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NEW INEQUALITIES FOR LOGARITHM VIA TAYLOR'S EXPANSION WITH INTEGRAL REMAINDER

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Abstract. In this paper we establish several new inequalities for logarithm by the use of Taylor's expansion with integral remainder. The case of two positive numbers and an analysis of which bound is better are also considered.

1. Introduction

In the recent paper [1] we established the following result:

$$(0\leq)\left(1-\nu\right)a+\nu b-a^{1-\nu}b^{\nu}\leq\nu\left(1-\nu\right)\left(b-a\right)\left(\ln b-\ln a\right) \tag{1.1}$$

for any $a,\ b>0$ and $\nu\in(0,1)$.

If we take in (1.1) $b=x+1$, $x>0$ and $a=1$, then we get

$$\ln\left(x+1\right)\geq\frac{1-\nu+\nu\left(x+1\right)-\left(x+1\right)^{\nu}}{\nu\left(1-\nu\right)x}\ (\geq 0) \tag{1.2}$$

for any $\nu\in(0,1)$ and, in particular

$$\ln\left(x+1\right)\geq\frac{2\left(\sqrt{x+1}-1\right)^{2}}{x} \tag{1.3}$$

for any $x>0$ and $\nu\in(0,1)$.

If we take in (1.1) $b=x$ and $a=1$ we also have

$$\frac{\ln x}{x-1}\geq\frac{1-\nu+\nu x-x^{\nu}}{\nu\left(1-\nu\right)\left(x-1\right)^{2}} \tag{1.4}$$

for any $x>0$, $x\neq1$ and $\nu\in(0,1)$.

Further, by choosing in (1.4) $\nu=\frac{1}{2}$ and perform the calculations, we get

$$\frac{\ln x}{x-1}\geq\frac{2}{\left(\sqrt{x}+1\right)^{2}} \tag{1.5}$$

for any $x>0$, $x\neq1$.

In the recent paper [5] we obtained the following inequalities for logarithm as well

$$\begin{aligned}0 &\leq \frac{x-1}{x}\leq\frac{2\left(x-1\right)}{x+1}\leq\ln x\leq\frac{x-1}{\sqrt{x}}\\ &\leq\frac{x-1}{x+1}+\frac{x^{2}-1}{4x}\leq\frac{x^{2}-1}{2x}\leq x-1\end{aligned} \tag{1.6}$$

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where $x \geq 1$ and

$$0 \leq \frac{x^2 - 1}{2x} - \ln x \leq \frac{1}{8} \frac{(x-1)^3 (x+1)}{x^2}, \tag{1.7}$$

$$0 \leq \ln x - \frac{2(x-1)}{x+1} \leq \frac{1}{8} \frac{(x-1)^3 (x+1)}{x^2}, \tag{1.8}$$

where $x \geq 1$.

There are also a number of inequalities for logarithm that are well know and widely used in literature, such as:

$$\frac{x-1}{x} \leq \ln x \leq x - 1 \text{ for } x > 0, \tag{1.9}$$

$$\frac{2x}{2+x} \leq \ln(1+x) \leq \frac{x}{\sqrt{x+1}} \text{ for } x \geq 0, \tag{1.10}$$

$$x \leq -\ln(1-x) \leq \frac{x}{1-x}, \text{ for } x < 1,$$

$$\ln x \leq n\left(x^{1/n} - 1\right) \text{ for } n > 0 \text{ and } x > 0,$$

$$\ln(1 - |x|) \leq \ln(x+1) \leq -\ln(1 - |x|) \text{ for } |x| < 1,$$

and

$$-\frac{3}{2}x \leq \ln(1-x) \leq \frac{3}{2}x \text{ for } 0 < x \leq 0.5838,$$

see for instance

`http://functions.wolfram.com/ElementaryFunctions/Log/29/`

and [7].

A simple proof of the first inequality in (1.10) may be found, for instance, in [8], see also [9] where the following rational bounds are provided as well:

$$\frac{x\left(1 + \frac{5}{6}x\right)}{(1+x)\left(1 + \frac{1}{3}x\right)} \leq \ln(1+x) \leq \frac{x\left(1 + \frac{1}{6}x\right)}{1 + \frac{2}{3}x} \text{ for } x \geq 0.$$

In the recent paper [3] we established the following inequalities as well:

Theorem 1. *For any a, $b > 0$ we have for $n \geq 1$ that*

$$\frac{(b-a)^{2n}}{2n \max^{2n}\{a,b\}} \leq \sum_{k=1}^{2n-1} (-1)^{k-1} \frac{(b-a)^k}{ka^k} - \ln b + \ln a \leq \frac{(b-a)^{2n}}{2n \min^{2n}\{a,b\}} \tag{1.11}$$

and

$$\frac{(b-a)^{2n}}{2n \max^{2n}\{a,b\}} \leq \ln b - \ln a - \sum_{k=1}^{2n-1} \frac{(b-a)^k}{kb^k} \leq \frac{(b-a)^{2n}}{2n \min^{2n}\{a,b\}}. \tag{1.12}$$

Corollary 1. *For any a, $b > 0$ we have*

$$\frac{(b-a)^2}{2\max^2\{a,b\}} \leq \frac{b-a}{a} - \ln b + \ln a \leq \frac{(b-a)^2}{2\min^2\{a,b\}}, \tag{1.13}$$

$$\frac{(b-a)^2}{2\max^2\{a,b\}} \leq \ln b - \ln a - \frac{b-a}{b} \leq \frac{(b-a)^2}{2\min^2\{a,b\}}. \tag{1.14}$$

