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ON GENERALIZATION CONFORMABLE FRACTIONAL INTEGRAL INEQUALITIES

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Abstract. The main issues addressed in this paper are making generalization of Gronwall, Volterra and Pachpatte type inequalities for conformable differential equations. By using the Katugampola definition for conformable calculus we found some upper or lower bound for fractional derivatives and integrals. The established results are extensions of some existing Gronwall, Volterra and Pachpattetype inequalities in the previous published studies.

1. Introduction & Preliminaries

Until quite recently, the question of how to take non-integer order of derivative or integration was phenomenon among the mathematicians. However together with the development of mathematics knowledge, this question was answered via Fractional Calculus which is a generalization of ordinary differentiation and integration to arbitrary (non-integer) order. During three centuries, the theory of fractional calculus developed as a pure theoretical field, useful only for mathematicians, we refer to [10], see also [11]. In more recent times a new local, limit-based definition of a conformable derivative has been introduced in [1], [4], [8], with several follow-up papers [2], [3], [5]-[9]. In this study, we use the Katugampola derivative formulation of conformable derivative of order for $\alpha \in (0,1]$ and $t \in [0,\infty)$ given by

$$D^{\alpha}(f)(t)=\lim_{\varepsilon\to 0}\frac{f\left(te^{\varepsilon t^{-\alpha}}\right)-f(t)}{\varepsilon},\ D^{\alpha}(f)(0)=\lim_{t\to 0}D^{\alpha}(f)(t), \tag{1.1}$$

provided the limits exist (for detail see, [8]). If f is fully differentiable at t, then

$$D^{\alpha}(f)(t)=t^{1-\alpha}\frac{df}{dt}(t). \tag{1.2}$$

A function f is $\alpha-$differentiable at a point $t \geq 0$ if the limit in (1.1) exists and is finite. This definition yields the following results;

Theorem 1. *Let $\alpha \in (0,1]$ and f,g be $\alpha-$differentiable at a point $t>0$. Then*
i. $D^{\alpha}(af+bg)=aD^{\alpha}(f)+bD^{\alpha}(g)$, *for all* $a,b\in\mathbb{R}$,
ii. $D^{\alpha}(\lambda)=0$, *for all constant functions* $f(t)=\lambda$,
iii. $D^{\alpha}(fg)=fD^{\alpha}(g)+gD^{\alpha}(f)$,
iv. $D^{\alpha}\left(\frac{f}{g}\right)=\frac{fD^{\alpha}(g)-gD^{\alpha}(f)}{g^2}$
v. $D^{\alpha}(t^n)=nt^{n-\alpha}$ *for all* $n\in\mathbb{R}$
vi. $D^{\alpha}(f\circ g)(t)=f'(g(t))D^{\alpha}(g)(t)$ *for f is differentiable at $g(t)$.*

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Definition 1 (Conformable fractional integral)**.** *Let $\alpha \in (0,1]$ and $0 \leq a < b$. A function $f : [a,b] \rightarrow \mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral*

$$\int_a^b f(x)\, d_\alpha x := \int_a^b f(x)\, x^{\alpha-1} dx$$

exists and is finite. All α-fractional integrable on $[a,b]$ is indicated by $L_\alpha^1([a,b])$

Remark 1.

$$I_\alpha^a(f)(t) = I_1^a\left(t^{\alpha-1} f\right) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

We will also use the following important results, which can be derived from the results above.

Lemma 1. *Let the conformable differential operator D^α be given as in (1.1), where $\alpha \in (0,1]$ and $t \geq 0$, and assume the functions f and g are α-differentiable as needed. Then*

i. $D^\alpha(\ln t) = t^{-\alpha}$ *for* $t > 0$

ii. $D^\alpha\left[\int_a^t f(t,s)\, d_\alpha s\right] = f(t,t) + \int_a^t D^\alpha\left[f(t,s)\right] d_\alpha s$

iii. $\int_a^b f(x)\, D^\alpha(g)(x)\, d_\alpha x = fg|_a^b - \int_a^b g(x)\, D^\alpha(f)(x)\, d_\alpha x.$

The definition given in below is a generalization of the limit definition of the derivative for the case of a function with many variables.

