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GRONWALL TYPE INEQUALITIES FOR CONFORMABLE FRACTIONAL INTEGRALS

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Abstract. In this paper, some new generalized Gronwall-type inequalities are investigated for conformable differential equations. The established results are extensions of some existing Gronwall-type inequalities in the literature.

1. Introduction

Fractional Calculus is a generalization of ordinary differentiation and integration to arbitrary (non-integer) order. The subject is as old as the calculus of differentiation and goes back to times when Leibniz, Gauss, and Newton invented this kind of calculation. During three centuries, the theory of fractional calculus developed as a pure theoretical field, useful only for mathematicians, we refer to [10], see also [11]. Recently a new local, limit-based definition of a conformable derivative has been formulated [1], [4], [8], with several follow-up papers [2], [3], [5]-[9]. In this paper, we use the Katugampola derivative formulation of conformable derivative of order for $\alpha \in (0,1]$and $t \in [0,\infty)$ given by

$$D^{\alpha}(f)(t)=\lim_{\varepsilon\to 0}\frac{f\left(te^{\varepsilon t^{-\alpha}}\right)-f(t)}{\varepsilon},\ D^{\alpha}(f)(0)=\lim_{t\to 0}D^{\alpha}(f)(t), \tag{1.1}$$

provided the limits exist (for detail see, [8]). If f is fully differentiable at t, then

$$D^{\alpha}(f)(t)=t^{1-\alpha}\frac{df}{dt}(t). \tag{1.2}$$

A function f is $\alpha-$differentiable at a point $t\geq 0$ if the limit in (1.1) exists and is finite. This definition yields the following results;

Theorem 1. *Let $\alpha\in(0,1]$ and f,g be $\alpha-$differentiable at a point $t>0$. Then*
i. $D^{\alpha}(af+bg)=aD^{\alpha}(f)+bD^{\alpha}(g)$, *for all* $a,b\in\mathbb{R}$,
ii. $D^{\alpha}(\lambda)=0$, *for all constant functions* $f(t)=\lambda$,
iii. $D^{\alpha}(fg)=fD^{\alpha}(g)+gD^{\alpha}(f)$,
iv. $D^{\alpha}\left(\frac{f}{g}\right)=\frac{fD^{\alpha}(g)-gD^{\alpha}(f)}{g^{2}}$
v. $D^{\alpha}(t^{n})=nt^{n-\alpha}$ *for all* $n\in\mathbb{R}$
vi. $D^{\alpha}(f\circ g)(t)=f'(g(t))D^{\alpha}(g)(t)$ *for f is differentiable at $g(t)$.*

Definition 1 (Conformable fractional integral)**.** *Let $\alpha\in(0,1]$ and $0\leq a<b$. A function $f:[a,b]\to\mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral*

$$\int_a^b f(x)\,d_{\alpha}x:=\int_a^b f(x)\,x^{\alpha-1}dx$$

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exists and is finite. All α-fractional integrable on $[a,b]$ is indicated by $L_{\alpha}^{1}([a,b])$

Remark 1.

$$I_{\alpha}^{a}(f)(t)=I_{1}^{a}\left(t^{\alpha-1}f\right)=\int_{a}^{t}\frac{f(x)}{x^{1-\alpha}}dx,$$

where the integral is the usual Riemann improper integral, and $\alpha\in(0,1]$.

We will also use the following important results, which can be derived from the results above.

Lemma 1. *Let the conformable differential operator D^{α} be given as in (1.1), where $\alpha\in(0,1]$ and $t\geq 0$, and assume the functions f and g are α-differentiable as needed. Then*

i. $D^{\alpha}(\ln t)=t^{-\alpha}$ *for* $t>0$

ii. $D^{\alpha}\left[\int_{a}^{t}f(t,s)\,d_{\alpha}s\right]=f(t,t)+\int_{a}^{t}D^{\alpha}\left[f(t,s)\right]d_{\alpha}s$

iii. $\int_{a}^{b}f(x)\,D^{\alpha}(g)(x)\,d_{\alpha}x=\left.fg\right|_{a}^{b}-\int_{a}^{b}g(x)\,D^{\alpha}(f)(x)\,d_{\alpha}x.$

In this paper, some new generalized Gronwall-type inequalities are investigated for conformable differential equations. The established results are extensions of some existing Gronwall-type inequalities in the literature.