We have the following upper bounds [3]:

Theorem 2. *For any a, $b>0$ we have for $n\geq 1$ that*

$$(0\leq)\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}(b-a)^k}{ka^k}-\ln b+\ln a\leq\frac{|b-a|^{2n-1}\left|b^{2n-1}-a^{2n-1}\right|}{(2n-1)\,b^{2n-1}a^{2n-1}} \tag{1.15}$$

and

$$(0\leq)\ln b-\ln a-\sum_{k=1}^{2n-1}\frac{(b-a)^k}{kb^k}\leq\frac{|b-a|^{2n-1}\left|b^{2n-1}-a^{2n-1}\right|}{(2n-1)\,b^{2n-1}a^{2n-1}}. \tag{1.16}$$

Corollary 2. *For any a, $b>0$ we have the simpler inequalities*

$$(0\leq)\frac{b-a}{a}-\ln b+\ln a\leq\frac{(b-a)^2}{ab} \tag{1.17}$$

and

$$(0\leq)\ln b-\ln a-\frac{b-a}{b}\leq\frac{(b-a)^2}{ab}. \tag{1.18}$$

We also have [3]:

Theorem 3. *Let p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. For any a, $b>0$ we have for $n\geq 1$ that*

$$\begin{aligned}(0\leq)&\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}(b-a)^k}{ka^k}-\ln b+\ln a\\&\leq\frac{|b-a|^{2n-1+1/p}\left|b^{2nq-1}-a^{2nq-1}\right|^{1/q}}{[(2n-1)\,p+1]^{1/p}(2nq-1)^{1/q}(ba)^{2n-1/q}}\end{aligned} \tag{1.19}$$

and

$$\begin{aligned}(0\leq)&\ln b-\ln a-\sum_{k=1}^{2n-1}\frac{(b-a)^k}{kb^k}\\&\leq\frac{|b-a|^{2n-1+1/p}\left|b^{2nq-1}-a^{2nq-1}\right|^{1/q}}{[(2n-1)\,p+1]^{1/p}(2nq-1)^{1/q}(ba)^{2n-1/q}}.\end{aligned} \tag{1.20}$$

Corollary 3. *Let p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. For any a, $b>0$ we have*

$$(0\leq)\frac{b-a}{a}-\ln b+\ln a\leq\frac{|b-a|^{1+1/p}\left|b^{2q-1}-a^{2q-1}\right|^{1/q}}{[(p+1)]^{1/p}(2q-1)^{1/q}(ba)^{2-1/q}} \tag{1.21}$$

and

$$(0\leq)\ln b-\ln a-\frac{b-a}{b}\leq\frac{|b-a|^{1+1/p}\left|b^{2q-1}-a^{2q-1}\right|^{1/q}}{[(p+1)]^{1/p}(2q-1)^{1/q}(ba)^{2-1/q}}. \tag{1.22}$$

In this paper we establish some bounds for the quantities

$$\ln b-\ln a-\frac{b-a}{b}+\frac{1}{b}\sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k(k-1)}\frac{(b-a)^k}{a^{k-1}}$$

and

$$\frac{b-a}{a}-\frac{1}{a}\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}\frac{(b-a)^k}{b^{k-1}}-\ln b+\ln a$$

when a, $b>0$ and $n\geq 1$. The quantities above without sum-terms are of interest as well. The case $a=1$ and $b=x$ is explored and various local and global bounds in the case that x is located in a bounded interval are also provided. By performing some numerical experiments it is also shown that the obtained upper bounds can not be compared in general, meaning that some time one is better than the other.

2. Some New Inequalities for Two Numbers

The following theorem is well known in the literature as Taylor's theorem with the integral remainder.

Theorem 4. *Let $I\subset\mathbb{R}$ be a closed interval, $a\in I$ and let m be a positive integer. If $f:I\longrightarrow\mathbb{R}$ is such that $f^{(m)}$ is absolutely continuous on I, then for each $x\in I$*

$$f(x)=T_m(f;a,x)+R_m(f;a,x) \tag{2.1}$$

where $T_m(f;a,x)$ is Taylor's polynomial, i.e.,

$$T_m(f;a,x):=\sum_{k=0}^{m}\frac{(x-a)^k}{k!}f^{(k)}(a).$$

(Note that $f^{(0)}:=f$ and $0!:=1$), and the remainder is given by

$$R_m(f;a,x):=\frac{1}{m!}\int_a^x (x-t)^m f^{(m+1)}(t)\,dt.$$

We have the following representation result:

Lemma 1. *For any a, $b>0$ we have*

$$\ln b-\ln a-\frac{b-a}{b}=\frac{1}{b}\int_a^b\frac{b-t}{t}dt,\ [4] \tag{2.2}$$

and for any $m\geq 2$ and any a, $b>0$

$$\ln b-\ln a-\frac{b-a}{b}+\frac{1}{b}\sum_{k=2}^{m}\frac{(-1)^{k-1}}{k(k-1)}\frac{(b-a)^k}{a^{k-1}}=\frac{(-1)^{m-1}}{mb}\int_a^b\frac{(b-t)^m}{t^m}dt. \tag{2.3}$$

Proof. Consider the function $f:(0,\infty)\longrightarrow\mathbb{R}$, $f(x)=x\ln x$, then

$$f'(x)=\ln x+1 \text{ and } f''(x)=\frac{1}{x}$$

and, in general, for $m\geq 2$ we have

$$f^{(m)}(x)=\frac{(-1)^m(m-2)!}{x^{m-1}},$$

where $0!:=1$.