Definition 2. *Let f be a function with n variables $t_1, ..., t_n$ and the conformable partial derivative of f of order $\alpha \in (0,1]$ in x_i is defined as follows*

$$\frac{\partial^\alpha}{\partial t_i^\alpha} f(t_1, ..., t_n) = \lim_{\varepsilon \rightarrow 0} \frac{f(t_1, ..., t_{i-1}, t_i e^{\varepsilon t_i^{-\alpha}}, ..., t_n) - f(t_1, ..., t_n)}{\varepsilon}. \tag{1.3}$$

The below theorem is the generalization of Theorem 2.10 of [3] which the detailed proof can be found in [12].

Theorem 2. *Assume that $f(t,s)$ is function for which $\partial_t^\alpha\left[\partial_s^\beta f(t,s)\right]$ and $\partial_s^\beta\left[\partial_t^\alpha f(t,s)\right]$ exist and are continuous over the domain $D \subset \mathbb{R}^2$, then*

$$\partial_t^\alpha\left[\partial_s^\beta f(t,s)\right] = \partial_s^\beta\left[\partial_t^\alpha f(t,s)\right]. \tag{1.4}$$

This prospective study was designed to investigate the new generalization of Gronwall, Volterra and Pachpatte type inequalities for conformable differential equations. The established results are extensions of some existing Gronwall, Volterra and Pachpatte type inequalities in the literature.

2. Main Findings & Cumulative Results

Throughout this paper, all the functions which appear in the inequalities are assumed to be real-valued and all the integrals involved exist on the respective domains of their definitions, and $C(M,S)$ and $C^1(M,S)$denote the class of all continuous functions and the first order conformable derivative, respectively, defined on set M with range in the set S.

Theorem 3. *Let $k, y, x, g \in C(\mathbb{R}^+, \mathbb{R}^+)$, $r \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and assume that r is non-decreasing with $r(t) \le t$ for $t \ge 0$. If $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies*

$$u(t) \le k(t) + y(t) \int_0^{r(t)} [x(s)u(s) + g(s)] d_\alpha s, \quad t \ge 0, \tag{2.1}$$

then

$$u(t) \le k(t) + y(t) \int_0^t e^{\int_{r(\tau)}^{r(t)} x(s)y(s)d_\alpha s} [x(r(\tau))\, k(r(\tau)) + g(r(\tau))] D^\alpha r(\tau) d_\alpha \tau, \quad t \ge 0. \tag{2.2}$$

Proof. If we set

$$z(t) = \int_0^{r(t)} [x(s)u(s) + g(s)] d_\alpha s$$

then, by using conformable rules we see that

$$\begin{aligned} D^\alpha z(t) &= [x(r(t))\, u(r(t)) + g(r(t))] D^\alpha r(t) \\ &\le \{x(r(t))\, [k(r(t)) + y(r(t))\, z(r(t))] + g(r(t))\} D^\alpha r(t) \\ &\le \{x(r(t))\, [k(r(t)) + y(r(t))\, z(t)] + g(r(t))\} D^\alpha r(t). \end{aligned}$$

Thus, we have

$$D^\alpha z(t) - x(r(t))\, y(r(t))\, z(t) D^\alpha r(t) \le [x(r(t))\, k(r(t)) + g(r(t))] D^\alpha r(t).$$

Multiplying the above inequality by $e^{-\int_0^{r(t)} x(s)y(s)d_\alpha s}$, we obtain that

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(z(t) e^{-\int_0^{r(t)} x(s)y(s)d_\alpha s} \right) \le e^{-\int_0^{r(t)} x(s)y(s)d_\alpha s} [x(r(t))\, k(r(t)) + g(r(t))] D^\alpha r(t).$$

Integrating this from 0 to t yields

$$\begin{aligned} z(t) &\le e^{\int_0^{r(t)} x(s)y(s)d_\alpha s} \int_0^t e^{-\int_0^{r(\tau)} x(s)y(s)d_\alpha s} [x(r(\tau))\, k(r(\tau)) + g(r(\tau))] D^\alpha r(\tau) d_\alpha \tau \\ &= \int_0^t e^{\int_{r(\tau)}^{r(t)} x(s)y(s)d_\alpha s} [x(r(\tau))\, k(r(\tau)) + g(r(\tau))] D^\alpha r(\tau) d_\alpha \tau \end{aligned}$$

and hence the claim follows because of $u(t) \le k(t) + y(t)z(t)$. The proof is complete. $\square$

Remark 2. *If we take $g(t) = 0$ in Theorem 3, then Theorem 3 reduces to Theorem 4 is proved by Sarikaya in* [12].