2. Main Results

Troughout this paper, all the functions which appear in the inequalities are assumed to be real-valued and all the integrals involved exist on the respective domains of their definitions, and $C(M,S)$ and $C^{1}(M,S)$denote the class of all continuous functions and the first order conformable derivative, respectively, defined on set M with range in the set S.

Firstly, we start with the following definition, which is a generalization of the limit definition of the derivative for the case of a function with many variables.

Definition 2. *Let f be a function with n variables $t_1,...,t_n$ and the conformable partial derivative of f of order $\alpha\in(0,1]$ in x_i is defined as follows*

$$\frac{\partial^{\alpha}}{\partial t_{i}^{\alpha}}f(t_{1},...,t_{n})=\lim_{\varepsilon\rightarrow 0}\frac{f(t_{1},...,t_{i-1},t_{i}e^{\varepsilon t_{i}^{-\alpha}},...,t_{n})-f(t_{1},...,t_{n})}{\varepsilon}. \tag{2.1}$$

The first result is the generalization of Theorem 2.10 of [3].

Theorem 2. *Assume that $f(t,s)$ is function for which $\partial_{t}^{\alpha}\left[\partial_{s}^{\beta}f(t,s)\right]$ and $\partial_{s}^{\beta}\left[\partial_{t}^{\alpha}f(t,s)\right]$ exist and are continuos over the domain $D\subset\mathbb{R}^{2}$, then*

$$\partial_{t}^{\alpha}\left[\partial_{s}^{\beta}f(t,s)\right]=\partial_{s}^{\beta}\left[\partial_{t}^{\alpha}f(t,s)\right]. \tag{2.2}$$

Proof. By using the (1.1), it follows that

$$\begin{aligned}\partial_{t}^{\alpha}\left[\partial_{s}^{\beta}f(t,s)\right] &= \partial_{t}^{\alpha}\left[\lim_{\varepsilon\rightarrow 0}\frac{f\left(t,se^{\varepsilon s^{-\beta}}\right)-f(t,s)}{\varepsilon}\right]\\ &= \partial_{t}^{\alpha}\left[\lim_{\varepsilon\rightarrow 0}\frac{f\left(t,s+\varepsilon s^{1-\beta}+O(\varepsilon^{2})\right)-f(t,s)}{\varepsilon}\right].\end{aligned}$$

Making the change of variable $k = \varepsilon s^{1-\beta}(1+O(\varepsilon))$, we get

$$\partial_t^\alpha\left[\partial_s^\beta f(t,s)\right] = \partial_t^\alpha\left[\lim_{k\to 0}\frac{f(t,s+k)-f(t,s)}{\frac{ks^{\beta-1}}{1+O(\varepsilon)}}\right].$$

Since f is diffentiable in s-direction, we obtain

$$\partial_t^\alpha\left[\partial_s^\alpha f(t,s)\right] = s^{1-\beta}\partial_t^\alpha\left[\frac{\partial}{\partial s}f(t,s)\right]. \tag{2.3}$$

Again by definition (1.1), it follows that

$$\partial_t^\alpha\left[\partial_s^\alpha f(t,s)\right] = s^{1-\beta}\lim_{\varepsilon\to 0}\frac{\frac{\partial}{\partial s}f\left(te^{\varepsilon t^{-\alpha}},s\right)-\frac{\partial}{\partial s}f(t,s)}{\varepsilon}.$$