If we use Taylor's representation (2.1) for $m=1$ we have

$$f(x)=f(a)+(x-a)f'(a)+\int_a^x(x-t)f''(t)\,dt$$

for any x, $a\in I$.

If we write this equality for $f(x)=x\ln x$ and $x=b$ we get

$$b\ln b=a\ln a+(b-a)(\ln a+1)+\int_a^b\frac{b-t}{t}dt$$

namely

$$b\ln b = b\ln a + b - a + \int_a^b \frac{b-t}{t}dt$$

for any $a,\ b > 0$ that is equivalent to (2.2).

If we use Taylor's representation (2.1) for $m \geq 1$ we have

$$f(x) = f(a) + (x-a) f'(a) + \sum_{k=2}^{m} \frac{(x-a)^k}{k!} f^{(k)}(a) + \frac{1}{m!}\int_a^x (x-t)^m f^{(m+1)}(t)\,dt$$

for any $x, a \in I$.

If we write this equality for $f(x) = x\ln x$ and $x = b$ we get

$$\begin{aligned} b\ln b &= a\ln a + (b-a)(\ln a + 1) + \sum_{k=2}^{m} \frac{(-1)^k}{k(k-1)} \frac{(b-a)^k}{a^{k-1}} \\ &\quad + \frac{(-1)^{m-1}}{m}\int_a^b \frac{(b-t)^m}{t^m}dt \end{aligned}$$

namely

$$\begin{aligned} b\ln b &= b\ln a + b - a + \sum_{k=2}^{m} \frac{(-1)^k}{k(k-1)} \frac{(b-a)^k}{a^{k-1}} \\ &\quad + \frac{(-1)^{m-1}}{m}\int_a^b \frac{(b-t)^m}{t^m}dt \end{aligned}$$

for any $a,\ b > 0$ that is equivalent to (2.3). □

We have the following bounds:

Theorem 5. *For any $a,\ b > 0$ we have that*

$$\frac{1}{2b\max\{a,b\}}(b-a)^2 \leq \ln b - \ln a - \frac{b-a}{b} \leq \frac{1}{2b\min\{a,b\}}(b-a)^2 \tag{2.4}$$

and

$$\frac{1}{2a\max\{a,b\}}(b-a)^2 \leq \frac{b-a}{a} - \ln b + \ln a \leq \frac{1}{2a\min\{a,b\}}(b-a)^2. \tag{2.5}$$

If $n \geq 1$, then for any $a,\ b > 0$ we have that

$$\begin{aligned} &\frac{(b-a)^{2n+2}}{(2n+1)(2n+2)\,b\max^{2n+1}\{a,b\}} \\ &\leq \ln b - \ln a - \frac{b-a}{b} + \frac{1}{b}\sum_{k=2}^{2n+1} \frac{(-1)^{k-1}}{k(k-1)} \frac{(b-a)^k}{a^{k-1}} \\ &\leq \frac{(b-a)^{2n+2}}{(2n+1)(2n+2)\,b\min^{2n+1}\{a,b\}} \end{aligned} \tag{2.6}$$

and

$$\frac{(b-a)^{2n+2}}{(2n+1)(2n+2)\,a\max^{2n+1}\{a,b\}} \tag{2.7}$$
$$\leq \frac{b-a}{a}-\frac{1}{a}\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}\frac{(b-a)^k}{b^{k-1}}-\ln b+\ln a$$
$$\leq \frac{(b-a)^{2n+2}}{(2n+1)(2n+2)\,a\min^{2n+1}\{a,b\}}.$$

Proof. Let $b>a>0$, then

$$\frac{1}{a}\int_a^b (b-t)\,dt \geq \int_a^b \frac{b-t}{t}dt \geq \frac{1}{b}\int_a^b (b-t)\,dt$$

giving that

$$\frac{1}{2a}(b-a)^2 \geq \int_a^b \frac{b-t}{t}dt \geq \frac{1}{2b}(b-a)^2. \tag{2.8}$$

Let $a>b>0$, then

$$\frac{1}{b}\int_b^a (t-b)\,dt \geq \int_a^b \frac{b-t}{t}dt = \int_b^a \frac{t-b}{t}dt \geq \frac{1}{a}\int_b^a (t-b)\,dt$$

giving that

$$\frac{1}{2b}(b-a)^2 \geq \int_a^b \frac{b-t}{t}dt \geq \frac{1}{2a}(b-a)^2. \tag{2.9}$$

Therefore, by (2.4) and (2.5) we get

$$\frac{1}{2\min\{a,b\}}(b-a)^2 \geq \int_a^b \frac{b-t}{t}dt \geq \frac{1}{2\max\{a,b\}}(b-a)^2,$$

for any $a,\ b>0$.

By utilising the equality (2.2) we get the desired result (2.4).

Let $m=2n+1$ with $n\geq 1$. Then by (2.3) we have

$$\ln b-\ln a-\frac{b-a}{b}+\frac{1}{b}\sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k(k-1)}\frac{(b-a)^k}{a^{k-1}} \tag{2.10}$$
$$=\frac{1}{(2n+1)\,b}\int_a^b \frac{(b-t)^{2n+1}}{t^{2n+1}}dt.$$

Let $b>a>0$, then

$$\frac{(b-a)^{2n+2}}{a^{2n+1}(2n+2)} \geq \int_a^b \frac{(b-t)^{2n+1}}{t^{2n+1}}dt \geq \frac{(b-a)^{2n+2}}{b^{2n+1}(2n+2)}. \tag{2.11}$$

