Received 02/08/16

SOME NEW REVERSES AND REFINEMENTS OF INEQUALITIES FOR RELATIVE OPERATOR ENTROPY

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Abstract. In this paper we obtain new inequalities for relative operator entropy $S(A|B)$ in the case of operators satisfying the condition $mA \leq B \leq MA$, with $0 < m < M$.

1. Introduction

Kamei and Fujii [6], [7] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B) := A^{\frac{1}{2}} \left(\ln \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) \right) A^{\frac{1}{2}}, \tag{1.1}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [12].

In general, we can define for positive operators A, B

$$S(A|B) := s\text{-}\lim_{\varepsilon \to 0+} S(A + \varepsilon 1_H | B)$$

if it exists, here 1_H is the identity operator.

For the entropy function $\eta(t) = -t \ln t$, the *operator entropy* has the following expression:

$$\eta(A) = -A \ln A = S(A|1_H) \geq 0$$

for positive contraction A. This shows that the relative operator entropy (1.1) is a relative version of the operator entropy.

Following [8, p. 149-p. 155], we recall some important properties of relative operator entropy for A and B positive invertible operators:

(i) We have the equalities

$$S(A|B) = -A^{1/2} \left(\ln A^{1/2} B^{-1} A^{1/2} \right) A^{1/2} = B^{1/2} \eta \left(B^{-1/2} A B^{-1/2} \right) B^{1/2}; \tag{1.2}$$

(ii) We have the inequalities

$$S(A|B) \leq A (\ln \|B\| - \ln A) \text{ and } S(A|B) \leq B - A; \tag{1.3}$$

(iii) For any C, D positive invertible operators we have that

$$S(A + B|C + D) \geq S(A|C) + S(B|D);$$

(iv) If $B \leq C$ then

$$S(A|B) \leq S(A|C);$$

(v) If $B_n \downarrow B$ then

$$S(A|B_n) \downarrow S(A|B);$$

1991 *Mathematics Subject Classification.* 47A63, 47A30,

Key words and phrases. Inequalities for Logarithm, Relative operator entropy, Operator entropy.

(vi) For $\alpha > 0$ we have

$$S(\alpha A|\alpha B) = \alpha S(A|B);$$

(vii) For every operator T we have

$$T^* S(A|B) T \le S(T^*AT|T^*BT).$$

The relative operator entropy is *jointly concave*, namely, for any positive invertible operators A, B, C, D we have

$$S(tA + (1-t)B|tC + (1-t)D) \ge tS(A|C) + (1-t)S(B|D)$$

for any $t \in [0,1]$.

For other results on the relative operator entropy see [1], [4], [9], [10], [11] and [13].

Observe that, if we replace in (1.2) B with A, then we get

$$\begin{aligned} S(B|A) &= A^{1/2}\eta\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} \\ &= A^{1/2}\left(-A^{-1/2}BA^{-1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}, \end{aligned}$$

therefore we have

$$A^{1/2}\left(A^{-1/2}BA^{-1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2} = -S(B|A) \tag{1.4}$$

for positive invertible operators A and B.

It is well know that, in general $S(A|B)$ is not equal to $S(B|A)$.

In [15], A. Uhlmann has shown that the relative operator entropy $S(A|B)$ can be represented as the strong limit

$$S(A|B) = s\text{-}\lim_{t\to 0}\frac{A\sharp_t B - A}{t}, \tag{1.5}$$

where

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2},\ \nu \in [0,1]$$

is the *weighted geometric mean* of positive invertible operators A and B. For $\nu = \frac{1}{2}$ we denote $A\sharp B$.

This definition of the weighted geometric mean can be extended for any real number ν with $\nu \neq 0$.

For $t > 0$ and the positive invertible operators A, B we define the *Tsallis relative operator entropy* (see also [3]) by

$$T_t(A|B) := \frac{A\sharp_t B - A}{t}.$$

The following result providing upper and lower bounds for relative operator entropy in terms of $T_t(\cdot|\cdot)$ has been obtained in [6] for $0 < t \le 1$. However, it hods for any $t > 0$.

Theorem 1. *Let A, B be two positive invertible operators, then for any $t > 0$ we have*

$$T_t(A|B)(A\sharp_t B)^{-1} A \le S(A|B) \le T_t(A|B). \tag{1.6}$$

In particular, we have for $t = 1$ that

$$\left(1_H - AB^{-1}\right) A \leq S\left(A|B\right) \leq B - A, \text{ [6]} \tag{1.7}$$

and for $t = 2$ that

$$\frac{1}{2}\left(1_H - \left(AB^{-1}\right)^2\right) A \leq S\left(A|B\right) \leq \frac{1}{2}\left(BA^{-1}B - A\right). \tag{1.8}$$

The case $t = \frac{1}{2}$ is of interest as well. Since in this case we have

$$T_{1/2}\left(A|B\right) := 2\left(A\sharp B - A\right)$$

and

$$T_{1/2}\left(A|B\right)\left(A\sharp_{1/2}B\right)^{-1} A = 2\left(1_H - A\left(A\sharp B\right)^{-1}\right) A,$$

hence by (1.6) we get

$$2\left(1_H - A\left(A\sharp B\right)^{-1}\right) A \leq S\left(A|B\right) \leq 2\left(A\sharp B - A\right) \leq B - A. \tag{1.9}$$

Motivated by the above results, in this paper we obtain new inequalities for the relative operator entropy in the case of operators satisfying the condition $mA \leq B \leq MA$, with $0 < m < M$.