Similarly, after making the change of variable, we have

$$\partial_t^\alpha\left[\partial_s^\alpha f(t,s)\right] = s^{1-\beta}t^{1-\alpha}\lim_{h\to 0}\frac{\frac{\partial}{\partial s}f(t+h,s)-\frac{\partial}{\partial s}f(t,s)}{\varepsilon}.$$

Since f is diffentiable in t-direction, we obtain

$$\partial_t^\alpha\left[\partial_s^\alpha f(t,s)\right] = s^{1-\beta}t^{1-\alpha}\frac{\partial^2}{\partial t\partial s}f(t,s). \tag{2.4}$$

Since f is continuous, by using the Clairaut's theorem for partial derivatives, it follows that

$$\frac{\partial^2}{\partial s\partial t}f(t,s) = \frac{\partial^2}{\partial t\partial s}f(t,s).$$

Therefore the equation (2.4) becomes

$$\partial_t^\alpha\left[\partial_s^\alpha f(t,s)\right] = s^{1-\beta}t^{1-\alpha}\frac{\partial^2}{\partial t\partial s}f(t,s) = s^{1-\beta}t^{1-\alpha}\lim_{k\to 0}\frac{\frac{\partial}{\partial t}f(t,s+k)-\frac{\partial}{\partial t}f(t,s)}{k}.$$

Thus, taking $k = \varepsilon s^{1-\beta}(1+O(\varepsilon))$ and laler $h = \varepsilon t^{1-\alpha}(1+O(\varepsilon))$ we arrive at

$$\partial_t^\alpha\left[\partial_s^\alpha f(t,s)\right] = \partial_s^\alpha\left[\lim_{k\to 0}\frac{\frac{\partial}{\partial t}f(t,s+k)-\frac{\partial}{\partial t}f(t,s)}{k}\right] = \partial_s^\alpha\left[\partial_t^\alpha f(t,s)\right]$$

which completes the proof. □

Theorem 3. *Let* $k\in C(\mathbb{R}^+,\mathbb{R}^+)$, $y\in C(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$, $r\in C^1(\mathbb{R}^+,\mathbb{R}^+)$ *with* $(t,s)\to\partial_t^\alpha y(t,s)\in C(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$. *Assume in additional that* r *is nondecreasing and* $r(t)\le t$ *for* $t\ge 0$. *If* $u\in C(\mathbb{R}^+,\mathbb{R}^+)$ *satisfies*

$$u(t)\le k(t)+\int_0^{r(t)}y(t,s)\,u(s)d_\alpha s,\quad t\ge 0, \tag{2.5}$$

then

$$u(t)\le k(t)+e^{\int_0^{r(t)}y(t,s)d_\alpha s}\int_0^t e^{-\int_0^{r(\tau)}y(\tau,s)d_\alpha s}\frac{\partial^\alpha}{\partial\tau^\alpha}\left(\int_0^{r(\tau)}y(\tau,s)\,k(s)d_\alpha s\right)d_\alpha\tau,\quad t\ge 0. \tag{2.6}$$

Proof. If we set

$$z(t)=\int_0^{r(t)}y(t,s)\,u(s)d_\alpha s$$

then our assumptions on y and r imply that z is nondecreasing on $\mathbb{R}^+$. Thus, for $t \geq 0$, by using Lemma 1 (ii), we get

$$\begin{aligned} D^{\alpha}z(t) &= y(t, r(t))\, u(r(t)) D^{\alpha}r(t) + \int_0^{r(t)} \left[\frac{\partial^{\alpha}}{\partial t^{\alpha}} y(t, s)\right] u(s) d_{\alpha}s \\ &\leq y(t, r(t)) \left[k(r(t)) + z(r(t))\right] D^{\alpha}r(t) + \int_0^{r(t)} \left[\frac{\partial^{\alpha}}{\partial t^{\alpha}} y(t, s)\right] \left[k(s) + z(s)\right] d_{\alpha}s \\ &\leq y(t, r(t)) \left[k(r(t)) + z(t)\right] D^{\alpha}r(t) + \int_0^{r(t)} \left[\frac{\partial^{\alpha}}{\partial t^{\alpha}} y(t, s)\right] k(s) d_{\alpha}s + z(t) \int_0^{r(t)} \frac{\partial^{\alpha}}{\partial t^{\alpha}} y(t, s)\, d_{\alpha}s \end{aligned}$$