If $a>b>0$, then

$$\int_a^b \frac{(b-t)^{2n+1}}{t^{2n+1}}dt = \int_b^a \frac{(t-b)^{2n+1}}{t^{2n+1}}dt$$

and

$$\frac{(b-a)^{2n+2}}{b^{2n+1}(2n+2)} \geq \int_b^a \frac{(t-b)^{2n+1}}{t^{2n+1}}dt \geq \frac{(b-a)^{2n+2}}{a^{2n+1}(2n+2)}. \tag{2.12}$$

Using (2.11) and (2.12) we get

$$\frac{(b-a)^{2n+2}}{\min^{2n+1}\{a,b\}(2n+2)} \geq \int_a^b \frac{(b-t)^{2n+1}}{t^{2n+1}}dt \geq \frac{(b-a)^{2n+2}}{\max^{2n+1}\{a,b\}(2n+2)} \tag{2.13}$$

for any a, $b>0$.

Finally, on utilising the representation (2.10) and the inequality (2.13) we get the desired result (2.6).

The inequality (2.7) follows from (2.6) by replacing a with b. □

Corollary 4. *For any a, $b>0$ we have that*

$$\begin{aligned}&\frac{(b-a)^4}{12b\max^3\{a,b\}}\\ &\leq \ln b-\ln a-\frac{b-a}{b}-\frac{1}{2}\frac{(b-a)^2}{ab}+\frac{1}{6}\frac{(b-a)^3}{a^2b}\\ &\leq \frac{(b-a)^4}{12b\min^3\{a,b\}}\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}&\frac{(b-a)^4}{12a\max^3\{a,b\}}\\ &\leq \frac{b-a}{a}-\frac{(b-a)^2}{2ab}-\frac{1}{6}\frac{(b-a)^3}{ab^2}-\ln b+\ln a\\ &\leq \frac{(b-a)^4}{12a\min^3\{a,b\}}.\end{aligned} \tag{2.15}$$

Remark 1. *Since the lower bounds in (2.6) and (2.7) are positive, then we have*

$$\begin{aligned}&\frac{b-a}{b}+\frac{1}{b}\sum_{k=2}^{2n+1}\frac{(-1)^k}{k(k-1)}\frac{(b-a)^k}{a^{k-1}}\\ &\leq \ln b-\ln a\\ &\leq \frac{b-a}{a}-\frac{1}{a}\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}\frac{(b-a)^k}{b^{k-1}}\end{aligned} \tag{2.16}$$

for any a, $b>0$ and $n\geq 1$.

In particular, we have

$$\begin{aligned}&\frac{b-a}{b}+\frac{1}{2}\frac{(b-a)^2}{ab}-\frac{1}{6}\frac{(b-a)^3}{a^2b}\\ &\leq \ln b-\ln a\\ &\leq \frac{b-a}{a}-\frac{(b-a)^2}{2ab}-\frac{1}{6}\frac{(b-a)^3}{ab^2}\end{aligned} \tag{2.17}$$

for any a, $b>0$.

We have the following upper bounds:

Theorem 6. *For any a, $b>0$ we have that*

$$(0\leq)\ln b-\ln a-\frac{b-a}{b}\leq\frac{(b-a)^{2}}{bL(a,b)}\leq\frac{(b-a)^{2}}{b\sqrt{ab}} \tag{2.18}$$

and

$$(0\leq)\frac{b-a}{a}-\ln b+\ln a\leq\frac{(b-a)^{2}}{aL(a,b)}\leq\frac{(b-a)^{2}}{a\sqrt{ab}} \tag{2.19}$$

where $L(a,b)$ is the logarithmic mean

$$L(a,b):=\begin{cases}\frac{b-a}{\ln b-\ln a} & \text{if } a\neq b,\\ \\ b & \text{if } a=b.\end{cases}$$

If $n\geq 1$, then for any a, $b>0$ we have that

$$(0\leq)\ln b-\ln a-\frac{b-a}{b}+\frac{1}{b}\sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k(k-1)}\frac{(b-a)^{k}}{a^{k-1}} \tag{2.20}$$
$$\leq\frac{1}{2n(2n+1)}|b-a|^{2n+1}\frac{\left|b^{2n}-a^{2n}\right|}{a^{2n}b^{2n+1}}$$

and

$$(0\leq)\frac{b-a}{a}-\frac{1}{a}\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}\frac{(b-a)^{k}}{b^{k-1}}-\ln b+\ln a \tag{2.21}$$
$$\leq\frac{1}{2n(2n+1)}|b-a|^{2n+1}\frac{\left|b^{2n}-a^{2n}\right|}{a^{2n+1}b^{2n}}.$$

Proof. Let $b>a>0$, then

$$\int_{a}^{b}\frac{b-t}{t}dt\leq(b-a)(\ln b-\ln a)=\frac{(b-a)^{2}}{L(a,b)}$$

and the same inequality for $a>b>0$. This proves the first inequality in (2.18). The second part is well known as

$$L(a,b)\geq G(a,b),$$

where $G(a,b):=\sqrt{ab}$ is the geometric mean of a, $b>0$.