2. Inequalities for Log-function

We have:

Theorem 2. *For any a, $b > 0$ we have the inequalities*

$$\frac{1}{2b\min\{a,b\}}\left(b-a\right)^2 \geq \ln b - \ln a - \frac{b-a}{b} \geq \frac{1}{2b\max\{a,b\}}\left(b-a\right)^2 \tag{2.1}$$

and

$$\frac{1}{2a\min\{a,b\}}\left(b-a\right)^2 \geq \frac{b-a}{a} - \ln b + \ln a \geq \frac{1}{2a\max\{a,b\}}\left(b-a\right)^2. \tag{2.2}$$

Proof. We have

$$\int_a^b \frac{b-t}{t}dt = b\int_a^b \frac{1}{t}dt - \int_a^b dt = b\left(\ln b - \ln a\right) - \left(b-a\right)$$

giving that

$$\ln b - \ln a - \frac{b-a}{b} = \frac{1}{b}\int_a^b \frac{b-t}{t}dt \tag{2.3}$$

for any a, $b > 0$.

Let $b > a > 0$, then

$$\frac{1}{a}\int_a^b \left(b-t\right)dt \geq \int_a^b \frac{b-t}{t}dt \geq \frac{1}{b}\int_a^b \left(b-t\right)dt$$

giving that

$$\frac{1}{2a}\left(b-a\right)^2 \geq \int_a^b \frac{b-t}{t}dt \geq \frac{1}{2b}\left(b-a\right)^2. \tag{2.4}$$

Let $a > b > 0$, then

$$\frac{1}{b}\int_b^a \left(t-b\right)dt \geq \int_a^b \frac{b-t}{t}dt = \int_b^a \frac{t-b}{t}dt \geq \frac{1}{a}\int_b^a \left(t-b\right)dt$$

giving that

$$\frac{1}{2b}(b-a)^2 \geq \int_a^b \frac{b-t}{t}dt \geq \frac{1}{2a}(b-a)^2. \tag{2.5}$$

Therefore, by (2.4) and (2.5) we get

$$\frac{1}{2\min\{a,b\}}(b-a)^2 \geq \int_a^b \frac{b-t}{t}dt \geq \frac{1}{2\max\{a,b\}}(b-a)^2,$$

for any $a,\ b>0$.

By utilising the equality (2.3) we get the desired result (2.1). □

Corollary 1. *For any $y>0$ we have*

$$\frac{1}{2y\min\{1,y\}}(y-1)^2 \geq \ln y - \frac{y-1}{y} \geq \frac{1}{2y\max\{1,y\}}(y-1)^2 \tag{2.6}$$

and

$$\frac{1}{2\min\{1,y\}}(y-1)^2 \geq y-1-\ln y \geq \frac{1}{2\max\{1,y\}}(y-1)^2. \tag{2.7}$$

Remark 1. *Since for any $a,\ b>0$ we have $\max\{a,b\}\min\{a,b\}=ab$, then (2.1) and (2.2) can also be written as*

$$\frac{1}{2a}\max\{a,b\}\left(\frac{b-a}{b}\right)^2 \geq \ln b - \ln a - \frac{b-a}{b} \geq \frac{1}{2a}\min\{a,b\}\left(\frac{b-a}{b}\right)^2 \tag{2.8}$$

and

$$\frac{1}{2b}\max\{a,b\}\left(\frac{b-a}{a}\right)^2 \geq \frac{b-a}{a} - \ln b + \ln a \geq \frac{1}{2b}\min\{a,b\}\left(\frac{b-a}{a}\right)^2 \tag{2.9}$$

for any $a,\ b>0$.

The inequalities can also be written as

$$\frac{1}{2}\max\{1,y\}\left(\frac{y-1}{y}\right)^2 \geq \ln y - \frac{y-1}{y} \geq \frac{1}{2}\min\{1,y\}\left(\frac{y-1}{y}\right)^2 \tag{2.10}$$

and

$$\frac{1}{2y}\max\{1,y\}(y-1)^2 \geq y-1-\ln y \geq \frac{1}{2y}\min\{1,y\}(y-1)^2, \tag{2.11}$$

for any $y>0$.

In the recent paper [2] we obtained the following inequalities that provide upper and lower bounds for the quantity $\ln b - \ln a - \frac{b-a}{b}$:

$$\frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}} \geq \frac{b-a}{a} - \ln b + \ln a \geq \frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}}, \tag{2.12}$$

where $a,\ b>0$ and

$$\frac{(b-a)^2}{ab} \geq \frac{b-a}{a} - \ln b + \ln a \tag{2.13}$$

for any $a,\ b>0$.

It is natural to ask, which of the upper bounds for the quantity

$$\frac{b-a}{a} - \ln b + \ln a$$

as provided by (2.2), (2.12) and (2.13) is better?

It has been shown in [2] that neither of the upper bounds in (2.12) and (2.13) is always best.

Consider now the difference

$$D_1(a,b) := \frac{1}{2a\min\{a,b\}}(b-a)^2 - \frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}}$$
$$= \frac{1}{2}\frac{(b-a)^2}{a\min^2\{a,b\}}(\min\{a,b\}-a) \le 0,$$

which shows that upper bound in (2.2) is always better than the upper bound in (2.12).