Corollary 1. *Assume y, x, k are as in Theorem 3 and $r(t) = \frac{t^\alpha}{\alpha}$. If $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (2.1), then*

$$u(t) \le k(t) + y(t) \int_0^t e^{\int_{\frac{\tau^\alpha}{\alpha}}^{\frac{t^\alpha}{\alpha}} x(s)y(s)d_\alpha s} [x(\tau)\, k(\tau) + g(\tau)] d_\alpha \tau, \quad t \ge 0.$$

Theorem 4. *Let $k, y, x, g \in C(\mathbb{R}^+, \mathbb{R}^+)$, $r \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and assume that r is non-decreasing with $r(t) \le t$ for $t \ge 0$. If $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies*

$$u(t) \le k(t) + \sum_{i=1}^n y_i(t) \int_0^{r(t)} [x_i(s)u(s) + g_i(s)] d_\alpha s, \quad t \ge 0, \tag{2.3}$$

then
(2.4)
$$u(t) \leq k(t)+Y(t)\int_0^t e^{\int_{r(\tau)}^{r(t)}\sum_{i=1}^n x_i(s)y(s)d_\alpha s}\sum_{i=1}^n [x_i\left(r(\tau)\right)k(r(\tau))+g_i(r(\tau))]D^\alpha r(\tau)d_\alpha\tau,\ \ t\geq 0.$$

where $Y(t)=\sup_{i=1,\ldots,n} y_i(t)$.

Proof. The inequality (2.3) implies that

$$u(t)\leq k(t)+Y(t)\int_0^{r(t)}\sum_{i=1}^n [x_i(s)u(s)+g_i(s)]d_\alpha s.$$

Now an application of Theorem 3 provides the desired inequality (2.4). □

Theorem 5. *Let* $v,y,h\in C\left(\mathbb{R}^+,\mathbb{R}^+\right),\ r,p\in C^1\left(\mathbb{R}^+,\mathbb{R}^+\right)$ *and assume that* p *is non-decreasing with* $p(x)\leq x$ *for* $x\geq 0$. *If* $u\in C\left(\mathbb{R}^+,\mathbb{R}^+\right)$ *satisfies*

$$u(t)\geq v(x)+y(t)\int_{p(x)}^{r(t)} h(s)v(s)d_\alpha s,\quad 0\leq x\leq t, \tag{2.5}$$

then

$$u(t)\geq v(x)e^{y(t)\int_{p(\chi)}^{r(\tau)}h(\chi)d_\alpha \chi},\quad 0\leq x\leq t, \tag{2.6}$$

Proof. Denote

$$z(x)=u(t)-y(t)\int_{p(x)}^{r(t)}h(s)v(s)d_\alpha s$$

hence, by using conformable rules we have

$$\begin{aligned} D^\alpha z(x) &= -y(t)h(p(x))v(p(x))D^\alpha p(x) \\ &\geq -y(t)h(p(x))z(p(x))D^\alpha p(x) \\ &\geq -y(t)h(p(x))z(x)D^\alpha p(x). \end{aligned}$$

Thus, we have

$$D^\alpha z(x)+y(t)h(p(x))z(x)D^\alpha p(x)\geq 0.$$

Multiplying the above inequality by $e^{y(t)\int_{p(x)}^{r(t)}h(s)d_\alpha s}$, we obtain that

$$\frac{\partial^\alpha}{\partial x^\alpha}\left(z(x)e^{y(t)\int_{p(x)}^{r(t)}h(s)d_\alpha s}\right)\geq 0.$$