Let $n\geq 1$ and a, $b>0$. If $b>a>0$, then

$$\int_{a}^{b}\frac{(b-t)^{2n+1}}{t^{2n+1}}dt\leq(b-a)^{2n+1}\int_{a}^{b}t^{-2n-1}dt$$
$$=\frac{1}{2n}(b-a)^{2n+1}\frac{b^{2n}-a^{2n}}{a^{2n}b^{2n}}.$$

If $a>b>0$, then, similarly

$$\int_{a}^{b}\frac{(b-t)^{2n+1}}{t^{2n+1}}dt\leq\frac{1}{2n}(a-b)^{2n+1}\frac{a^{2n}-b^{2n}}{a^{2n}b^{2n}}.$$

Therefore for any $n\geq 1$ and a, $b>0$ we have

$$\int_{a}^{b}\frac{(b-t)^{2n+1}}{t^{2n+1}}dt\leq\frac{1}{2n}|b-a|^{2n+1}\frac{\left|b^{2n}-a^{2n}\right|}{a^{2n}b^{2n}}.$$

Using (2.10) we have (2.20). □

Corollary 5. *For any a, $b>0$ we have that*

$$(0\le)\ln b-\ln a-\frac{b-a}{b}-\frac{1}{2}\frac{(b-a)^2}{ab}+\frac{1}{6}\frac{(b-a)^3}{a^2b}\le\frac{1}{6}(b-a)^4\frac{b^2+ba+a^2}{a^2b^3}\tag{2.22}$$

and

$$(0\le)\frac{b-a}{a}-\frac{(b-a)^2}{2ab}-\frac{1}{6}\frac{(b-a)^3}{ab^2}-\ln b+\ln a\le\frac{1}{6}(b-a)^4\frac{b^2+ba+a^2}{a^3b^2}.\tag{2.23}$$

We also have:

Theorem 7. *Let p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. For any a, $b>0$ we have*

$$(0\le)\ln b-\ln a-\frac{b-a}{b}\le\frac{1}{(p+1)^{1/p}(q-1)^{1/q}}\frac{|b-a|^{1+1/p}\left|b^{q-1}-a^{q-1}\right|^{1/q}}{b^{2-1/q}a^{1-1/q}}\tag{2.24}$$

and

$$(0\le)\frac{b-a}{a}-\ln b+\ln a\le\frac{1}{(p+1)^{1/p}(q-1)^{1/q}}\frac{|b-a|^{1+1/p}\left|b^{q-1}-a^{q-1}\right|^{1/q}}{b^{1-1/q}a^{2-1/q}}.\tag{2.25}$$

If $n\ge1$ and a, $b>0$, then we have

$$(0\le)\ln b-\ln a-\frac{b-a}{b}+\frac{1}{b}\sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k(k-1)}\frac{(b-a)^k}{a^{k-1}}\le\frac{1}{(2n+1)((2n+1)p+1)^{1/p}((2n+1)q-1)^{1/q}}\times\frac{|b-a|^{2n+1+1/p}\left|b^{(2n+1)q-1}-a^{(2n+1)q-1}\right|^{1/q}}{b^{2n+2-1/q}a^{2n+1-1/q}}\tag{2.26}$$

and

$$(0\le)\frac{b-a}{a}-\frac{1}{a}\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}\frac{(b-a)^k}{b^{k-1}}-\ln b+\ln a\le\frac{1}{(2n+1)((2n+1)p+1)^{1/p}((2n+1)q-1)^{1/q}}\times\frac{|b-a|^{2n+1+1/p}\left|b^{(2n+1)q-1}-a^{(2n+1)q-1}\right|^{1/q}}{b^{2n+1-1/q}a^{2n+2-1/q}}.\tag{2.27}$$

Proof. Using Hölder's integral inequality for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have for $b>a>0$ that

$$\int_a^b \frac{b-t}{t}dt \le \left(\int_a^b (b-t)^p\,dt\right)^{1/p}\left(\int_a^b t^{-q}dt\right)^{1/q}$$
$$=\left(\frac{(b-a)^{p+1}}{p+1}\right)^{1/p}\left(\frac{b^{-q+1}-a^{-q+1}}{-q+1}\right)^{1/q}$$
$$=\frac{(b-a)^{1+1/p}}{(p+1)^{1/p}}\left(\frac{b^{q-1}-a^{q-1}}{(q-1)\,b^{q-1}a^{q-1}}\right)^{1/q}$$
$$=\frac{1}{(p+1)^{1/p}(q-1)^{1/q}}\frac{(b-a)^{1+1/p}\left(b^{q-1}-a^{q-1}\right)^{1/q}}{b^{1-1/q}a^{1-1/q}}.$$

If $a>b>0$, then in a similar way, we also have

$$\int_a^b \frac{b-t}{t}dt \le \frac{1}{(p+1)^{1/p}(q-1)^{1/q}}\frac{(a-b)^{1+1/p}\left(a^{q-1}-b^{q-1}\right)^{1/q}}{b^{1-1/q}a^{1-1/q}}.$$

Therefore, for any $a,\ b>0$ we have

$$\int_a^b \frac{b-t}{t}dt \le \frac{1}{(p+1)^{1/p}(q-1)^{1/q}}\frac{|b-a|^{1+1/p}\left|b^{q-1}-a^{q-1}\right|^{1/q}}{b^{1-1/q}a^{1-1/q}}.$$

Using the representation (2.2) we deduce the desired result (2.24).