Consider the difference

$$D_2(a,b) := \frac{1}{2a\min\{a,b\}}(b-a)^2 - \frac{(b-a)^2}{ab}$$
$$= \frac{1}{2ab\min\{a,b\}}(b-a)^2(b-2\min\{a,b\}),$$

which can take both positive and negative values for a, $b > 0$, showing that neither of the bounds (2.2) and (2.13) is always best.

Now, consider the difference

$$d(a,b) := \frac{1}{2a\max\{a,b\}}(b-a)^2 - \frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}}$$
$$= \frac{1}{2a\max^2\{a,b\}}(b-a)^2(\max\{a,b\}-a) \ge 0,$$

which shows that lower bound in (2.2) is always better than the lower bound in (2.12).

Corollary 2. *If $y \in [k,K] \subset (0,\infty)$, then we have the local inequalities*

$$\frac{1}{2\min\{1,k\}}\frac{(y-1)^2}{y} \ge \ln y - \frac{y-1}{y} \ge \frac{1}{2\max\{1,K\}}\frac{(y-1)^2}{y}, \tag{2.14}$$

$$\frac{1}{2\min\{1,k\}}(y-1)^2 \ge y-1-\ln y \ge \frac{1}{2\max\{1,K\}}(y-1)^2, \tag{2.15}$$

$$\frac{1}{2}\max\{1,K\}\left(\frac{y-1}{y}\right)^2 \ge \ln y - \frac{y-1}{y} \ge \frac{1}{2}\min\{1,k\}\left(\frac{y-1}{y}\right)^2 \tag{2.16}$$

and

$$\frac{1}{2}\max\{1,K\}\frac{(y-1)^2}{y} \ge y-1-\ln y \ge \frac{1}{2}\min\{1,k\}\frac{(y-1)^2}{y}. \tag{2.17}$$

Proof. If $y \in [k,K] \subset (0,\infty)$, then by analyzing all possible locations of the interval $[k,K]$ and 1 we have

$$\min\{1,k\} \le \min\{1,y\} \le \min\{1,K\}$$

and

$$\max\{1,k\} \le \max\{1,y\} \le \max\{1,K\}.$$

By using the inequalities (2.6) and (2.7) we have

$$\begin{aligned}
&\frac{1}{2y\min\{1,k\}}(y-1)^2 \geq \\
&\frac{1}{2y\min\{1,y\}}(y-1)^2 \geq \ln y - \frac{y-1}{y} \geq \frac{1}{2y\max\{1,y\}}(y-1)^2 \\
&\geq \frac{1}{2y\max\{1,K\}}(y-1)^2
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2\min\{1,k\}}(y-1)^2 \geq \\
&\frac{1}{2\min\{1,y\}}(y-1)^2 \geq y-1-\ln y \geq \frac{1}{2\max\{1,y\}}(y-1)^2 \\
&\geq \frac{1}{2\max\{1,K\}}(y-1)^2
\end{aligned}$$

for any $y \in [k,K]$, that prove (2.14) and (2.15).

The inequalities (2.16) and (2.17) follows by (2.16) and (2.17). □

If we consider the function $f(y) = \frac{(y-1)^2}{y}$, $y>0$, then we observe that

$$f'(y) = \frac{y^2-1}{y^2} \text{ and } f''(y) = \frac{2}{y^3},$$

which shows that f is strictly decreasing on $(0,1)$, strictly increasing on $[1,\infty)$ and strictly convex for $y>0$. We also have $f\left(\frac{1}{y}\right) = f(y)$ for $y>0$.

By the properties of f we then have that

$$\max_{y\in[k,K]} \frac{(y-1)^2}{y} = \begin{cases} \frac{(k-1)^2}{k} \text{ if } K<1, \\ \max\left\{\frac{(k-1)^2}{k}, \frac{(K-1)^2}{K}\right\} \text{ if } k\leq 1\leq K, \\ \frac{(K-1)^2}{K} \text{ if } 1<k \end{cases} =: U(k,K) \tag{2.18}$$

and

$$\min_{y\in[k,K]} \frac{(y-1)^2}{y} = \begin{cases} \frac{(1-K)^2}{K} \text{ if } K<1, \\ 0 \text{ if } k\leq 1\leq K, \\ \frac{(k-1)^2}{k} \text{ if } 1<k \end{cases} =: u(k,K). \tag{2.19}$$

We can provide now some *global bounds* as follows.

From (2.14) we then get for any $y\in[k,K]$ that

$$\frac{1}{2\min\{1,k\}}U(k,K) \geq \ln y - \frac{y-1}{y} \geq \frac{1}{2\max\{1,K\}}u(k,K), \tag{2.20}$$

while from (2.17) we get for any $y\in[k,K]$ that

$$\frac{1}{2}\max\{1,K\}U(k,K) \geq y-1-\ln y \geq \frac{1}{2}\min\{1,k\}u(k,K). \tag{2.21}$$

Consider

$$Z(k,K) := \max_{y\in[k,K]} (y-1)^2 = \begin{cases} (1-k)^2 \text{ if } K<1, \\ \max\left\{(1-k)^2, (K-1)^2\right\} \text{ if } k \le 1 \le K, \\ (K-1)^2 \text{ if } 1<k \end{cases} \tag{2.22}$$

and

$$z(k,K) := \min_{y\in[k,K]} (y-1)^2 = \begin{cases} (1-K)^2 \text{ if } K<1, \\ 0 \text{ if } k \le 1 \le K, \\ (k-1)^2 \text{ if } 1<k. \end{cases} \tag{2.23}$$

By making use of (2.15) we get

$$\frac{1}{2\min\{1,k\}} Z(k,K) \ge y-1-\ln y \ge \frac{1}{2\max\{1,K\}} z(k,K), \tag{2.24}$$

for any $y \in [k,K]$.