Then if $q(x)=e^{y(t)\int_{p(x)}^{r(t)}h(s)d_\alpha s}$, we have $\frac{\partial^\alpha}{\partial x^\alpha}\left(zq\right)(x)\geq 0$ and so $(zq)(t)\geq (zq)(x)$ on $[0,t]$. Now $z(x)\geq v(x)$ and $z(t)=u(t)$ and we have the result given in (2.6). This result is the best possible in the sense that if equation (2.5) holds on $[0,t]$, then equation (2.5) holds on $[0,t]$. □

Theorem 6. *Let* $k,m,f,g\in C\left(\mathbb{R}^+,\mathbb{R}^+\right),\ y\in C\left(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+\right),\ r\in C^1\left(\mathbb{R}^+,\mathbb{R}^+\right)$ *with* $(t,s)\rightarrow\partial_t^\alpha y(t,s)\in C\left(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+\right)$. *Assume in additional that* r *is non-decreasing and* $r(t)\leq t$ *for* $t\geq 0$. *If* $u\in C\left(\mathbb{R}^+,\mathbb{R}^+\right)$ *satisfies*

$$u(t)\leq k(t)+m(t)\int_0^{r(t)} y\left(t,s\right)[f(s)u(s)+g(s)]d_\alpha s, \tag{2.7}$$

then

$$(2.8)\quad u(t) \le k(t) + m(t) e^{\int_0^{r(t)} y(t,s)m(s)f(s)d_\alpha s} \int_0^t e^{-\int_0^{r(\tau)} y(\tau,s)m(s)f(s)d_\alpha s} \times \frac{\partial^\alpha}{\partial \tau^\alpha}\left(\int_0^{r(\tau)} y(\tau,s)\left[f(s)k(s)+g(s)\right]d_\alpha s\right) d_\alpha \tau$$

for $t \ge 0$.

Proof. Let describe

$$z(t) = \int_0^{r(t)} y(t,s)\left[f(s)u(s)+g(s)\right]d_\alpha s$$

then our assumptions on y, f, g and r imply that z is non-decreasing on $\mathbb{R}^+$. Thus, for $t \ge 0$, by using Lemma 1 (ii), we get

$$\begin{aligned} D^\alpha z(t) &= y(t,r(t))\left[f(r(t))u(r(t))+g(r(t))\right]D^\alpha r(t) + \int_0^{r(t)}\left[\frac{\partial^\alpha}{\partial t^\alpha} y(t,s)\right]\left[f(s)u(s)+g(s)\right]d_\alpha s \\ &\le y(t,r(t))\left[f(r(t))\{k(r(t))+m(r(t))z(r(t))\}+g(r(t))\right]D^\alpha r(t) + \\ &+ \int_0^{r(t)}\left[\frac{\partial^\alpha}{\partial t^\alpha} y(t,s)\right]\left[f(s)\{k(s)+m(s)z(s)\}+g(s)\right]d_\alpha s \\ &\le y(t,r(t))\left[f(r(t))\{k(r(t))+m(r(t))z(t)\}+g(r(t))\right]D^\alpha r(t) + \\ &+ \int_0^{r(t)}\left[\frac{\partial^\alpha}{\partial t^\alpha} y(t,s)\right]\left[f(s)k(s)+g(s)\right]d_\alpha s + z(t)\int_0^{r(t)} \frac{\partial^\alpha}{\partial t^\alpha} y(t,s)\left[m(s)y(s)\right]d_\alpha s \end{aligned}$$

or, equivalently

$$D^\alpha z(t) - z(t)\frac{\partial^\alpha}{\partial t^\alpha}\left(\int_0^{r(t)} y(t,s)\, m(s)f(s)d_\alpha s\right) \le \frac{\partial^\alpha}{\partial t^\alpha}\left(\int_0^{r(t)} y(t,s)\left[f(s)k(s)+g(s)\right]d_\alpha s\right).$$

Multiplying the above inequality by $e^{-\int_0^{r(t)} y(t,s)m(s)f(s)d_\alpha s}$, we obtain that

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(z(t)e^{-\int_0^{r(t)} y(t,s)m(s)f(s)d_\alpha s}\right) \le e^{-\int_0^{r(t)} y(t,s)m(s)f(s)d_\alpha s}\frac{\partial^\alpha}{\partial t^\alpha}\left(\int_0^{r(t)} y(t,s)\left[f(s)k(s)+g(s)\right]d_\alpha s\right).$$