Using Hölder's integral inequality for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we also have for $b>a>0$ that

$$\int_a^b \frac{(b-t)^{2n+1}}{t^{2n+1}}dt$$
$$\le \left(\int_a^b (b-t)^{(2n+1)p}\,dt\right)^{1/p}\left(\int_a^b t^{-(2n+1)q}dt\right)^{1/q}$$
$$=\left(\frac{(b-a)^{(2n+1)p+1}}{(2n+1)\,p+1}\right)^{1/p}\left(\frac{b^{-(2n+1)q+1}-a^{-(2n+1)q+1}}{-(2n+1)\,q+1}\right)^{1/q}$$
$$=\frac{(b-a)^{2n+1+1/p}}{((2n+1)\,p+1)^{1/p}}\left(\frac{b^{(2n+1)q-1}-a^{(2n+1)q-1}}{((2n+1)\,q-1)\,b^{(2n+1)q-1}a^{(2n+1)q-1}}\right)^{1/q}$$
$$=\frac{(b-a)^{2n+1+1/p}}{((2n+1)\,p+1)^{1/p}}\frac{\left(b^{(2n+1)q-1}-a^{(2n+1)q-1}\right)^{1/q}}{((2n+1)\,q-1)^{1/q}\,b^{2n+1-1/q}a^{2n+1-1/q}}.$$

If $a>b>0$, then in a similar way we also have

$$\int_a^b \frac{(b-t)^{2n+1}}{t^{2n+1}}dt$$
$$\le \frac{(a-b)^{2n+1+1/p}}{((2n+1)\,p+1)^{1/p}}\frac{\left(a^{(2n+1)q-1}-b^{(2n+1)q-1}\right)^{1/q}}{((2n+1)\,q-1)^{1/q}\,b^{2n+1-1/q}a^{2n+1-1/q}}.$$

Therefore, we have

$$\int_a^b \frac{(b-t)^{2n+1}}{t^{2n+1}}dt$$
$$\leq \frac{|b-a|^{2n+1+1/p}}{((2n+1)p+1)^{1/p}} \frac{\left|b^{(2n+1)q-1} - a^{(2n+1)q-1}\right|^{1/q}}{((2n+1)q-1)^{1/q} b^{2n+1-1/q} a^{2n+1-1/q}}.$$

Using the representation (2.10) we deduce the desired inequality □

Corollary 6. *For any a, $b > 0$ we have that*

$$(0 \leq) \ln b - \ln a - \frac{b-a}{b} - \frac{1}{2}\frac{(b-a)^2}{ab} + \frac{1}{6}\frac{(b-a)^3}{a^2 b} \tag{2.28}$$
$$\leq \frac{1}{3(3p+1)^{1/p}(3q-1)^{1/q}} \frac{|b-a|^{3+1/p}\left|b^{3q-1} - a^{3q-1}\right|^{1/q}}{b^{4-1/q}a^{3-1/q}}$$

and

$$(0 \leq) \frac{b-a}{a} - \frac{(b-a)^2}{2ab} - \frac{1}{6}\frac{(b-a)^3}{ab^2} - \ln b + \ln a \tag{2.29}$$
$$\leq \frac{1}{3(3p+1)^{1/p}(3q-1)^{1/q}} \frac{|b-a|^{3+1/p}\left|b^{3q-1} - a^{3q-1}\right|^{1/q}}{b^{3-1/q}a^{4-1/q}}.$$

Remark 2. *The Euclidean case, namely when $p = q = 2$ produces in Theorem 7*

$$(0 \leq) \ln b - \ln a - \frac{b-a}{b} \leq \frac{1}{\sqrt{3}} \frac{(b-a)^2}{b\sqrt{ab}} \tag{2.30}$$

and

$$(0 \leq) \frac{b-a}{a} - \ln b + \ln a \leq \frac{1}{\sqrt{3}} \frac{(b-a)^2}{a\sqrt{ab}}. \tag{2.31}$$

If $n \geq 1$ and a, $b > 0$, then we have

$$(0 \leq) \ln b - \ln a - \frac{b-a}{b} + \frac{1}{b}\sum_{k=2}^{2n+1} \frac{(-1)^{k-1}}{k(k-1)} \frac{(b-a)^k}{a^{k-1}} \tag{2.32}$$
$$\leq \frac{1}{(2n+1)(4n+3)^{1/2}(4n+1)^{1/2}} \frac{|b-a|^{2n+3/2}\left|b^{4n+1} - a^{4n+1}\right|^{1/2}}{b^{2n+3/2}a^{2n+1/2}}$$

and

$$(0 \leq) \frac{b-a}{a} - \frac{1}{a}\sum_{k=2}^{2n+1} \frac{1}{k(k-1)} \frac{(b-a)^k}{b^{k-1}} - \ln b + \ln a \tag{2.33}$$
$$\leq \frac{1}{(2n+1)(4n+3)^{1/2}(4n+1)^{1/2}} \frac{|b-a|^{2n+3/2}\left|b^{4n+1} - a^{4n+1}\right|^{1/2}}{b^{2n+1/2}a^{2n+3/2}}.$$

We observe that, by the inequalities (2.4), (2.20) and (2.32) the quantity

$$\frac{b-a}{a} - \ln b + \ln a$$

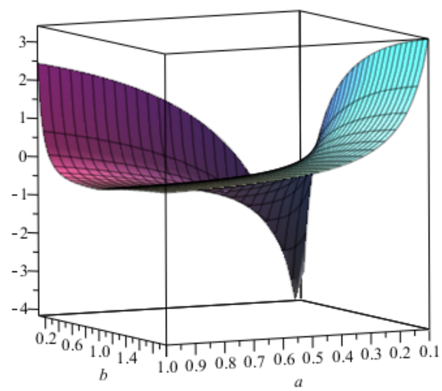

FIGURE 1. Plot of $D_1(a,b)$ on $[0.1,1]\times[0.1,2]$

has as upper bounds

$$A_1(a,b) := \frac{1}{2b\min\{a,b\}}(b-a)^2,\ A_2(a,b) = \frac{(b-a)^2}{bL(a,b)}$$

and

$$A_3(a,b) := \frac{1}{\sqrt{3}}\frac{(b-a)^2}{b\sqrt{ab}},$$

for a, $b > 0$.

