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JENSEN-OSTROWSKI INEQUALITIES AND INTEGRATION SCHEMES VIA THE DARBOUX EXPANSION

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ABSTRACT. Using Darboux's formula, which is a generalisation of Taylor's formula, we derive some Jensen-Ostrowski type inequalities. Applications for quadrature rules and f-divergence measures (specifically, for higher-order χ-divergence) are also given.

1. INTRODUCTION

In 1938, Ostrowski proved the following inequality [13]: Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b-a), \tag{1.1}$$

for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

Ostrowski's inequality has been developed for non-differentiable functions. In particular, when $x = (a+b)/2$, this inequality gives an error estimate to the midpoint rule: $\int_a^b f(t)\,dt \approx (b-a) f\left(\frac{a+b}{2}\right)$.

The midpoint rule is the simplest form of quadrature rules. Derivative-based quadrature rules have been developed due to the larger number of parameters which contributes to its precision and order of accuracy [2]. Wiersma [17] introduced a quadrature rule that is similar to the Euler-Maclaurin formula.

In Wang and Guo [16], the Euler-Maclaurin formula, or simply Euler's formula, is derived from Darboux's formula. Darboux's formula also generalises Taylor's formula (with integral remainder).

Proposition 1 (Darboux's formula)**.** *Let $f(z)$ be an analytic function along the straight line from a point a to the point z, and let $\varphi(t)$ be an arbitrary polynomial of degree n. Then,*

$$\begin{aligned} &\varphi^{(n)}(0)\{f(z) - f(a)\} \\ &= \sum_{m=1}^{n} (-1)^{m-1} (z-a)^m \{\varphi^{(n-m)}(1) f^{(m)}(z) - \varphi^{(n-m)}(0) f^{(m)}(a)\} \\ &+(-1)^n (z-a)^{n+1} \int_0^1 \varphi(t) f^{(n+1)}[a + (z-a)t]\,dt. \end{aligned} \tag{1.2}$$

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Taylor's formula (with integral remainder) is a special case, with $\varphi(t) = (t-1)^n$ [16].

In [5], inequalities are derived by utilising Taylor's formula:

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t)\, dt.$$

These inequalities also generalises Ostrowski's inequality and Jensen's inequality for general integrals (and are referred to as Jensen-Ostrowski type inequality). In particular, an Ostrowski type inequality in [5, p. 68] gives the following quadrature rule

$$\int_a^b f(t)\, dt \approx (b-a)f(\zeta) + \sum_{k=1}^{n} f^{(k)}(\zeta) \frac{(b-\zeta)^{k+1} - (a-\zeta)^{k+1}}{(k+1)!}, \tag{1.3}$$

for $\zeta \in [a,b]$ and the error estimate is given by

$$\|f^{(n+1)}\|_{[a,b],\infty} \frac{(\zeta-a)^{n+2} + (b-\zeta)^{n+2}}{(n+2)!}.$$

In this paper, we generalise the inequalities given in [5] by considering Darboux's formula in place of Taylor's formula. We consider the application of these inequalities to obtain numerical integration schemes. We also present the applications for f-divergence measures, specifically for the higher-order χ-divergence.

2. Preliminaries

This section serves as a reference point for the facts concerning Euler's formula as well as Jensen-Ostrowski type inequalities.

2.1. **Euler's formula.** The explicit expression for the Bernoulli polynomial is

$$\varphi_n(x) = \sum_{k=0}^{n} \binom{n}{k} \varphi_k x^{n-k} \tag{2.1}$$

where

$$\varphi_0 = 1, \quad \text{and} \quad \sum_{k=0}^{n-1} \frac{1}{k!(n-k)!} \varphi_k = 0 \quad (n \geq 2).$$

The Bernoulli numbers are given by

$$\varphi_0 = 1, \ \ \varphi_1 = -\frac{1}{2}, \ \ \varphi_{2k} = (-1)^{k-1} B_k, \text{ and } \ \varphi_{2k+1} = 0 \quad (k \geq 2). \tag{2.2}$$

The first ten Bernoulli numbers and the first seven Bernoulli polynomials are given in the following:

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30}, \quad B_5 = \frac{5}{66}$$

$$B_6 = \frac{691}{2730}, \quad B_7 = \frac{7}{6}, \quad B_8 = \frac{3617}{510}, \quad B_9 = \frac{43867}{798}, \quad B_{10} = \frac{174611}{330},$$

$$\varphi_0(x) = 1, \quad \varphi_1(x) = x - \frac{1}{2}, \quad \varphi_2(x) = x^2 - x + \frac{1}{6},$$
$$\varphi_3(x) = x(x-1)\left(x - \frac{1}{2}\right) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \quad \varphi_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30},$$
$$\varphi_5(x) = x(x-1)\left(x - \frac{1}{2}\right)\left(x^2 - x - \frac{1}{3}\right) = x^5 - \frac{5}{4}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x,$$
$$\varphi_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}.$$

Choosing the Bernoulli polynomial $\varphi_n(t)$ in place of $\varphi(t)$ and replacing n by $2n$ in Darboux's formula (1.2) gives Euler's formula:

$$\begin{aligned} f(z) - f(a) &= \frac{z-a}{2}\left[f'(z) + f'(a)\right] + \sum_{k=1}^{n} (-1)^k \frac{(z-a)^{2k}}{(2k)!} B_k [f^{(2k)}(z) - f^{(2k)}(a)] \\ &+ \frac{(z-a)^{2n+1}}{(2n)!} \int_0^1 \varphi_{2n}(t) f^{(2n+1)}((1-t)a + tz)\, dt. \end{aligned} \tag{2.3}$$

2.2. **Jensen-Ostrowski type inequalities.** Jensen-Ostrowski type inequalities are introduced by Dragomir in [8] and has been further developed in the following papers: [3], [4], [5], [8], [9], and [10]. In what follows, we recall a general form of the Jensen-Ostrowski inequalities [5].

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space with $\int_\Omega d\mu = 1$, consisting of a set Ω, a σ-algebra $\mathcal{A}$ of subsets of Ω, and a countably additive and positive measure μ on $\mathcal{A}$ with values in the set of extended real numbers. We have the following result:

Proposition 2. *Let $f : I \in \mathbb{R} \to \mathbb{C}$ (I interval of $\mathbb{R}$) be such that $f^{(n)}$ is absolutely continuous on I and $a \in \mathring{I}$. If $g : \Omega \to I$ is Lebesgue μ-measurable on Ω, $f \circ g$, $(g-a)^k$, $f^{(n+1)}((1-s)a + sg) \in L(\Omega, \mu)$ for all $k \in \{1, ..., n+1\}$, and $s \in [0,1]$, then we have*

$$\left| \int_\Omega f \circ g \, d\mu - f(a) - \sum_{k=1}^{n} f^{(k)}(a) \int_\Omega \frac{(g-a)^k}{k!}\, d\mu - \lambda \frac{1}{(n+1)!} \int_\Omega (g-a)^{n+1}\, d\mu \right|$$
$$\leq \frac{1}{(n+1)!} \left(\int_\Omega |g-a|^{n+1} \left\| f^{(n+1)}((1-\ell)a + \ell g) - \lambda \right\|_{[0,1],\infty} d\mu \right)$$
$$\leq \begin{cases} \frac{1}{(n+1)!} \| \, |g-a|^{n+1} \, \|_{\Omega,\infty} \left\| \left\| f^{(n+1)}((1-\ell)a + \ell g) - \lambda \right\|_{[0,1],\infty} \right\|_{\Omega,1}, \\ \frac{1}{(n+1)!} \| \, |g-a|^{n+1} \, \|_{\Omega,p} \left\| \left\| f^{(n+1)}((1-\ell)a + \ell g) - \lambda \right\|_{[0,1],\infty} \right\|_{\Omega,q}, \\ \qquad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1 \\ \frac{1}{(n+1)!} \| \, |g-a|^{n+1} \, \|_{\Omega,1} \left\| \left\| f^{(n+1)}((1-\ell)a + \ell g) - \lambda \right\|_{[0,1],\infty} \right\|_{\Omega,\infty}, \end{cases}$$

for any $\lambda \in \mathbb{C}$.