Consider the function $g(y) = \left(\frac{y-1}{y}\right)^2$, $y>0$, then we observe that

$$g'(y) = \frac{2(y-1)}{y^2} \text{ and } g''(y) = \frac{2(3-2y)}{y^4},$$

which shows that g is strictly decreasing on $(0,1)$, strictly increasing on $[1,\infty)$ strictly convex for $y \in (0, 3/2)$ and strictly concave on $(3/2, \infty)$.

Consider

$$W(k,K) := \max_{y\in[k,K]} \left(\frac{y-1}{y}\right)^2 = \begin{cases} \left(\frac{1-k}{k}\right)^2 \text{ if } K<1, \\ \max\left\{\left(\frac{1-k}{k}\right)^2, \left(\frac{K-1}{K}\right)^2\right\} \text{ if } k \le 1 \le K, \\ \left(\frac{K-1}{K}\right)^2 \text{ if } 1<k \end{cases} \tag{2.25}$$

and

$$w(k,K) := \min_{y\in[k,K]} \left(\frac{y-1}{y}\right)^2 = \begin{cases} \left(\frac{1-K}{K}\right)^2 \text{ if } K<1, \\ 0 \text{ if } k \le 1 \le K, \\ \left(\frac{k-1}{k}\right)^2 \text{ if } 1<k. \end{cases} \tag{2.26}$$

Then by (2.16) we get

$$\frac{1}{2}\max\{1,K\} W(k,K) \ge \ln y - \frac{y-1}{y} \ge \frac{1}{2}\min\{1,k\} w(k,K) \tag{2.27}$$

for any $y \in [k,K]$.

3. Operator Inequalities

We have the following:

Lemma 1. *Let $x \in [k, K]$ and $t > 0$, then we have*

$$
\begin{aligned}
(3.1)\qquad & \frac{1}{2\min\{1,k^t\}}\left(\frac{x^t-1}{t}-\frac{1-x^{-t}}{t}\right)\\
& \geq \ln x-\frac{1-x^{-t}}{t}\\
& \geq \frac{1}{2\max\{1,K^t\}}\left(\frac{x^t-1}{t}-\frac{1-x^{-t}}{t}\right)\geq 0
\end{aligned}
$$

and

$$
\begin{aligned}
(3.2)\qquad & \frac{1}{2}\max\left\{1,K^t\right\}t\left(\frac{1-x^{-t}}{t}\right)^2\\
& \geq \ln x-\frac{1-x^{-t}}{t}\geq \frac{1}{2}\min\left\{1,k^t\right\}t\left(\frac{1-x^{-t}}{t}\right)^2\geq 0.
\end{aligned}
$$

Proof. Let $y = x^t \in [k^t, K^t]$. By using the inequality (2.14) we have

$$
\begin{aligned}
\frac{1}{2\min\{1,k^t\}}\left(x^t+x^{-t}-2\right) &\geq t\ln x-\frac{x^t-1}{x^t}\\
&\geq \frac{1}{2\max\{1,K^t\}}\left(x^t+x^{-t}-2\right)\geq 0
\end{aligned}
$$

that is equivalent to (3.1).

From the inequality (2.16) we have for $y = x^t$

$$
\begin{aligned}
\frac{1}{2}\max\left\{1,K^t\right\}\left(1-2x^{-t}+x^{-2t}\right) &\geq t\ln x-\frac{x^t-1}{x^t}\\
&\geq \frac{1}{2}\min\left\{1,k^t\right\}\left(1-2x^{-t}+x^{-2t}\right)\geq 0
\end{aligned}
$$

that is equivalent to (3.2). □

We have:

Theorem 3. *Let A, B be two positive invertible operators and the constants $M > m > 0$ with the property that*

$$
(3.3)\qquad mA \leq B \leq MA.
$$

Then for any $t > 0$ we have

$$
\begin{aligned}
(3.4)\qquad & \frac{1}{2\min\{1,m^t\}}T_t(A|B)\left(A^{-1}-(A\sharp_t B)^{-1}\right)A\\
& \geq S(A|B)-T_t(A|B)(A\sharp_t B)^{-1}A\\
& \geq \frac{1}{2\max\{1,M^t\}}T_t(A|B)\left(A^{-1}-(A\sharp_t B)^{-1}\right)A\geq 0
\end{aligned}
$$

and

$$
\begin{aligned}
(3.5)\qquad & \frac{1}{2}\max\left\{1,M^t\right\}t\left(T_t(A|B)(A\sharp_t B)^{-1}\right)^2A\\
& \geq S(A|B)-T_t(A|B)(A\sharp_t B)^{-1}A\\
& \geq \frac{1}{2}\min\left\{1,m^t\right\}t\left(T_t(A|B)(A\sharp_t B)^{-1}\right)^2A\geq 0.
\end{aligned}
$$