It is therefore natural the to ask *how these bounds do compare*?

In order to answer this question we consider the simpler functions

$$B_1(a,b) := \frac{1}{2\min\{a,b\}},\ B_2(a,b) = \frac{1}{L(a,b)}$$

and

$$B_3(a,b) := \frac{1}{\sqrt{3}\sqrt{ab}}$$

for a, $b > 0$ and define the differences $D_1(a,b) := B_1(a,b) - B_2(a,b)$, $D_2(a,b) := B_2(a,b) - B_3(a,b)$ and $D_3(a,b) := B_1(a,b) - B_3(a,b)$.

The plot of $D_1(a,b)$ on the box $[0.1,1]\times[0.1,2]$ is depicted in Figure 1 and the plot of $D_3(a,b)$ on the box $[0.1,0.2]^2$ is depicted in Figure 2 showing that the bounds $A_1(a,b)$ and $A_2(a,b)$ and $A_1(a,b)$ and $A_3(a,b)$ do not compare in general, meaning that one is better than the other for different pairs of (a,b). Several numerical experiments show that $D_2(a,b) \geq 0$ suggesting that the bound $A_3(a,b)$ is better than $A_1(a,b)$. However we do not have an analytic proof of this fact and is left as an open question.

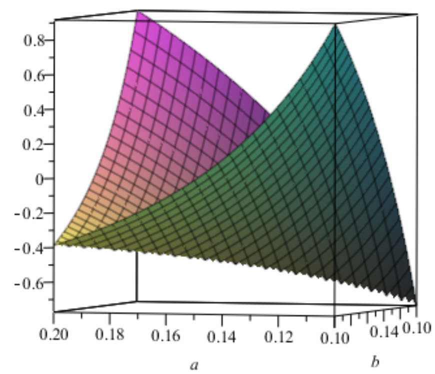

FIGURE 2. Plot of $D_2(a,b)$ on $[0.1,0.2]^2$

3. Some New Inequalities for One Number

From Theorem 5 we have, by taking $b = x \in (0,\infty)$ and $a = 1$, that

$$\frac{1}{2}\frac{(x-1)^2}{x\max\{1,x\}} \le \ln x - \frac{x-1}{x} \le \frac{1}{2}\frac{(x-1)^2}{x\min\{1,x\}} \tag{3.1}$$

and

$$\frac{1}{2}\frac{(x-1)^2}{\max\{1,x\}} \le x - 1 - \ln x \le \frac{1}{2}\frac{(x-1)^2}{\min\{1,x\}} \tag{3.2}$$

that have been obtained in [4] as well.

If $n \ge 1$, then for any $x > 0$ we have from (2.6) and (2.7) that

$$\begin{aligned} &\frac{(x-1)^{2n+2}}{(2n+1)(2n+2)\,x\max^{2n+1}\{1,x\}} \\ &\le \ln x - \frac{x-1}{x} + \frac{1}{x}\sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k(k-1)}(x-1)^k \\ &\le \frac{(x-1)^{2n+2}}{(2n+1)(2n+2)\,x\min^{2n+1}\{1,x\}} \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \frac{(x-1)^{2n+2}}{(2n+1)(2n+2)\max^{2n+1}\{1,x\}} &\le x - 1 - \sum_{k=2}^{2n+1}\frac{1}{k(k-1)}\frac{(x-1)^k}{x^{k-1}} - \ln x \\ &\le \frac{(x-1)^{2n+2}}{(2n+1)(2n+2)\min^{2n+1}\{1,x\}}. \end{aligned} \tag{3.4}$$

For $n=1$ we get from (3.3) and (3.4) that

$$\frac{(x-1)^4}{12x\max^3\{1,x\}} \le \ln x - \frac{x-1}{x} - \frac{1}{2}\frac{(x-1)^2}{x} + \frac{1}{6}\frac{(x-1)^3}{x} \le \frac{(x-1)^4}{12x\min^3\{1,x\}} \tag{3.5}$$

and

$$\frac{(x-1)^4}{12\max^3\{1,x\}} \le x-1-\frac{1}{2}\frac{(x-1)^2}{x} - \frac{1}{6}\frac{(x-1)^3}{x^2} - \ln x \le \frac{(x-1)^4}{12\min^3\{1,x\}} \tag{3.6}$$

for any $x>0$.

From (2.16) and (2.17) we have

$$\frac{x-1}{x} + \frac{1}{x}\sum_{k=2}^{2n+1}\frac{(-1)^k}{k(k-1)}(x-1)^k \le \ln x \le x-1-\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}\frac{(x-1)^k}{x^{k-1}} \tag{3.7}$$

for any $x>0$ and $n\ge 1$.

In particular, we have

$$\frac{x-1}{x} + \frac{1}{2}\frac{(x-1)^2}{x} - \frac{1}{6}\frac{(x-1)^3}{x} \le \ln x \le x-1-\frac{(x-1)^2}{2x} - \frac{1}{6}\frac{(x-1)^3}{x^2} \tag{3.8}$$

for any $x>0$.