Here (and throughout the text), ℓ denotes the identity function on $[0,1]$, namely $\ell(t) = t$, for $t \in [0,1]$. We also use the notation

$$\|k\|_{\Omega,p} := \begin{cases} \left(\int_\Omega |k(t)|^p \, d\mu(t) \right)^{1/p}, & p \geq 1, \ k \in L_p(\Omega, \mu); \\ \operatorname*{ess\,sup}_{t \in \Omega} |k(t)|, & p = \infty, \ k \in L_\infty(\Omega, \mu); \end{cases}$$

and

$$\|f\|_{[0,1],p} := \begin{cases} \left(\int_0^1 |f(s)|^p \, ds\right)^{1/p}, & p \geq 1, \ f \in L_p([0,1]); \\ \operatorname*{ess\,sup}_{s\in[0,1]} |f(s)|, & p = \infty, \ f \in L_\infty([0,1]). \end{cases}$$

Inequalities of Jensen and Ostrowski type are obtained by setting $x = \int_\Omega g \, d\mu$ and $\lambda = 0$, respectively, in Proposition 2.

Proposition 3. *Under the assumptions of Proposition 2, we have the following Ostrowski type inequality:*

$$\left| \int_\Omega f \circ g \, d\mu - f(\zeta) - \sum_{k=1}^{n} f^{(k)}(\zeta) \int_\Omega \frac{(g-\zeta)^k}{k!} \, d\mu \right| \leq \frac{1}{(n+1)!} \|f^{(n+1)}\|_{I,\infty} \int_\Omega |g-\zeta|^{n+1} \, d\mu. \tag{2.4}$$

We also have the following Jensen type inequality:

$$\left| \int_\Omega f \circ g \, d\mu - f\left(\int_\Omega g \, d\mu\right) - \sum_{k=2}^{n} f^{(k)}\left(\int_\Omega g \, d\mu\right) \int_\Omega \frac{\left(g - \int_\Omega g \, d\mu\right)^k}{k!} \, d\mu \right| \leq \frac{1}{(n+1)!} \|f^{(n+1)}\|_{I,\infty} \int_\Omega \left| g - \int_\Omega g \, d\mu \right|^{n+1} \, d\mu. \tag{2.5}$$

3. Identities

Throughout the paper, we consider the measure space $(\Omega, \mathcal{A}, \mu)$ with $\int_\Omega d\mu = 1$, consisting of a set Ω, a σ-algebra $\mathcal{A}$ of subsets of λ, and a countably additive and positive measure μ on $\mathcal{A}$ with values in the set of extended real numbers.

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{C}$ (I interval of $\mathbb{R}$) be such that $f^{(n)}$ is absolutely continuous on I and $a \in \mathring{I}$. Let $\varphi(t)$ be an arbitrary polynomial of degree n. If $g : \Omega \to I$ is Lebesgue μ-measurable on Ω, $f \circ g$, $(g-a)^m$, $(g-a)^m (f^{(m)} \circ g)$, $f^{(n+1)}((1-t)a + tg) \in L(\Omega, \mu)$ for all $m \in \{1, ..., n+1\}$, and $t \in [0,1]$, then we have*

$$\int_\Omega f \circ g \, d\mu - f(a) = P_{n,\varphi}(a, \lambda) + R_{n,\varphi}(a, \lambda), \tag{3.1}$$

for all $\lambda \in \mathbb{C}$, where $P_{n,\varphi}(a,\lambda) = P_{n,\varphi}(a,\lambda; f, g)$ is defined by

$$\begin{aligned} P_{n,\varphi}(a,\lambda) = {} & \frac{1}{\varphi^{(n)}(0)} \sum_{m=1}^{n} (-1)^{m-1} \left\{ \varphi^{(n-m)}(1) \int_\Omega (g-a)^m (f^{(m)} \circ g) \, d\mu \right. \\ & \left. - \varphi^{(n-m)}(0) f^{(m)}(a) \int_\Omega (g-a)^m \, d\mu \right\} + \frac{(-1)^n \lambda}{\varphi^{(n)}(0)} \int_0^1 \varphi(t) \, dt \int_\Omega (g-a)^{n+1} \, d\mu, \end{aligned} \tag{3.2}$$

and $R_{n,\varphi}(a,\lambda) = R_{n,\varphi}(a,\lambda; f, g)$ *is defined by*

$$
\begin{aligned}
&(3.3)\qquad R_{n,\varphi}(a,\lambda))\\
&= (-1)^n \frac{1}{\varphi^{(n)}(0)} \int_\Omega (g-a)^{n+1} \left(\int_0^1 \varphi(t) \left[f^{(n+1)}[(1-t)a+tg] - \lambda \right] dt \right) d\mu \\
&= (-1)^n \frac{1}{\varphi^{(n)}(0)} \int_0^1 \varphi(t) \int_\Omega (g-a)^{n+1} \left(\left[f^{(n+1)}[(1-t)a+tg] - \lambda \right] d\mu \right) dt.
\end{aligned}
$$

Proof. From Proposition 1, we have

$$
\begin{aligned}
f(z) - f(a) &= \frac{1}{\varphi^{(n)}(0)} \sum_{m=1}^{n} (-1)^{m-1} (z-a)^m \{ \varphi^{(n-m)}(1) f^{(m)}(z) \\
&\quad - \varphi^{(n-m)}(0) f^{(m)}(a) \} + \frac{\lambda (-1)^n (z-a)^{n+1}}{\varphi^{(n)}(0)} \int_0^1 \varphi(t)\, dt \\
&\quad + \frac{(-1)^n (z-a)^{n+1}}{\varphi^{(n)}(0)} \int_0^1 \varphi(t) \left[f^{(n+1)}[(1-t)a+tz] - \lambda \right] dt.
\end{aligned}
$$

By replacing z with $g(t)$ and integrating on Ω, we have

$$
\begin{aligned}
&\int_\Omega f \circ g \, d\mu - f(a) \\
&= \frac{1}{\varphi^{(n)}(0)} \sum_{m=1}^{n} (-1)^{m-1} \left\{ \varphi^{(n-m)}(1) \int_\Omega (g-a)^m (f^{(m)} \circ g)\, d\mu \right. \\
&\quad \left. - \varphi^{(n-m)}(0) f^{(m)}(a) \int_\Omega (g-a)^m \, d\mu \right\} + \frac{(-1)^n \lambda}{\varphi^{(n)}(0)} \int_0^1 \varphi(t)\, dt \int_\Omega (g-a)^{n+1}\, d\mu \\
&\quad + \frac{(-1)^n}{\varphi^{(n)}(0)} \int_\Omega (g-a)^{n+1} \left(\int_0^1 \varphi(t) \left[f^{(n+1)}[(1-t)a+tg] - \lambda \right] dt \right) d\mu.
\end{aligned}
$$