Proof. Since $mA \leq B \leq MA$ and A is invertible, then by multiplying both sides with $A^{-1/2}$ we get $m1_H \leq A^{-1/2}BA^{-1/2} \leq M$. Denote $X = A^{-1/2}BA^{-1/2}$ and by using the functional calculus for X that has its spectrum contained in the interval $[m, M]$ and the inequality (3.1), we get

$$\begin{aligned}
(3.6)\qquad & \frac{1}{2\min\{1, m^t\}} \\
& \left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^t - 1_H}{t} - \frac{1_H - \left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right) \\
& \geq \ln\left(A^{-1/2}BA^{-1/2}\right) - \frac{1_H - \left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t} \\
& \geq \frac{1}{2\max\{1, M^t\}} \\
& \left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^t - 1_H}{t} - \frac{1_H - \left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right) \\
& \geq 0
\end{aligned}$$

for any $t > 0$.

Now, if we multiply both sides of (3.6) by $A^{1/2}$, then we get

$$\begin{aligned}
(3.7)\qquad & \frac{1}{2\min\{1, m^t\}} \\
& A^{1/2}\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^t - 1_H}{t} - \frac{1_H - \left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right)A^{1/2} \\
& \geq A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2} - A^{1/2}\frac{1_H - \left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}A^{1/2} \\
& \geq \frac{1}{2\max\{1, M^t\}} \\
& A^{1/2}\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^t - 1_H}{t} - \frac{1_H - \left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right)A^{1/2} \\
& \geq 0
\end{aligned}$$

for any $t > 0$.

Observe that

$$A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} = S\left(A|B\right),$$

$$A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^t - 1}{t}A^{1/2} = \frac{A\sharp_t B - A}{t} = T_t\left(A|B\right),$$

$$
\begin{aligned}
(3.8)\quad & A^{1/2}\frac{1_H-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}A^{1/2}\\
&=A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^{t}\left(A^{-1/2}BA^{-1/2}\right)^{-t}-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}A^{1/2}\\
&=A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^{t}-1_H}{t}\left(A^{-1/2}BA^{-1/2}\right)^{-t}A^{1/2}\\
&=A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^{t}-1_H}{t}A^{1/2}A^{-1/2}\left(A^{-1/2}BA^{-1/2}\right)^{-t}A^{-1/2}A\\
&=T_t\left(A|B\right)\left(A\sharp_t B\right)^{-1}A
\end{aligned}
$$

and then by (3.7) we get

$$
\begin{aligned}
&\frac{1}{2\min\left\{1,m^t\right\}}T_t\left(A|B\right)\left(1_H-\left(A\sharp_t B\right)^{-1}A\right)\\
&\geq S\left(A|B\right)-T_t\left(A|B\right)\left(A\sharp_t B\right)^{-1}A\\
&\geq\frac{1}{2\max\left\{1,M^t\right\}}T_t\left(A|B\right)\left(1_H-\left(A\sharp_t B\right)^{-1}A\right)\geq 0
\end{aligned}
$$

that is equivalent to (3.4).

From the inequality (3.2) we also have

$$
\begin{aligned}
(3.9)\quad & \frac{1}{2}\max\left\{1,M^t\right\}t\left(\frac{1_H-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right)^2\\
&\geq\ln\left(A^{-1/2}BA^{-1/2}\right)-\frac{1_H-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\\
&\geq\frac{1}{2}\min\left\{1,m^t\right\}t\left(\frac{1_H-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right)^2\geq 0.
\end{aligned}
$$

Now, if we multiply both sides of (3.9) by $A^{1/2}$, then we get

$$
\begin{aligned}
(3.10)\quad & \frac{1}{2}\max\left\{1,M^t\right\}tA^{1/2}\left(\frac{1_H-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right)^2A^{1/2}\\
&\geq A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}-A^{1/2}\frac{1_H-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}A^{1/2}\\
&\geq\frac{1}{2}\min\left\{1,m^t\right\}tA^{1/2}\left(\frac{1_H-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right)^2A^{1/2}\geq 0.
\end{aligned}
$$

From (3.8) we have, by multiplying both sides by $A^{-1/2}$, that

$$
\frac{1_H-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}=A^{-1/2}T_t\left(A|B\right)\left(A\sharp_t B\right)^{-1}A^{1/2}.
$$

Then

$$\begin{aligned}
&A^{1/2}\left(\frac{1_H-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right)^2 A^{1/2}\\
&=A^{1/2}\left(A^{-1/2}T_t\left(A|B\right)\left(A\natural_t B\right)^{-1}A^{1/2}\right)^2 A^{1/2}\\
&=A^{1/2}A^{-1/2}T_t\left(A|B\right)\left(A\natural_t B\right)^{-1}A^{1/2}A^{-1/2}T_t\left(A|B\right)\left(A\natural_t B\right)^{-1}A^{1/2}A^{1/2}\\
&=T_t\left(A|B\right)\left(A\natural_t B\right)^{-1}T_t\left(A|B\right)\left(A\natural_t B\right)^{-1}A\\
&=\left(T_t\left(A|B\right)\left(A\natural_t B\right)^{-1}\right)^2 A,
\end{aligned}$$

which together with (3.10) produces the desired result (3.5). □

There are some particular inequalities of interest as follows.