From (2.18) and (2.19) we have

$$(0\le)\ln x - \frac{x-1}{x} \le \frac{(x-1)^2}{xL(1,x)} \le \frac{(x-1)^2}{x\sqrt{x}} \tag{3.9}$$

and

$$(0\le)\, x-1-\ln x \le \frac{(x-1)^2}{L(1,x)} \le \frac{(x-1)^2}{\sqrt{x}} \tag{3.10}$$

for any $x>0$, while from (2.20) and (2.21) we have that

$$\begin{aligned}(0\le)\ln x - \frac{x-1}{x} + \frac{1}{x}\sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k(k-1)}(x-1)^k \\ \le \frac{1}{2n(2n+1)}\frac{|x-1|^{2n+1}\left|x^{2n}-1\right|}{x^{2n+1}}\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}(0\le)\, x-1-\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}\frac{(x-1)^k}{x^{k-1}} - \ln x \\ \le \frac{1}{2n(2n+1)}\frac{|x-1|^{2n+1}\left|x^{2n}-1\right|}{x^{2n}}.\end{aligned} \tag{3.12}$$

From Theorem 7 we have for $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ that

$$\begin{aligned}(0\le)\ln x - \frac{x-1}{x} \\ \le \frac{1}{(p+1)^{1/p}(q-1)^{1/q}}\frac{|x-1|^{1+1/p}\left|x^{q-1}-1\right|^{1/q}}{x^{2-1/q}}\end{aligned} \tag{3.13}$$

and

$$(0 \leq) \, x - 1 - \ln x \leq \frac{1}{(p+1)^{1/p} (q-1)^{1/q}} \frac{|x-1|^{1+1/p} \left|x^{q-1} - 1\right|^{1/q}}{x^{1-1/q}} \tag{3.14}$$

for any $x > 0$.

If $n \geq 1$, then we have

$$\begin{aligned}(0 \leq) \, \ln x - \frac{x-1}{x} + \frac{1}{x} \sum_{k=2}^{2n+1} \frac{(-1)^{k-1}}{k(k-1)} (x-1)^k & \\ \leq \frac{1}{(2n+1)((2n+1)p+1)^{1/p}((2n+1)q-1)^{1/q}} & \\ \times \frac{|x-1|^{2n+1+1/p} \left|x^{(2n+1)q-1} - 1\right|^{1/q}}{x^{2n+2-1/q}} &\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}(0 \leq) \, x - 1 - \sum_{k=2}^{2n+1} \frac{1}{k(k-1)} \frac{(x-1)^k}{x^{k-1}} - \ln x & \\ \leq \frac{1}{(2n+1)((2n+1)p+1)^{1/p}((2n+1)q-1)^{1/q}} & \\ \times \frac{|x-1|^{2n+1+1/p} \left|x^{(2n+1)q-1} - 1\right|^{1/q}}{x^{2n+1-1/q}} &\end{aligned} \tag{3.16}$$

for any $x > 0$.

We consider the functions γ, $\Gamma : (0, \infty) \to (0, \infty)$ defined by

$$\gamma(x) := \frac{1}{2} \frac{(x-1)^2}{x \max\{1, x\}} \text{ and } \Gamma(x) := \frac{1}{2} \frac{(x-1)^2}{x \min\{1, x\}}.$$

We observe that

$$\gamma(x) = \frac{1}{2} \begin{cases} \frac{(x-1)^2}{x} & \text{if } x \in (0,1), \\ \frac{(x-1)^2}{x^2} & \text{if } x \in [1, \infty) \end{cases} \quad \text{and } \Gamma(x) = \frac{1}{2} \begin{cases} \frac{(x-1)^2}{x^2} & \text{if } x \in (0,1), \\ \frac{(x-1)^2}{x} & \text{if } x \in [1, \infty). \end{cases}$$

It is easy to see that both functions are continuous on $(0, \infty)$, strictly decreasing on $(0, 1)$ and strictly increasing on $(1, \infty)$ with $\gamma(0) = \Gamma(0)$. We have

$$\lim_{x \to 0+} \gamma(x) = \lim_{x \to 0+} \Gamma(x) = \infty, \quad \lim_{x \to \infty} \gamma(x) = \frac{1}{2}, \quad \lim_{x \to \infty} \Gamma(x) = \infty.$$

Therefore for $[m, M] \subset (0, \infty)$ we have

$$\gamma_{m,M} := \min_{x \in [m,M]} \gamma(x) = \begin{cases} \gamma(M) \text{ if } M < 1 \\ 0 \text{ if } m \leq 1 \leq M \\ \gamma(m) \text{ if } 1 < m \end{cases} = \frac{1}{2} \begin{cases} \frac{(M-1)^2}{M} \text{if } M < 1 \\ 0 \text{ if } m \leq 1 \leq M \\ \frac{(m-1)^2}{m^2} \text{ if } 1 < m \end{cases} \tag{3.17}$$

and

$$\Gamma_{m,M} := \max_{x\in[m,M]} \Gamma(x) = \begin{cases} \Gamma(m) \text{ if } M < 1 \\ \max\{\Gamma(m), \Gamma(M)\} \text{ if } m \le 1 \le M \\ \Gamma(M) \text{ if } 1 < m \end{cases} \tag{3.18}$$

$$= \frac{1}{2}\begin{cases} \frac{(m-1)^2}{m^2} \text{if } M < 1 \\ \max\left\{\frac{(m-1)^2}{m^2}, \frac{(M-1)^2}{M}\right\} \text{ if } m \le 1 \le M \\ \frac{(M-1)^2}{M} \text{ if } 1 < m. \end{cases}$$

By the inequality (3.1) we can state then the following result:

Proposition 1. *For any $x \in [m, M] \subset (0, \infty)$ we have the global bounds*

$$\gamma_{m,M} \le \ln x - \frac{x-1}{x} \le \Gamma_{m,M}$$

where $\gamma_{m,M}$ is defined by (3.17) and $\Gamma_{m,M}$ is defined by (3.18).

Similar results may be stated by utilising other inequalities provided above, however the details are omitted.

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