The last equality in (3.3) follows by Fubini's theorem. □

Lemma 2. *Let* $f : I \subset \mathbb{R} \to \mathbb{C}$ *(*I *interval of* $\mathbb{R}$*) be such that* $f^{(n)}$ *is absolutely continuous on* I *and* $a \in \mathring{I}$. *Let* $\varphi_{2n}(t)$ *be the Bernoulli polynomials. If* $g : \Omega \to I$ *is Lebesgue* μ*-measurable on* Ω, $f \circ g$, $(g-a)^m$, $(g-a)^m (f^{(m)} \circ g)$, $f^{(2n+1)}((1-t)a+tg) \in L(\Omega,\mu)$ *for all* $m \in \{1, \ldots, 2n+1\}$*, and* $t \in [0,1]$*, then we have*

$$
\int_\Omega f \circ g \, d\mu - f(a) = P_n(a,\lambda) + R_n(a,\lambda), \tag{3.4}
$$

for all $\lambda \in \mathbb{C}$*, where* $P_n(a,\lambda) = P_n(a,\lambda; f, g)$ *is defined by*

$$
\begin{aligned}
(3.5)\qquad P_n(a,\lambda) &= \int_\Omega \frac{(g-a)}{2} \left[f'(a) + f' \circ g \right] d\mu \\
&\quad + \int_\Omega \sum_{k=1}^{n} \frac{(-1)^k B_k (g-a)^{2k}}{(2k)!} [f^{(2k)} \circ g - f^{(2k)}(a)]\, d\mu \\
&\quad + \lambda \int_0^1 \varphi_{2n}(t)\, dt \int_\Omega \frac{(g-a)^{2n+1}}{(2n)!}\, d\mu
\end{aligned}
$$

and $R_n(a,\lambda) = R_n(a,\lambda; f, g)$ *is defined by*

$$
\begin{aligned}
R_n(a,\lambda) &= \int_\Omega \frac{(g-a)^{2n+1}}{(2n)!}\left[\int_0^1 \varphi_{2n}(t)[f^{(2n+1)}((1-t)a+tg)-\lambda]\,dt\right]d\mu, \\
&= \int_0^1 \varphi_{2n}(t)\int_\Omega \frac{(g-a)^{2n+1}}{(2n)!}\left[f^{(2n+1)}((1-t)a+tg)-\lambda\right]d\mu\,dt. \qquad (3.6)
\end{aligned}
$$

The proof follows by the Euler's formula (2.3) and similar arguments to those in the proof of Lemma 1. We omit the proof.

Remark 1. Recall that $B_1 = \frac{1}{6}$ and $\varphi_2(t) = t^2 - t + \frac{1}{6}$; and note that $\int_0^1 \varphi_2(t)\,dt = 0$. Take $n = 1$ in Lemma 2, we have

$$
\begin{aligned}
&\int_\Omega f\circ g\,d\mu - f(a) \qquad (3.7)\\
&= \int_\Omega \frac{(g-a)}{2}[f'(a)+f'\circ g]\,d\mu - \frac{1}{12}\int_\Omega (g-a)^2[f''\circ g - f''(a)]\,d\mu \\
&+ \int_\Omega \frac{(g-a)^3}{2}\left[\int_0^1 \left(t^2-t+\frac{1}{6}\right)[f^{(3)}((1-t)a+tg)-\lambda]\,dt\right]d\mu.
\end{aligned}
$$

4. Jensen-Ostrowski inequalities

Utilising the identities obtain in Section 3, we derive some inequalities of Jensen-Ostrowski inequalities.

Theorem 1. *Let* $f : I \in \mathbb{R} \to \mathbb{C}$ *(I interval of $\mathbb{R}$) be such that* $f^{(n)}$ *is absolutely continuous on* I *and* $a \in \mathring{I}$. *Let* $\varphi(t)$ *be an arbitrary polynomial of degree* n. *If* $g : \Omega \to I$ *is Lebesgue* μ*-measurable on* Ω, $f\circ g$, $(g-a)^m$, $(g-a)^m(f^{(m)}\circ g)$, $f^{(n+1)}((1-t)a+tg) \in L(\Omega,\mu)$ *for all* $m \in \{1,...,n+1\}$, *and* $t\in[0,1]$, *then we have*

$$
\begin{aligned}
&\left|\int_\Omega f\circ g\,d\mu - f(a) - P_{n,\varphi}(a,\lambda)\right| \qquad (4.1)\\
&\le \int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\,dt\left(\int_\Omega |g-a|^{n+1}\,\|f_{n+1,\lambda,g}\|_{[0,1],\infty}\,d\mu\right) \\
&\le \left(\int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\,dt\right)\begin{cases} \left\|\,|g-a|^{n+1}\right\|_{\Omega,\infty}\left\|\|f_{n+1,g}(a,\lambda)\|_{[0,1],\infty}\right\|_{\Omega,1}, \\ \left\|\,|g-a|^{n+1}\right\|_{\Omega,p}\left\|\|f_{n+1,g}(a,\lambda)\|_{[0,1],\infty}\right\|_{\Omega,q}, \\ \qquad\qquad p>1,\quad \frac{1}{p}+\frac{1}{q}=1, \\ \left\|\,|g-a|^{n+1}\right\|_{\Omega,1}\left\|\|f_{n+1,g}(a,\lambda)\|_{[0,1],\infty}\right\|_{\Omega,\infty}, \end{cases}
\end{aligned}
$$

for any $\lambda\in\mathbb{C}$, *where* $f_{n+1,g}(a,\lambda) = f^{(n+1)}[(1-\ell)a+\ell g]-\lambda$. *Here,* $P_{n,\varphi}(a,\lambda)$ *is as defined in* (3.2).

Proof. Taking the modulus in (3.1), we have

$$\begin{aligned}
&\left|\int_\Omega f\circ g\,d\mu - f(a) - P_{n,\varphi}(a,\lambda)\right| \\
&\le \frac{1}{|\varphi^{(n)}(0)|}\int_0^1 |\varphi(t)|\left(\int_\Omega |g-a|^{n+1}\left|f^{(n+1)}[(1-t)a+tg]-\lambda\right| d\mu\right) dt \\
&\le \frac{1}{|\varphi^{(n)}(0)|}\int_0^1 |\varphi(t)|\,dt\left(\int_\Omega |g-a|^{n+1}\left\|f^{(n+1)}[(1-\ell)a+\ell g]-\lambda\right\|_{[0,1],\infty} d\mu\right)
\end{aligned}$$

for any $\lambda\in\mathbb{C}$. We obtain the desired result by applying Hölder's inequality. □

Corollary 1. *Under the assumptions of Theorem 1, we have*

$$\begin{aligned}
&\left|\int_\Omega f\circ g\,d\mu - f(a) - P_{n,\varphi}(a,0)\right| \\
&\le \left(\int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\,dt\right)\|f^{(n+1)}\|_{I,\infty}\left(\int_\Omega |g-a|^{n+1}\,d\mu\right).
\end{aligned} \tag{4.2}$$

We also have the following Jensen type inequality

$$\begin{aligned}
&\left|\int_\Omega f\circ g\,d\mu - f\left(\int_\Omega g\,d\mu\right) - P_{n,\varphi}\left(\int_\Omega g\,d\mu, 0\right)\right| \\
&\le \left(\int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\,dt\right)\|f^{(n+1)}\|_{I,\infty}\left(\int_\Omega \left|g-\int_\Omega g\,d\mu\right|^{n+1} d\mu\right).
\end{aligned} \tag{4.3}$$

Here, $P_{n,\varphi}(a,\lambda)$ *is as defined in* (3.2).