For $t=1$ we get from (3.4) and (3.5) that

$$\begin{aligned}
&\frac{1}{2\min\{1,m\}}\left(B-A\right)\left(A^{-1}-B^{-1}\right)A\\
&\geq S\left(A|B\right)-\left(1_H-AB^{-1}\right)A\\
&\geq \frac{1}{2\max\{1,M\}}\left(B-A\right)\left(A^{-1}-B^{-1}\right)A\geq 0
\end{aligned}\tag{3.11}$$

and

$$\begin{aligned}
&\frac{1}{2}\max\{1,M\}\left(1_H-AB^{-1}\right)^2 A\\
&\geq S\left(A|B\right)-\left(1_H-AB^{-1}\right)A\\
&\geq \frac{1}{2}\min\{1,m\}\left(1_H-AB^{-1}\right)^2 A\geq 0.
\end{aligned}\tag{3.12}$$

For $t=1/2$ we get from (3.4) and (3.5) that

$$\begin{aligned}
&\frac{1}{\min\{1,\sqrt{m}\}}\left(A\sharp B-A\right)\left(A^{-1}-\left(A\sharp B\right)^{-1}\right)A\\
&\geq S\left(A|B\right)-2\left(1_H-A\left(A\sharp B\right)^{-1}\right)A\\
&\geq \frac{1}{\max\left\{1,\sqrt{M}\right\}}\left(A\sharp B-A\right)\left(A^{-1}-\left(A\sharp B\right)^{-1}\right)A\geq 0
\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}
&\max\left\{1,\sqrt{M}\right\}\left(1_H-A\left(A\sharp B\right)^{-1}\right)^2 A\\
&\geq S\left(A|B\right)-2\left(1_H-A\left(A\sharp B\right)^{-1}\right)A\\
&\geq \min\left\{1,\sqrt{m}\right\}\left(1_H-A\left(A\sharp B\right)^{-1}\right)^2 A\geq 0.
\end{aligned}\tag{3.14}$$

For $t=2$ we get from (3.4) and (3.5) that

$$\begin{aligned}\frac{1}{4\min\{1,m^2\}}&\left(BA^{-1}B-A\right)\left(A^{-1}-B^{-1}AB^{-1}\right)A \\ &\geq S(A|B)-\frac{1}{2}\left(1_H-\left(AB^{-1}\right)^2\right)A \\ &\geq \frac{1}{4\max\{1,M^2\}}\left(BA^{-1}B-A\right)\left(A^{-1}-B^{-1}AB^{-1}\right)A\geq 0\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\frac{1}{4}\max\left\{1,M^2\right\}&\left(1_H-\left(AB^{-1}\right)^2\right)^2 A \\ &\geq S(A|B)-\frac{1}{2}\left(1_H-\left(AB^{-1}\right)^2\right)A \\ &\geq \frac{1}{4}\min\left\{1,m^2\right\}\left(1_H-\left(AB^{-1}\right)^2\right)^2 A\geq 0.\end{aligned} \tag{3.16}$$

We have the following:

Lemma 2. *Let $x\in[m,M]$ and $t>0$, then we have*

$$\frac{1}{2\min\{1,m^t\}}t\left(\frac{x^t-1}{t}\right)^2\geq\frac{x^t-1}{t}-\ln x\geq\frac{1}{2\max\{1,M^t\}}t\left(\frac{x^t-1}{t}\right)^2 \tag{3.17}$$

and

$$\begin{aligned}\frac{1}{2}\max\left\{1,M^t\right\}&\left(\frac{x^t-1}{t}-\frac{1-x^{-t}}{t}\right) \\ &\geq\frac{x^t-1}{t}-\ln x \\ &\geq\frac{1}{2}\min\left\{1,m^t\right\}\left(\frac{x^t-1}{t}-\frac{1-x^{-t}}{t}\right).\end{aligned} \tag{3.18}$$

Proof. Let $y=x^t\in[m^t,M^t]$. By using the inequality (2.15) we have (3.17) and by (2.17) we have (3.18). □

We also have:

Theorem 4. *Let A, B be two positive invertible operators and the constants $M>m>0$ with the property (3.3). Then for any $t>0$ we have*

$$\begin{aligned}\frac{1}{2\min\{1,m^t\}}&tT_t(A|B)A^{-1}T_t(A|B) \\ &\geq T_t(A|B)-S(A|B) \\ &\geq\frac{1}{2\max\{1,M^t\}}tT_t(A|B)A^{-1}T_t(A|B)\geq 0\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\frac{1}{2}\max\left\{1,M^t\right\}&T_t(A|B)\left(1_H-(A\sharp_t B)^{-1}A\right) \\ &\geq T_t(A|B)-S(A|B) \\ &\geq\frac{1}{2}\min\left\{1,m^t\right\}T_t(A|B)\left(1_H-(A\sharp_t B)^{-1}A\right)\geq 0.\end{aligned} \tag{3.20}$$

Proof. If we use the inequality (3.17) for the selfadjoint operator $X = A^{-1/2}BA^{-1/2}$ that has its spectrum contained in the interval $[m, M]$, then we get

$$\begin{aligned}
&\frac{1}{2\min\{1,m^t\}}t\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}\right)^2\\
&\geq \frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}-\ln\left(A^{-1/2}BA^{-1/2}\right)\\
&\geq \frac{1}{2\max\{1,M^t\}}t\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}\right)^2\geq 0
\end{aligned}$$

for any $t > 0$.

