)}^a (f) \right)(t) = f(t).$$

Proof. Since f is continuous, then $I_{(\alpha,a)}^a (f)$ is clearly differentiable. Therefore, by using *vii* of Theorem 2, we get

$$\begin{aligned} D_\alpha^a \left(I_{(\alpha,a)}^a (f) \right)(t) &= \frac{(t - a)}{(t^\alpha - a)} \frac{d}{dt} I_{(\alpha,a)}^a (f)(t) \\ &= \frac{(t - a)}{(t^\alpha - a)} \frac{d}{dt} \left(\int_a^t f(x) \frac{(x^\alpha - a)}{x - a} dx \right) \\ &= \frac{(t - a)}{(t^\alpha - a)} f(t) \frac{(t^\alpha - a)}{t - a} = f(t). \end{aligned}$$

This completes the proof. □

Theorem 9. *Let $\alpha \in (0, 1]$ and $0 \le a < b$. Also, let $f : (a, b) \to \mathbb{R}$ be differentiable function. Then, for all $t > a$ we have*

$$I_{(\alpha,a)}^a \left(D_\alpha^a (f) \right)(t) = f(t) - f(a).$$

Proof. Since f is differentiable , then, by using vii of Theorem 2, we get

$$\begin{aligned} I_{(\alpha,a)}^{a}\left(D_{\alpha}^{a}\left(f\right)\right)\left(t\right) &= \int_{a}^{t}\frac{\left(x^{\alpha}-a\right)}{x-a}D_{\alpha}^{a}\left(f\right)\left(x\right)dx \\ &= \int_{a}^{t}\frac{\left(x^{\alpha}-a\right)}{x-a}f'\left(x\right)\frac{\left(x^{\alpha}-a\right)}{x-a}dx \\ &= f\left(t\right)-f\left(a\right). \end{aligned}$$

□

Theorem 10. *Let $\alpha \in (0,1]$ and $0 \leq a < b$. Also, let $f, g : [a,b] \to \mathbb{R}$ be continuous functions. Then*

i. $\int_a^b \left[f(x) + g(x)\right] d_\alpha^a x = \int_a^b f(x)\, d_\alpha^a x + \int_a^b g(x)\, d_\alpha^a x$
ii. $\int_a^b \lambda f(x)\, d_\alpha^a x = \lambda \int_a^b f(x)\, d_\alpha^a x$, $\lambda \in \mathbb{R}$
iii. $\int_a^b f(x)\, d_\alpha^a x = -\int_b^a f(x)\, d_\alpha^a x$
iv. $\int_a^b f(x)\, d_\alpha^a x = \int_a^c f(x)\, d_\alpha^a x + \int_c^b f(x)\, d_\alpha^a x$
v. $\int_a^a f(x)\, d_\alpha^a x = 0$
vi. if $f(x) \geq 0$ *for all* $x \in [a,b]$ *, then* $\int_a^b f(x)\, d_\alpha^a x \geq 0$
vii. $\left|\int_a^b f(x)\, d_\alpha^a x\right| \leq \int_a^b |f(x)|\, d_\alpha^a x$ *for* $x^\alpha > a$.

Proof. The relations follow from Definition 2 and Theorem 9, analogous properties of (α, a)-fractional integral, and the properties of section 2 for the a-conformable fractional derivative. □

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