Proof. Let $\lambda=0$ in (3.1), and take the modulus to obtain

$$\begin{aligned}
&\left|\int_\Omega f\circ g\,d\mu - f(a) - P_{n,\varphi}(a,0)\right| \\
&\le \int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\,dt\left(\int_\Omega |g-a|^{n+1}\left\|f^{(n+1)}[(1-\ell)a+\ell g]\right\|_{[0,1],\infty} d\mu\right).
\end{aligned} \tag{4.4}$$

For any $t\in\Omega$ and almost every $s\in[0,1]$, we have

$$|f^{(n+1)}((1-s)a+sg(t))| \le \operatorname*{ess\,sup}_{u\in I}|f^{(n+1)}(u)| = \|f^{(n+1)}\|_{I,\infty}.$$

Therefore, we have

$$\begin{aligned}
\left\|f^{(n+1)}((1-\ell)a+\ell g)\right\|_{[0,1],\infty} &\le \operatorname*{ess\,sup}_{s\in[0,1],\ t\in\Omega}\|f^{(n+1)}((1-s)a+sg(t))\| \\
&\le \|f^{(n+1)}\|_{I,\infty}.
\end{aligned} \tag{4.5}$$

The desired inequality follows from (4.4) and (4.5). □

Utilising (2.3) and applying similar arguments to those in Theorem 1 and Corollary 1, we have the following results. We omit the proofs.

Theorem 2. *Let* $f:I\subset\mathbb{R}\to\mathbb{C}$ *(*I *interval of* $\mathbb{R}$*) be such that* $f^{(n)}$ *is absolutely continuous on* I *and* $a\in\mathring{I}$*. Let* $\varphi_{2n}(t)$ *be the Bernoulli polynomials. If* $g:\Omega\to I$ *is Lebesgue* μ*-measurable on* Ω *and* $f\circ g$*,* $(g-a)^m$*,* $(g-a)^m(f^{(m)}\circ$

g), $f^{(2n+1)}((1-t)a+tg) \in L(\Omega,\mu)$ *for all* $m \in \{1,\dots,2n+1\}$*, and* $t \in [0,1]$*, then we have*

$$\left|\int_\Omega f\circ g\,d\mu - f(a) - P_n(a,\lambda)\right| \tag{4.6}$$
$$\le \int_0^1 \frac{|\varphi_{2n}(t)|}{(2n)!}\,dt \int_\Omega |g-a|^{2n+1}\,\|f_{2n+1,g}(a,\lambda)\|_{[0,1],\infty}\,d\mu$$
$$\le \left(\int_0^1 \frac{|\varphi_{2n}(t)|}{(2n)!}\,dt\right)\begin{cases} \left\||g-a|^{2n+1}\right\|_{\Omega,\infty}\left\|\|f_{2n+1,g}(a,\lambda)\|_{[0,1],\infty}\right\|_{\Omega,1}, \\ \left\||g-a|^{2n+1}\right\|_{\Omega,p}\left\|\|f_{2n+1,g}(a,\lambda)\|_{[0,1],\infty}\right\|_{\Omega,q}, \\ \qquad \frac{1}{p}+\frac{1}{q}=1,\ p>1, \\ \left\||g-a|^{2n+1}\right\|_{\Omega,1}\left\|\|f_{2n+1,g}(a,\lambda)\|_{[0,1],\infty}\right\|_{\Omega,\infty}, \end{cases}$$

for any $\lambda \in \mathbb{C}$*, where* $f_{2n+1,g}(a,\lambda) = f^{(2n+1)}((1-\ell)a+\ell g) - \lambda$*. Here,* $P_n(a,\lambda)$ *is as defined in* (3.5).

Corollary 2. *Under the assumptions of Theorem 2, we have*

$$\left|\int_\Omega f\circ g\,d\mu - f(a) - P_n(a,0)\right| \tag{4.7}$$
$$\le \left(\int_0^1 \frac{|\varphi_{2n}(t)|}{(2n)!}\,dt\right)\|f^{(2n+1)}\|_{I,\infty}\left(\int_\Omega |g-a|^{2n+1}\,d\mu\right).$$

and

$$\left|\int_\Omega f\circ g\,d\mu - f\left(\int_\Omega g\,d\mu\right) - P_n\left(\int_\Omega g\,d\mu, 0\right)\right| \tag{4.8}$$
$$\le \left(\int_0^1 \frac{|\varphi_{2n}(t)|}{(2n)!}\,dt\right)\|f^{(2n+1)}\|_{I,\infty}\left(\int_\Omega \left|g - \int_\Omega g\,d\mu\right|^{2n+1}\,d\mu\right).$$

Here, $P_n(a,\lambda)$ *is as defined in* (3.5).

Remark 2. Set $n=1$ in Corollary 2, we have

$$\left|\int_\Omega f\circ g\,d\mu - f(a) - \int_\Omega \frac{(g-a)}{2}\left[f'(a)+f'\circ g\right]d\mu \right. \tag{4.9}$$
$$\left. + \frac{1}{12}\int_\Omega (g-a)^2[f''\circ g - f''(a)]\,d\mu\right| \le \frac{\|f'''\|_{I,\infty}}{18\sqrt{3}}\int_\Omega |g-a|^3\,d\mu.$$

The following terminology introduced in [8] will be required for alternate Jensen-Ostrowski inequality results. For $\gamma,\Gamma \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions [8]

$$\bar{U}_{[a,b]}(\gamma,\Gamma) := \left\{h:[a,b]\to\mathbb{C}\,\middle|\,\mathrm{Re}\left[(\Gamma-h(t))(\overline{h(t)}-\overline{\gamma})\right] \ge 0 \text{ for a.e. } t\in[a,b]\right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{h:[a,b]\to\mathbb{C}\,\middle|\,\left|h(t)-\frac{\gamma+\Gamma}{2}\right| \le \frac{1}{2}|\Gamma-\gamma| \text{ for a.e. } t\in[a,b]\right\}.$$

The following representation results may be stated [8].

Proposition 4. *For any* $\gamma,\Gamma\in\mathbb{C}$ *and* $\gamma\ne\Gamma$*, we have*

(i) $\bar{U}_{[a,b]}(\gamma,\Gamma)$ *and* $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ *are nonempty, convex and closed sets;*

(ii) $\bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$; *and*

(iii) $\bar{U}_{[a,b]}(\gamma,\Gamma) = \{h : [a,b] \to \mathbb{C} \mid (\mathrm{Re}(\Gamma) - \mathrm{Re}(h(t)))(\mathrm{Re}(h(t)) - \mathrm{Re}(\gamma)) + (\mathrm{Im}(\Gamma) - \mathrm{Im}(h(t)))(\mathrm{Im}(h(t)) - \mathrm{Im}(\gamma)) \geq 0$ *for a.e.* $t \in [a,b]\}$.

We have the following Jensen-Ostrowski inequality for functions with bounded higher $(n+1)$-th derivatives:

Theorem 3. *Let* $f : I \in \mathbb{R} \to \mathbb{C}$ *(*I *interval of* $\mathbb{R}$*) be such that* $f^{(n)}$ *is absolutely continuous on* I *and* $a \in \mathring{I}$. *For some* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, *assume that* $f^{(n+1)} \in \bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$. *If* $g : \Omega \to I$ *is Lebesgue* μ*-measurable on* Ω, $f \circ g$, $(g-a)^m$, $(g-a)^m (f^{(m)} \circ g)$, $f^{(n+1)}((1-t)a + tg) \in L(\Omega,\mu)$ *for all* $m \in \{1,...,n+1\}$, *and* $t \in [0,1]$, *then*

$$\left| \int_\Omega f \circ g \, d\mu - f(a) - P_{n,\varphi}\left(a, \frac{\gamma+\Gamma}{2}\right) \right| \leq \frac{|\Gamma-\gamma|}{2} \int_\Omega |g-a|^{n+1} \, d\mu \int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|} \, dt. \tag{4.10}$$

Here, $P_{n,\varphi}(a,\lambda)$ *is as defined in* (3.2).