If we multiply both sides of this inequality by $A^{1/2}$ we get

$$\begin{aligned}
&\frac{1}{2\min\{1,m^t\}}tA^{1/2}\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}\right)^2A^{1/2}\\
&\geq A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}A^{1/2}-A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}\\
&\geq \frac{1}{2\max\{1,M^t\}}tA^{1/2}\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}\right)^2A^{1/2}\geq 0
\end{aligned}\tag{3.21}$$

for any $t > 0$.

Since

$$A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}A^{1/2}=T_t\left(A|B\right),$$

then

$$\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}=A^{-1/2}T_t\left(A|B\right)A^{-1/2}$$

and

$$\begin{aligned}
&A^{1/2}\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}\right)^2A^{1/2}\\
&=A^{1/2}A^{-1/2}T_t\left(A|B\right)A^{-1/2}A^{-1/2}T_t\left(A|B\right)A^{-1/2}A^{1/2}\\
&=T_t\left(A|B\right)A^{-1}T_t\left(A|B\right)
\end{aligned}$$

for any $t > 0$.

By making use of (3.21) we then get (3.19).

By using inequality (3.18) we have

$$\begin{aligned}
&\frac{1}{2}\max\{1,M^t\}\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}-\frac{1-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right)\\
&\geq \frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}-\ln\left(A^{-1/2}BA^{-1/2}\right)\\
&\geq \frac{1}{2}\min\{1,m^t\}\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}-\frac{1-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right)\\
&\geq 0,
\end{aligned}$$

for any $t>0$.

If we multiply both sides of this inequality by $A^{1/2}$ we get

$$\begin{aligned}
&\frac{1}{2}\max\left\{1,M^{t}\right\}A^{1/2}\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^{t}-1}{t}-\frac{1-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right)A^{1/2}\\
&\geq A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^{t}-1}{t}A^{1/2}-A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}\\
&\geq \frac{1}{2}\min\left\{1,m^{t}\right\}A^{1/2}\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^{t}-1}{t}-\frac{1-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right)A^{1/2}\\
&\geq 0
\end{aligned}$$

for any $t>0$, and the inequality (3.20) is obtained. □

For $t=1$ we get from (3.19) and (3.20) that

$$\begin{aligned}
&\frac{1}{2\min\{1,m\}}(B-A)A^{-1}(B-A)\\
&\geq B-A-S(A|B)\\
&\geq \frac{1}{2\max\{1,M\}}(B-A)A^{-1}(B-A)\geq 0
\end{aligned}\tag{3.22}$$

and

$$\begin{aligned}
&\frac{1}{2}\max\{1,M\}(B-A)\left(1_{H}-B^{-1}A\right)\\
&\geq B-A-S(A|B)\\
&\geq \frac{1}{2}\min\{1,m\}(B-A)\left(1_{H}-B^{-1}A\right)\geq 0.
\end{aligned}\tag{3.23}$$

For $t=1/2$ we get from (3.19) and (3.20) that

$$\begin{aligned}
&\frac{1}{\min\{1,\sqrt{m}\}}(A\sharp B-A)A^{-1}(A\sharp B-A)\\
&\geq 2(A\sharp B-A)-S(A|B)\\
&\geq \frac{1}{\max\left\{1,\sqrt{M}\right\}}(A\sharp B-A)A^{-1}(A\sharp B-A)\geq 0
\end{aligned}\tag{3.24}$$

and

$$\begin{aligned}
&\max\left\{1,\sqrt{M}\right\}(A\sharp B-A)\left(1_{H}-(A\sharp B)^{-1}A\right)\\
&\geq 2(A\sharp B-A)-S(A|B)\\
&\geq \min\left\{1,\sqrt{m}\right\}(A\sharp B-A)\left(1_{H}-(A\sharp B)^{-1}A\right)\geq 0.
\end{aligned}\tag{3.25}$$

For $t=2$ we get from (3.19) and (3.20) that

$$\begin{aligned}
&\frac{1}{4\min\{1,m^{2}\}}\left(BA^{-1}B-A\right)A^{-1}\left(BA^{-1}B-A\right)\\
&\geq \frac{1}{2}\left(BA^{-1}B-A\right)-S(A|B)\\
&\geq \frac{1}{4\max\{1,M^{2}\}}\left(BA^{-1}B-A\right)A^{-1}\left(BA^{-1}B-A\right)\geq 0
\end{aligned}\tag{3.26}$$

and

$$\begin{aligned}(3.27)\qquad & \frac{1}{4}\max\left\{1,M^2\right\}\left(BA^{-1}B-A\right)\left(1_H-\left(B^{-1}A\right)^2\right)\\ & \geq \frac{1}{2}\left(BA^{-1}B-A\right)-S\left(A|B\right)\\ & \geq \frac{1}{4}\min\left\{1,m^2\right\}\left(BA^{-1}B-A\right)\left(1_H-\left(B^{-1}A\right)^2\right)\geq 0.\end{aligned}$$