or, equivalently

$$D^{\alpha}z(t) - z(t) \frac{\partial^{\alpha}}{\partial t^{\alpha}} \left(\int_0^{r(t)} y(t, s)\, d_{\alpha}s\right) \leq \frac{\partial^{\alpha}}{\partial t^{\alpha}} \left(\int_0^{r(t)} y(t, s)\, k(s) d_{\alpha}s\right).$$

Multiplying the above inequality by $e^{-\int_0^{r(t)} y(t,s) d_{\alpha}s}$, we obtain that

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} \left(z(t) e^{-\int_0^{r(t)} y(t,s) d_{\alpha}s}\right) \leq e^{-\int_0^{r(t)} y(t,s) d_{\alpha}s} \frac{\partial^{\alpha}}{\partial t^{\alpha}} \left(\int_0^{r(t)} y(t, s)\, k(s) d_{\alpha}s\right).$$

Integrating this from 0 to t yields

$$z(t) \leq e^{\int_0^{r(t)} y(t,s) d_{\alpha}s} \int_0^t e^{-\int_0^{r(\tau)} y(\tau,s) d_{\alpha}s} \frac{\partial^{\alpha}}{\partial \tau^{\alpha}} \left(\int_0^{r(\tau)} y(\tau, s)\, k(s) d_{\alpha}s\right) d_{\alpha}\tau.$$

Combine the above inequality with $u(t) \leq k(t) + z(t)$ this imply (2.4). The proof is complete. □

Corollary 1. *Assume y, r are as in Theorem 3 and $k(t) = k > 0$. If $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (2.5), then*

$$u(t) \leq k e^{\int_0^{r(t)} y(t,s) d_{\alpha}s}, \quad t \geq 0$$

Proof. Applying Theorem 3 for $k(t) = k$ and , we arrive at

$$\begin{aligned} u(t) &\leq k + k e^{\int_0^{r(t)} y(t,s) d_{\alpha}s} \int_0^t e^{-\int_0^{r(\tau)} y(\tau,s) d_{\alpha}s} \frac{\partial^{\alpha}}{\partial \tau^{\alpha}} \left(\int_0^{r(\tau)} y(\tau, s)\, d_{\alpha}s\right) d_{\alpha}\tau \\ &= k + k e^{\int_0^{r(t)} y(t,s) d_{\alpha}s} \left(1 - e^{-\int_0^{r(t)} y(t,s) d_{\alpha}s}\right) \\ &= k e^{\int_0^{r(t)} y(t,s) d_{\alpha}s}, \ t \geq 0. \end{aligned}$$

□

Remark 2. *If we take $r(t) = t$ in Corollary 1, then the inequality given by Corollary 1 reduces to Gronwall's inequality for conformable integrals in* [1].

Theorem 4. *Let $k, y, x \in C(\mathbb{R}^+, \mathbb{R}^+)$, $r \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and assume that r is nondecreasing with $r(t) \leq t$ for $t \geq 0$. If $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies*

$$u(t) \leq k(t) + y(t) \int_0^{r(t)} x(s) u(s) d_{\alpha}s, \quad t \geq 0, \tag{2.7}$$

then

$$u(t) \leq k(t) + y(t) \int_0^t e^{\int_{r(\tau)}^{r(t)} x(s)y(s)d_\alpha s} x\left(r(\tau)\right) k(r(\tau)) D^\alpha r(\tau) d_\alpha \tau, \quad t \geq 0. \tag{2.8}$$