Proof. Let $\lambda = (\gamma+\Gamma)/2$ in (3.1), we have

$$\int_\Omega f \circ g \, d\mu - f(a) - P_{n,\varphi}\left(a, \frac{\gamma+\Gamma}{2}\right) = (-1)^n \frac{1}{\varphi^{(n)}(0)} \int_\Omega (g-a)^{n+1} \left(\int_0^1 \varphi(t) \left[f^{(n+1)}[(1-t)a+tg] - \frac{\gamma+\Gamma}{2} \right] dt \right) d\mu.$$

Since $f^{(n+1)} \in \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$, we have

$$\left| f^{(n+1)}((1-t)a+tg) - \frac{\gamma+\Gamma}{2} \right| \leq \frac{1}{2}|\Gamma-\gamma|, \tag{4.11}$$

for almost every $t \in [0,1]$ and $s \in \Omega$. Multiply (4.11) with $|\varphi(t)| > 0$ and integrate over $[0,1]$, we obtain

$$\int_0^1 |\varphi(t)| \left| f^{(n+1)}((1-t)a+tg) - \frac{\gamma+\Gamma}{2} \right| dt \leq \frac{1}{2}|\Gamma-\gamma| \int_0^1 |\varphi(t)| \, dt,$$

for any $s \in \Omega$. Now, we have

$$\begin{aligned} &\left| \int_\Omega f \circ g \, d\mu - f(a) - P_{n,\varphi}\left(a, \frac{\gamma+\Gamma}{2}\right) \right| \\ &\leq \frac{1}{|\varphi^{(n)}(0)|} \int_\Omega |g-a|^{n+1} \left(\int_0^1 |\varphi(t)| \left| f^{(n+1)}[(1-t)a+tg] - \frac{\gamma+\Gamma}{2} \right| dt \right) d\mu \\ &\leq \frac{|\Gamma-\gamma|}{2|\varphi^{(n)}(0)|} \int_\Omega |g-a|^{n+1} \, d\mu \int_0^1 |\varphi(t)| \, dt. \end{aligned}$$

This completes the proof. □

Similarly, we have the following via Euler's formula (2.3) and Lemma 2:

Theorem 4. *Let* $f : I \in \mathbb{R} \to \mathbb{C}$ *(*I *interval of* $\mathbb{R}$*) be such that* $f^{(n)}$ *is absolutely continuous on* I *and* $a \in \mathring{I}$. *For some* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, *assume that* $f^{(2n+1)} \in \bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$. *If* $g : \Omega \to I$ *is Lebesgue* μ*-measurable on* Ω *and* $f \circ$

$g,\ (g-a)^m,\ (g-a)^m(f^{(m)}\circ g),\ f^{(2n+1)}((1-t)a+tg)\in L(\Omega,\mu)$ for all $m\in\{1,\dots,2n+1\}$, and $t\in[0,1]$, then we have

$$\left|\int_\Omega f\circ g\,d\mu - f(a) - P_n\left(a,\frac{\gamma+\Gamma}{2}\right)\right| \leq \frac{|\Gamma-\gamma|}{2((2n)!)}\int_\Omega |g-a|^{2n+1}\,d\mu\int_0^1|\varphi_{2n}(t)|\,dt. \tag{4.12}$$

Here, $P_n(a,\lambda)$ is as defined in (3.5).

5. Quadrature rules

In this section, we present quadrature rules based on the inequalities presented in Section 4. Composite rules using any of the given rules (in this section) may be stated by the usual manner of breaking up the interval [a, b] into some number n of subintervals, computing an approximation for each subinterval, then adding up all the results.

Let $g:[a,b]\to[a,b]$ defined by $g(t)=t$ and $\mu(t)=t/(b-a)$ in Corollary 1. We have the following quadrature rule:

$$\begin{aligned}\int_a^b f(t)\,dt &\approx (b-a)f(x)+\sum_{m=1}^n(-1)^{m-1}\left\{\frac{\varphi^{(n-m)}(1)}{\varphi^{(n)}(0)}\int_a^b(t-x)^m f^{(m)}(t)\,dt\right.\\ &\qquad\left.-\frac{\varphi^{(n-m)}(0)}{\varphi^{(n)}(0)}f^{(m)}(x)\int_a^b(t-x)^m\,dt\right\}\\ &= (b-a)f(x)+\sum_{m=1}^n(-1)^{m-1}\left\{\frac{\varphi^{(n-m)}(1)}{\varphi^{(n)}(0)}\int_a^b(t-x)^m f^{(m)}(t)\,dt\right.\\ &\qquad\left.-\frac{\varphi^{(n-m)}(0)}{\varphi^{(n)}(0)}f^{(m)}(x)\left(\frac{(b-x)^{m+1}-(a-x)^{m+1}}{m+1}\right)\right\},\end{aligned}$$

with the following error estimate:

$$\begin{aligned}&\int_0^1\frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\,dt\left(\int_a^b|t-x|^{n+1}\,dt\right)\|f^{(n+1)}\|_{[a,b],\infty}\\ &=\int_0^1\frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\,dt\left(\frac{(x-a)^{n+2}+(b-x)^{n+2}}{n+2}\right)\|f^{(n+1)}\|_{[a,b],\infty},\end{aligned}$$

for $x\in[a,b]$.

Similarly, Corollary 2 gives us

$$\begin{aligned}&\left|\frac{3}{2}\int_a^b f(t)\,dt-(b-a)f(x)-\frac{1}{2}[(b-x)f(b)-(a-x)f(a)]\right.\\ &\left.-\frac{f'(x)}{4}\left[(b-x)^2-(a-x)^2\right]-\int_a^b\sum_{k=1}^n\frac{(-1)^kB_k(t-x)^{2k}}{(2k)!}[f^{(2k)}(t)-f^{(2k)}(x)]\,dt\right|\\ &\leq\frac{1}{(2n)!}\left(\int_0^1|\varphi_{2n}(t)|\,dt\right)\|f^{(2n+1)}\|_{[a,b],\infty}\frac{(x-a)^{2n+2}+(b-x)^{2n+2}}{2n+2},\end{aligned}$$

for all $x \in [a,b]$, thus we have the following quadrature rule:

$$\begin{aligned}\int_a^b f(t)\,dt &\approx \frac{2}{3}(b-a)f(x)+\frac{1}{3}\left[(b-x)f(b)-(a-x)f(a)\right]\\ &+\frac{f'(x)}{6}\left[(b-x)^2-(a-x)^2\right]\\ &+\frac{2}{3}\int_a^b\sum_{k=1}^{n}\frac{(-1)^k B_k(t-x)^{2k}}{(2k)!}[f^{(2k)}(t)-f^{(2k)}(x)]\,dt\end{aligned}$$

for $x \in [a,b]$.