4. Some Global Bounds

For $[m,M]\subset(0,\infty)$ and $t>0$ and by the use of (2.18) we define

$$U_t\left(m,M\right):=U\left(m^t,M^t\right)=\begin{cases}\frac{\left(m^t-1\right)^2}{m^t} \text{ if } M<1,\\ \max\left\{\frac{\left(m^t-1\right)^2}{m^t},\frac{\left(M^t-1\right)^2}{M^t}\right\} \text{ if } m\leq 1\leq M,\\ \frac{\left(M^t-1\right)^2}{M^t} \text{ if } 1<m\end{cases}\tag{4.1}$$

and by (2.19)

$$u_t\left(m,M\right):=u\left(m^t,M^t\right)=\begin{cases}\frac{\left(1-M^t\right)^2}{M^t} \text{ if } M<1,\\ 0 \text{ if } m\leq 1\leq M,\\ \frac{\left(m^t-1\right)^2}{m^t} \text{ if } 1<m.\end{cases}\tag{4.2}$$

By (2.20) and (2.21) we have for $y=x^t\in[m^t,M^t]$ and $t>0$ that

$$\frac{1}{2t\min\left\{1,m^t\right\}}U_t\left(m,M\right)\geq \ln x-\frac{1-x^{-t}}{t}\geq\frac{1}{2t\max\left\{1,M^t\right\}}u_t\left(m,M\right),\tag{4.3}$$

and

$$\frac{1}{2t}\max\left\{1,M^t\right\}U_t\left(m,M\right)\geq\frac{x^t-1}{t}-\ln x\geq\frac{1}{2t}\min\left\{1,m^t\right\}u_t\left(m,M\right),\tag{4.4}$$

where $x\in[m,M]$ and $t>0$.

Using (2.22) and (2.23) we define

$$\begin{aligned}(4.5)\qquad Z_t\left(m,M\right)&:=Z\left(m^t,M^t\right)\\ &=\begin{cases}\left(1-m^t\right)^2 \text{ if } M<1,\\ \max\left\{\left(1-m^t\right)^2,\left(M^t-1\right)^2\right\} \text{ if } m\leq 1\leq M,\\ \left(M^t-1\right)^2 \text{ if } 1<m\end{cases}\end{aligned}$$

and

$$z_t\left(m,M\right):=z\left(m^t,M^t\right)=\begin{cases}\left(1-M^t\right)^2 \text{ if } M<1,\\ 0 \text{ if } m\leq 1\leq M,\\ \left(m^t-1\right)^2 \text{ if } 1<m.\end{cases}\tag{4.6}$$

By (2.24) we have for $y=x^t\in[m^t,M^t]$ and $t>0$ that

$$\frac{1}{2t\min\left\{1,m^t\right\}}Z_t\left(m,M\right)\geq\frac{x^t-1}{t}-\ln x\geq\frac{1}{2t\max\left\{1,M^t\right\}}z_t\left(m,M\right),\tag{4.7}$$

where $x\in[m,M]$ and $t>0$.

Utilising (2.25) and (2.26) we can define

$$W_t(m,M) := W\left(m^t, M^t\right) = \begin{cases} \left(\frac{1-m^t}{m^t}\right)^2 \text{ if } M<1, \\ \max\left\{\left(\frac{1-m^t}{m^t}\right)^2, \left(\frac{M^t-1}{M^t}\right)^2\right\} \text{ if } m \le 1 \le M, \\ \left(\frac{M^t-1}{M^t}\right)^2 \text{ if } 1<m \end{cases} \tag{4.8}$$

and

$$w_t(m,M) := W\left(m^t, M^t\right) = \begin{cases} \left(\frac{1-M^t}{M^t}\right)^2 \text{ if } M<1, \\ 0 \text{ if } m \le 1 \le M, \\ \left(\frac{m^t-1}{m^t}\right)^2 \text{ if } 1<m. \end{cases} \tag{4.9}$$

By (2.24) we have for $y = x^t \in [m^t, M^t]$ and $t>0$ that

$$\frac{1}{2t}\max\left\{1, M^t\right\} W_t(m,M) \ge \ln x - \frac{1-x^{-t}}{t} \ge \frac{1}{2t}\min\left\{1, m^t\right\} w_t(m,M), \tag{4.10}$$

where $x \in [m,M]$ and $t>0$.

Theorem 5. *Let A, B be two positive invertible operators and the constants $M > m > 0$ with the property (3.3). Then for any $t>0$ we have*

$$\begin{aligned} \frac{1}{2t\min\{1,m^t\}} U_t(m,M) A &\ge S(A|B) - T_t(A|B)(A\sharp_t B)^{-1} A \\ &\ge \frac{1}{2t\max\{1,M^t\}} u_t(m,M) A, \end{aligned}$$

$$\begin{aligned} \frac{1}{2t}\max\left\{1,M^t\right\} W_t(m,M) A &\ge S(A|B) - T_t(A|B)(A\sharp_t B)^{-1} A \\ &\ge \frac{1}{2t}\min\left\{1,m^t\right\} w_t(m,M) A, \end{aligned}$$

$$\begin{aligned} \frac{1}{2t\min\{1,m^t\}} Z_t(m,M) A &\ge T_t(A|B) - S(A|B) \\ &\ge \frac{1}{2t\max\{1,M^t\}} z_t(m,M) A \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2t}\max\left\{1,M^t\right\} U_t(m,M) A &\ge T_t(A|B) - S(A|B) \\ &\ge \frac{1}{2t}\min\left\{1,m^t\right\} u_t(m,M) A. \end{aligned}$$

The proof follows by the inequalities (4.4), (4.5), (4.7) and (4.10) in a similar way as the one from the proof of Theorem 3 and we omit the details.

For $t=1$, $t=1/2$ and $t=2$ one can obtain some particular inequalities of interest, however the details are not provided here.

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