Proof. If we set

$$z(t) = \int_0^{r(t)} x(s)u(s)d_\alpha s$$

then, by using conformable rules we see that

$$\begin{aligned} D^\alpha z(t) &= x\left(r(t)\right) u(r(t)) D^\alpha r(t) \\ &\leq x\left(r(t)\right) \left[k(r(t)) + y\left(r(t)\right) z(r(t))\right] D^\alpha r(t) \\ &\leq x\left(r(t)\right) \left[k(r(t)) + y\left(r(t)\right) z(t)\right] D^\alpha r(t). \end{aligned}$$

Thus, we have

$$D^\alpha z(t) - x\left(r(t)\right) y\left(r(t)\right) z(t) D^\alpha r(t) \leq x\left(r(t)\right) k(r(t)) D^\alpha r(t).$$

Multiplying the above inequality by $e^{-\int_0^{r(t)} x(s)y(s)d_\alpha s}$, we obtain that

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(z(t) e^{-\int_0^{r(t)} x(s)y(s)d_\alpha s}\right) \leq e^{-\int_0^{r(t)} x(s)y(s)d_\alpha s} x\left(r(t)\right) k(r(t)) D^\alpha r(t).$$

Integrating this from 0 to t yields

$$\begin{aligned} z(t) &\leq e^{\int_0^{r(t)} x(s)y(s)d_\alpha s} \int_0^t e^{-\int_0^{r(\tau)} x(s)y(s)d_\alpha s} x\left(r(\tau)\right) k(r(\tau)) D^\alpha r(\tau) d_\alpha \tau \\ &= \int_0^t e^{\int_{r(\tau)}^{r(t)} x(s)y(s)d_\alpha s} x\left(r(\tau)\right) k(r(\tau)) D^\alpha r(\tau) d_\alpha \tau \end{aligned}$$

and hence the claim follows because of $u(t) \leq k(t) + y(t)z(t)$. The proof is complete. □

Corollary 2. *Assume y, x, k are as in Theorem 4 and $r(t) = t$. If $u \in C\left(\mathbb{R}^+, \mathbb{R}^+\right)$ satisfies (2.7), then*

$$u(t) \leq k(t) + y(t) \int_0^t e^{\int_\tau^t x(s)y(s)d_\alpha s} x\left(\tau\right) k(\tau) d_\alpha \tau, \quad t \geq 0.$$

Remark 3. *If we take $y(t) = t$ in Corollary 2, then the inequality given by Corollary 2 reduces to Gronwall's inequality for conformable integrals in* [2].

References

[1] T. Abdeljawad, On conformable fractional calculus, Journal of Computational and Applied Mathematics 279 (2015) 57–66.

[2] D. R. Anderson and D. J. Ulness, Results for conformable differential equations, preprint, 2016.

[3] A. Atangana, D. Baleanu, and A. Alsaedi, New properties of conformable derivative, Open Math. 2015; 13: 889–898.

[4] R. Khalil, M. Al horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, Journal of Computational Apllied Mathematics, 264 (2014), 65-70.

[5] O.S. Iyiola and E.R.Nwaeze, Some new results on the new conformable fractional calculus with application using D'Alambert approach, Progr. Fract. Differ. Appl., 2(2), 115-122, 2016.

[6] M. Abu Hammad, R. Khalil, Conformable fractional heat differential equations, International Journal of Differential Equations and Applications 13(3), 2014, 177-183.

[7] M. Abu Hammad, R. Khalil, Abel's formula and wronskian for conformable fractional differential equations, International Journal of Differential Equations and Applications 13(3), 2014, 177-183.
[8] U. Katugampola, A new fractional derivative with classical properties, ArXiv:1410.6535v2.
[9] A. Zheng, Y. Feng and W. Wang, The Hyers-Ulam stability of the conformable fractional differential equation, Mathematica Aeterna, Vol. 5, 2015, no. 3, 485-492.
[10] A. A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier B.V., Amsterdam, Netherlands, 2006.
[11] S.G. Samko, A.A. Kilbas and O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordonand Breach, Yverdon et alibi, 1993.

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