When $n=1$, we have

$$\begin{aligned}&\left|\frac{5}{3}\int_a^b f(t)\,dt-(b-a)f(x)-\frac{2}{3}\left[(b-x)f(b)-(a-x)f(a)\right]\right.\\ &-\frac{f'(x)}{4}\left[(b-x)^2-(a-x)^2\right]+\frac{1}{12}[(b-x)^2f'(b)-(a-x)^2f'(a)]\\ &\left.-\frac{f''(x)}{36}[(b-x)^3-(a-x)^3]\right|\le\frac{1}{72\sqrt{3}}\|f'''\|_{[a,b],\infty}[(x-a)^4+(b-x)^4],\end{aligned}\tag{5.1}$$

for $x \in [a,b]$, thus we have the following quadrature rule:

$$\begin{aligned}\int_a^b f(t)\,dt &\approx \frac{3}{5}(b-a)f(x)+\frac{2}{5}\left[(b-x)f(b)-(a-x)f(a)\right]\\ &+\frac{3}{20}f'(x)\left[(b-x)^2-(a-x)^2\right]-\frac{1}{20}[(b-x)^2f'(b)-(a-x)^2f'(a)]\\ &+\frac{1}{60}f''(x)[(b-x)^3-(a-x)^3].\end{aligned}$$

6. Applications for f-Divergence

Assume that a set Ω and the σ-finite measure μ are given. Consider the set of all probability densities on μ to be

$$\mathcal{P}:=\left\{p|p:\Omega\to\mathbb{R},\,p(t)\ge 0,\ \int_\Omega p(t)\,d\mu(t)=1\right\}.$$

We recall the definition of some divergence measures which we use in this text.

Definition 1. Let $p,q\in\mathcal{P}$ and $k\ge 2$.

1. The Kullback-Leibler divergence [11]:

$$D_{KL}(p,q):=\int_\Omega p(t)\log\left[\frac{p(t)}{q(t)}\right]d\mu(t)\,,\quad p,q\in\mathcal{P}.\tag{6.1}$$

2. The χ^2-divergence:

$$D_{\chi^2}(p,q):=\int_\Omega p(t)\left[\left(\frac{q(t)}{p(t)}\right)^2-1\right]d\mu(t)\,,\quad p,q\in\mathcal{P}.\tag{6.2}$$

3. Higher order χ-divergence [1]:

$$D_{\chi^k}(p,q) := \int_\Omega \frac{(q(t)-p(t))^k}{p^{k-1}(t)} d\mu(t) = \int_\Omega \left(\frac{q(t)}{p(t)} - 1\right)^k p(t)\, d\mu(t), \tag{6.3}$$

$$D_{|\chi|^k}(p,q) := \int_\Omega \frac{|q(t)-p(t)|^k}{p^{k-1}(t)} d\mu(t) = \int_\Omega \left|\frac{q(t)}{p(t)} - 1\right|^k p(t)\, d\mu(t). \tag{6.4}$$

Furthermore, (6.3) and (6.4) can be generalised as follows [12]:

$$D_{\chi^k,a}(p,q) := \int_\Omega \frac{(q(t)-ap(t))^k}{p^{k-1}(t)} d\mu(t) = \int_\Omega \left(\frac{q(t)}{p(t)} - a\right)^k p(t)\, d\mu(t), \tag{6.5}$$

$$D_{|\chi|^k,a}(p,q) := \int_\Omega \frac{|q(t)-ap(t)|^k}{p^{k-1}(t)} d\mu(t) = \int_\Omega \left|\frac{q(t)}{p(t)} - a\right|^k p(t)\, d\mu(t). \tag{6.6}$$

4. Csiszár f-divergence [6]:

$$I_f(p,q) := \int_\Omega p(t) f\left[\frac{q(t)}{p(t)}\right] d\mu(t), \quad p,q \in \mathcal{P}, \tag{6.7}$$

where f is convex on $(0,\infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u=1$.

Remark 3. (1) We note that when $k=2$, (6.3) coincides with (6.2).
(2) The Kullback-Leibler divergence and the χ^2-divergence are particular instances of Csiszár f-divergence. For the basic properties of Csiszár f-divergence, we refer the readers to [6], [7], and [15].

Example 1. (i) Let $f:(0,\infty)\to\mathbb{R}$ be defined by $f(t) = -t\log(t)$, and $f'(t) = \log(t)+1$. We have

$$I_f(p,q) = \int_\Omega p(t)\frac{q(t)}{p(t)} \log\left[\frac{q(t)}{p(t)}\right] d\mu(t) = \int_\Omega q(t)\log\left[\frac{q(t)}{p(t)}\right] d\mu(t) = D_{KL}(q,p),$$

and

$$\begin{aligned} I_{f'}(p,q) &= \int_\Omega p(t)\left[\log\left[\frac{q(t)}{p(t)}\right] + 1\right] d\mu(t) \\ &= -\int_\Omega p(t)\log\left[\frac{p(t)}{q(t)}\right] d\mu(t) + 1 = -D_{KL}(p,q)+1. \end{aligned}$$

(ii) Let $g:(0,\infty)\to\mathbb{R}$ be defined by $g(t) = -\log(t)$ and $g'(t) = -1/t$. We have

$$I_g(p,q) = -\int_\Omega p(t)\log\left[\frac{q(t)}{p(t)}\right] d\mu(t) = \int_\Omega p(t)\log\left[\frac{p(t)}{q(t)}\right] d\mu(t) = D_{KL}(p,q),$$

and

$$I_{g'}(p,q) = -\int_\Omega \frac{p^2(t)}{q(t)}\, d\mu(t) = -\left[D_{\chi^2}(q,p)+1\right].$$

Note the use of the following identity:

$$D_{\chi^2}(q,p) = \int_\Omega q(t)\left[\left(\frac{p(t)}{q(t)}\right)^2 - 1\right] d\mu(t) = \int_\Omega \frac{p^2(t)}{q(t)}\, d\mu(t) - 1. \tag{6.8}$$

Proposition 5. *Let $f:(0,\infty)\to\mathbb{R}$ be a convex function with the property that $f(1)=0$. Let $\varphi(t)$ be an arbitrary polynomial of degree n. Assume that $p,q\in\mathcal{P}$ and there exists constants $0<r<1<R<\infty$ such that*

$$r\le\frac{q(t)}{p(t)}\le R,\quad \text{for } \mu\text{-a.e. } t\in\Omega. \tag{6.9}$$

If $a\in[r,R]$ and $f^{(n)}$ is absolutely continuous on $[r,R]$, then we have the inequalities

$$\begin{aligned}&\Bigg|I_f(p,q)-f(a)+\frac{1}{\varphi^{(n)}(0)}\sum_{m=1}^{n}(-1)^{m-1}\Big\{\varphi^{(n-m)}(0)f^{(m)}(a)D_{\chi^m,a}(p,q)\\&-\varphi^{(n-m)}(1)\int_\Omega\frac{(q(t)-ap(t))^m}{p^{m-1}(t)}f^{(m)}\left(\frac{q(t)}{p(t)}\right)d\mu\Big\}\Bigg|\\&\le\left(\int_0^1\frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\,dt\right)\|f^{(n+1)}\|_{[r,R],\infty}D_{|\chi|^{n+1},a}(p,q).\end{aligned}$$

Proof. We choose $g(t)=q(t)/p(t)$ in Corollary 1, and note that $\int_\Omega p(t)d\mu=1$. The proof is straightforward and therefore we omit the details. □

Proposition 6. *Let $f:(0,\infty)\to\mathbb{R}$ be a convex function with the property that $f(1)=0$. Let $\varphi_{2n}(t)$ be the Bernoulli polynomials. Assume that $p,q\in\mathcal{P}$ and there exists constants $0<r<1<R<\infty$ such that*

$$r\le\frac{q(t)}{p(t)}\le R,\quad \text{for } \mu\text{-a.e. } t\in\Omega. \tag{6.10}$$