Integrating this from 0 to t yields

$$z(t) \le e^{\int_0^{r(t)} y(t,s)m(s)f(s)d_\alpha s}\int_0^t e^{-\int_0^{r(\tau)} y(\tau,s)m(s)f(s)d_\alpha s}\frac{\partial^\alpha}{\partial \tau^\alpha}\left(\int_0^{r(\tau)} y(\tau,s)\left[f(s)k(s)+g(s)\right]d_\alpha s\right)d_\alpha \tau.$$

Combine the above inequality with $u(t) \le k(t) + m(t)z(t)$ this imply (2.8). The proof is complete. □

Remark 3. *If we take* $r(t) = t$, $k(t) = k$ *(a constant),* $m(t) = 1$, $f(s) = 1$ *and* $g(s) = 0$ *in Theorem 6, then the inequality given by Theorem 6 reduces to Gronwall's inequality for conformable integrals in* [1].

Theorem 7. *Let $f, g \in C(\mathbb{R}^+, \mathbb{R}^+)$, $r \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and assume that r is non-decreasing with $r(t) \leq t$ for $t \geq 0$. If $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies*

$$u(t) \leq u_0 + \int_0^{r(t)} f(s)u(s)d_\alpha s + \int_0^{r(t)} f(s)\left[\int_0^{p(s)} g(n)u(n)d_\alpha n\right] d_\alpha s, \quad t \geq 0, \tag{2.9}$$

then

$$u(t) = u_0 + u_0 \int_0^t f(s) e^{\int_0^{p(s)}[f(n)+g(n)]d_\alpha n} d_\alpha s, \quad t \geq 0. \tag{2.10}$$

Proof. Let denote $z(t)$ the right hand side of inequality (2.9). Then $u(t) \leq z(t)$ and $z(0) = u_0$ and

$$\begin{aligned} D^\alpha z(t) &= f(r(t))u(r(t))D^\alpha r(t) + f(r(t))D^\alpha r(t)\int_0^{r(t)} g(n)u(n)d_\alpha n \\ &\leq f(r(t))z(t)D^\alpha r(t) + f(r(t))D^\alpha r(t)\int_0^{r(t)} g(n)z(n)d_\alpha n \\ &\leq f(r(t))D^\alpha r(t)\left[z(t) + \int_0^{r(t)} g(n)z(n)d_\alpha n\right]. \end{aligned}$$

Define a function $m(t)$ by

$$m(t) = z(t) + \int_0^{r(t)} g(n)z(n)d_\alpha n$$

then $m(0) = z(0) = u_0$, $D^\alpha z(t) \leq f(r(t))D^\alpha r(t)m(t)$, $z(t) \leq m(t)$ and

$$\begin{aligned} D^\alpha m(t) &= D^\alpha z(t) + g(r(t))z(r(t))D^\alpha r(t) \\ &\leq D^\alpha z(t) + g(r(t))z(t)D^\alpha r(t). \end{aligned}$$

So we get

$$D^\alpha m(t) \leq [f(r(t)) + g(r(t))]m(t)D^\alpha r(t). \tag{2.11}$$

The inequality (2.11) implies the estimation of $m(t)$ such that

$$m(t) \leq u_0 e^{\int_0^{r(t)}[f(n)+g(n)]d_\alpha n}.$$

Then

$$D^\alpha z(t) \leq u_0 f(r(t))D^\alpha r(t) e^{\int_0^{r(t)}[f(n)+g(n)]d_\alpha n}. \tag{2.12}$$

Now by setting $r(t) = p(s)$ in (2.12) and integrating from 0 to t and substituting the bound $z(t)$ in $u(t) \leq z(t)$ we get

$$z(t) \leq u_0 + u_0 \int_0^t f(s) e^{\int_0^{p(s)}[f(n)+g(n)]d_\alpha n} d_\alpha s$$

which this proves our claim. □

3. Concluding Remark

The present study was designed to make the generalization of some inequalities for conformable differential equations. For this purpose we use the Katugampola derivative formulation of conformable derivative of order for $\alpha \in (0,1]$. The findings of this investigation complement those of earlier studies. In other words the present study confirms previous findings and contributes additional evidence by making generalization.

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