If $a\in[r,R]$ and $f^{(2n)}$ is absolutely continuous on $[r,R]$, then we have the inequalities

$$\begin{aligned}&\Bigg|I_f(p,q)-f(a)-\frac{f'(a)}{2}(1-a)-\frac{1}{2}\int_\Omega q(t)f'\left(\frac{q(t)}{p(t)}\right)d\mu+\frac{a}{2}I_{f'}(p,q)\\&-\sum_{k=1}^{n}\frac{(-1)^kB_k}{(2k)!}\left[\int_\Omega\frac{(q(t)-ap(t))^{2k}}{p^{2k-1}(t)}f^{(2k)}\left(\frac{q(t)}{p(t)}\right)d\mu-f^{(2k)}(a)D_{\chi^{2k},a}(p,q)\right]\Bigg|\\&\le\left(\int_0^1\frac{|\varphi_{2n}(t)|}{(2n)!}\,dt\right)\|f^{(2n+1)}\|_{[r,R],\infty}D_{|\chi|^{2n+1},a}(p,q),\end{aligned}$$

Proof. We choose $g(t)=q(t)/p(t)$ in Corollary 2, and note that $\int_\Omega p(t)d\mu=1$. The proof is straightforward and therefore we omit the details. □

Corollary 3. *Let $f:(0,\infty)\to\mathbb{R}$ be a convex function with the property that $f(1)=0$. Assume that $p,q\in\mathcal{P}$ and there exists constants $0<r<1<R<\infty$ such that $r\le\frac{q(t)}{p(t)}\le R$, for μ-a.e. $t\in\Omega$. If $a\in[r,R]$ and f'' is absolutely continuous on $[r,R]$, then we have the inequalities*

$$\begin{aligned}&\Bigg|I_f(p,q)-f(a)-\frac{f'(a)}{2}(1-a)-\frac{1}{2}\int_\Omega q(t)f'\left(\frac{q(t)}{p(t)}\right)d\mu+\frac{a}{2}I_{f'}(p,q)\\&+\frac{1}{12}\int_\Omega\frac{(q(t)-ap(t))^2}{p(t)}f''\left(\frac{q(t)}{p(t)}\right)d\mu-\frac{f''(a)}{12}D_{\chi^2,a}(p,q)\Bigg|\\&\le\frac{1}{18\sqrt{3}}\|f'''\|_{[r,R],\infty}D_{|\chi|^3,a}(p,q).\end{aligned}$$

Proof. We choose $g(t)=q(t)/p(t)$ in (4.9) and note that $\int_\Omega p(t)d\mu=1$. The proof is straightforward and therefore we omit the details. □

Example 2. We consider the convex function $f : (0, \infty) \to \mathbb{R}$, $f(t) = t\log(t)$. We have

$$f'(t) = \log(t) + 1 \quad \text{and} \quad f^{(k)}(t) = (-1)^k t^{-(k-1)}(k-2)!, \qquad \text{for } k \geq 2.$$

Thus, $\|f^{(k)}\|_{[r,R]} = r^{-(k-1)}(k-2)!$. Recall from Example 1 Part (i) that $I_f(p,q) = D_{KL}(q,p)$. We also have

$$\begin{aligned}
&\int_\Omega \frac{(q(t)-ap(t))^m}{p^{m-1}(t)} f^{(m)}\left(\frac{q(t)}{p(t)}\right) d\mu \\
= \;& (-1)^m (m-2)! \int_\Omega \frac{(q(t)-ap(t))^m}{p^{m-1}(t)} \left(\frac{p(t)}{q(t)}\right)^{m-1} d\mu \\
= \;& (-1)^m (-a)^m (m-2)! \int_\Omega \frac{(p(t)-\frac{1}{a}q(t))^m}{q^{m-1}(t)} d\mu \\
= \;& a^m (m-2)! D_{\chi^m, \frac{1}{a}}(q,p).
\end{aligned}$$

Therefore, Proposition 5 gives us:

$$\begin{aligned}
&\Bigg| D_{KL}(q,p) - a\log(a) - \frac{1}{\varphi^{(n)}(0)} \sum_{m=1}^{n} (m-2)! \Bigg\{ \frac{\varphi^{(n-m)}(0)}{a^{m-1}} D_{\chi^m,a}(p,q) \\
&+ (-1)^{m-1} a^m \varphi^{(n-m)}(1) D_{\chi^m,\frac{1}{a}}(q,p) \Bigg\} \Bigg| \\
&\leq \frac{(n-1)!}{r^n} \left(\int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|} dt \right) D_{|\chi|^{n+1},a}(p,q).
\end{aligned} \tag{6.11}$$

In particular, when $a = 1$, we have

$$\begin{aligned}
&\Bigg| D_{KL}(q,p) - \sum_{m=1}^{n} (m-2)! \Bigg\{ \frac{\varphi^{(n-m)}(0)}{\varphi^{(n)}(0)} D_{\chi^m}(p,q) \\
&+ (-1)^{m-1} \frac{\varphi^{(n-m)}(1)}{\varphi^{(n)}(0)} D_{\chi^m}(q,p) \Bigg\} \Bigg| \\
&\leq \frac{(n-1)!}{r^n} \left(\int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|} dt \right) D_{|\chi|^{n+1}}(p,q).
\end{aligned} \tag{6.12}$$

We also have

$$\begin{aligned}
\int_\Omega q(t) f'\left(\frac{q(t)}{p(t)}\right) d\mu(t) &= \int_\Omega q(t) \left(\log\left(\frac{q(t)}{p(t)}\right) + 1\right) d\mu(t) \\
&= \int_\Omega q(t) \log\left(\frac{q(t)}{p(t)}\right) d\mu(t) + \int_\Omega q(t)\, d\mu(t) \\
&= D_{KL}(q,p) + 1,
\end{aligned}$$

and from Example 1 Part (i) that $I_{f'}(p,q) = -D_{KL}(p,q)+1$. Therefore, Proposition 6 gives us

$$\begin{aligned}
&\Bigg| D_{KL}(q,p) - a\log(a) - \frac{\log(a)+1}{2}(1-a) - \frac{1}{2}D_{KL}(q,p) - \frac{1}{2} - \frac{aD_{KL}(p,q)}{2} + \frac{a}{2} \\
&- \sum_{k=1}^{n} \frac{(-1)^k B_k}{(2k)!}(2k-2)! \left[\int_\Omega \frac{(q(t)-ap(t))^{2k}}{q^{2k-1}(t)}\, d\mu(t) - \frac{D_{\chi^{2k},a}(p,q)}{a^{2k-1}} \right] \Bigg| \\
&= \Bigg| D_{KL}(q,p) - \frac{1}{2}\log(a)(a+1) + (a-1) - \frac{1}{2}(D_{KL}(q,p) + aD_{KL}(p,q)) \\
&- \sum_{k=1}^{n} \frac{(-1)^k B_k}{4k^2-2k} \left[a^{2k} D_{\chi^{2k},\frac{1}{a}}(q,p) - \frac{D_{\chi^{2k},a}(p,q)}{a^{2k-1}} \right] \Bigg| \\
&\le \frac{(2n-1)!}{r^{2n}} \left(\int_0^1 \frac{|\varphi_{2n}(t)|}{(2n)!}\, dt \right) D_{|\chi|^{2n+1},a}(p,q) \\
&= \left(\int_0^1 \frac{|\varphi_{2n}(t)|}{2n}\, dt \right) \frac{D_{|\chi|^{2n+1},a}(p,q)}{r^{2n}}.
\end{aligned}$$

In particular, when $a = 1$, we have

$$\begin{aligned}
\text{(6.13)}\qquad &\left| D_{KL}(q,p) - D_{KL}(p,q) - \sum_{k=1}^{n} \frac{(-1)^k B_k}{4k^2-2k} \left[D_{\chi^{2k}}(q,p) - D_{\chi^{2k}}(p,q) \right] \right| \\
&\le \left(\int_0^1 \frac{|\varphi_{2n}(t)|}{2n}\, dt \right) \frac{D_{|\chi|^{2n+1}}(p,q)}{r^{2n}}.
\end{aligned}$$

We note that

$$\begin{aligned}
\int_\Omega \frac{(q(t)-ap(t))^2}{p(t)} f''\left(\frac{q(t)}{p(t)}\right) d\mu(t) &= \int_\Omega \frac{(q(t)-ap(t))^2}{q(t)}\, d\mu(t) \\
&= 1 - 2a + a^2 \int_\Omega \frac{p(t)^2}{q(t)}\, d\mu(t) \\
&= 1 - 2a + a^2(D_{\chi^2}(q,p)+1) \\
&= a^2 D_{\chi^2}(q,p) + (1-a)^2.
\end{aligned}$$

Note the use of (6.8). Thus, Corollary 3 gives us

$$\begin{aligned}
&\Bigg| D_{KL}(q,p) - \frac{1}{2}\log(a)(a+1) + (a-1) - \frac{1}{2}(D_{KL}(q,p) + aD_{KL}(p,q)) \\
&+ \frac{1}{12}\left[a^2 D_{\chi^2}(q,p) + (1-a)^2 - \frac{1}{a} D_{\chi^2,a}(p,q) \right] \Bigg| \\
&\le \frac{D_{|\chi|^3,a}(p,q)}{18\sqrt{3}r^2}.
\end{aligned}$$

In particular, when $a = 1$, we have

$$\text{(6.14)}\qquad \left| D_{KL}(q,p) - D_{KL}(p,q) + \frac{1}{6}\left[D_{\chi^2}(q,p) - D_{\chi^2}(p,q) \right] \right| \le \frac{D_{|\chi|^3}(p,q)}{9\sqrt{3}r^2}.$$

Example 3. We consider the convex function $g : (0,\infty) \to \mathbb{R}$, $g(t) = -\log(t)$. We have

$$g^{(k)}(t) = (-1)^k t^{-k}(k-1)!, \quad \text{for } k \ge 1.$$

Thus, $\|g^{(k)}\|_{[r,R]} = r^{-k}$. From Example 1 Part (ii) , we have $I_g(p,q) = D_{KL}(p,q)$. Proposition 5 gives us

$$\begin{aligned}(6.15)\qquad & \Bigg| D_{KL}(p,q) + \log(a) - \frac{1}{\varphi^{(n)}(0)} \sum_{m=1}^{n} (m-1)! \left\{ \frac{\varphi^{(n-m)}(0)}{a^m} D_{\chi^m,a}(p,q) \right. \\ & \left. -\varphi^{(n-m)}(1) \int_\Omega \left(1 - a\frac{p(t)}{q(t)}\right)^m p(t)\, d\mu \right\} \Bigg| \\ & \le \frac{n!}{r^{n+1}} \left(\int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\, dt \right) D_{|\chi|^{n+1},a}(p,q).\end{aligned}$$

In particular, when $a = 1$, we have

$$\begin{aligned}(6.16)\qquad & \Bigg| D_{KL}(p,q) - \frac{1}{\varphi^{(n)}(0)} \sum_{m=1}^{n} (m-1)! \left\{ \varphi^{(n-m)}(0) D_{\chi^m}(p,q) \right. \\ & \left. -\varphi^{(n-m)}(1) \int_\Omega \left(1 - \frac{p(t)}{q(t)}\right)^m p(t)\, d\mu \right\} \Bigg| \\ & \le \frac{n!}{r^{n+1}} \left(\int_0^1 \frac{|\varphi(t)|}{|\varphi^{(n)}(0)|}\, dt \right) D_{|\chi|^{n+1}}(p,q).\end{aligned}$$

We have

$$\int_\Omega q(t) f'\left(\frac{q(t)}{p(t)}\right) d\mu = -\int_\Omega q(t) \left(\frac{p(t)}{q(t)}\right) d\mu = -1$$

and $I_{g'}(p,q) = -[D_{\chi^2}(q,p) + 1]$ from Example 1 Part (ii). Proposition 6 gives us

$$\begin{aligned}(6.17)\qquad & \Bigg| D_{KL}(p,q) + \log(a) + \frac{1}{2a}(1-a) + \frac{1}{2} - \frac{a}{2}(D_{\chi^2}(q,p) + 1) \\ & - \sum_{k=1}^{n} \frac{(-1)^k B_k}{2k} \left[\int_\Omega \left(1 - a\frac{p(t)}{q(t)}\right)^{2k} p(t)\, d\mu - \frac{1}{a^{2k}} D_{\chi^{2k},a}(p,q) \right] \Bigg| \\ & \le \left(\int_0^1 |\varphi_{2n}(t)|\, dt \right) \frac{D_{|\chi|^{2n+1},a}(p,q)}{r^{2n+1}}.\end{aligned}$$

In particular, when $a = 1$, we have

$$\begin{aligned}(6.18)\qquad & \Bigg| D_{KL}(p,q) - \frac{1}{2} D_{\chi^2}(q,p) \\ & - \sum_{k=1}^{n} \frac{(-1)^k B_k}{2k} \left[\int_\Omega \left(1 - \frac{p(t)}{q(t)}\right)^{2k} p(t)\, d\mu - D_{\chi^{2k}}(p,q) \right] \Bigg| \\ & \le \left(\int_0^1 |\varphi_{2n}(t)|\, dt \right) \frac{D_{|\chi|^{2n+1}}(p,q)}{r^{2n+1}}.\end{aligned}$$

Corollary 3 gives us

$$\begin{aligned}(6.19)\qquad & \Bigg| D_{KL}(p,q) + \log(a) + \frac{1}{2a}(1-a) + \frac{1}{2} - \frac{a}{2}(D_{\chi^2}(q,p) + 1) \\ & + \frac{1}{12} \int_\Omega \left(1 - a\frac{p(t)}{q(t)}\right)^2 p(t)\, d\mu - \frac{1}{12a^2} D_{\chi^2,a}(p,q) \Bigg| \\ & \le \frac{D_{|\chi|^3,a}(p,q)}{9\sqrt{3}r^3}.\end{aligned}$$

In particular, when $a = 1$, we have

$$\left| D_{KL}(p,q) - \frac{2}{3} D_{\chi^2}(q,p) + \frac{1}{12}\left[-1 + \int_\Omega \left(\frac{p(t)}{q(t)} \right)^2 p(t)\, d\mu - D_{\chi^2}(p,q) \right] \right| \le \frac{D_{|\chi|^3}(p,q)}{9\sqrt{3} r^3}. \tag{6.20}$$

We note the use of

$$\begin{aligned} \int_\Omega \left(1 - \frac{p(t)}{q(t)} \right)^2 p(t)\, d\mu &= \int_\Omega \left(p(t) - 2\frac{(p(t))^2}{q(t)} + \left(\frac{p(t)}{q(t)} \right)^2 p(t) \right) d\mu \\ &= 1 - 2(D_{\chi^2}(q,p) + 1) + \int_\Omega \left(\frac{p(t)}{q(t)} \right)^2 p(t)\, d\mu \\ &= -1 - 2D_{\chi^2}(q,p) + \int_\Omega \left(\frac{p(t)}{q(t)} \right)^2 p(t)\, d\mu. \end{aligned}$$

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