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SOME REFINEMENTS AND REVERSES OF CALLEBAUT'S INEQUALITY FOR ISOTONIC FUNCTIONALS VIA A RESULT DUE TO CARTWRIGHT AND FIELD

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Abstract. In this paper we obtain some refinements and reverses of Callebaut's inequality for isotonic functionals via a result on Young's inequality due to Cartwright and Field.

1. Introduction

Let L be a *linear class* of real-valued functions $g : E \to \mathbb{R}$ having the properties

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L$, i.e., if $f_0(t) = 1$, $t \in E$ then $f_0 \in L$.

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$.
(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.
The mapping A is said to be *normalised if*
(A3) $A(\mathbf{1}) = 1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [3], [16] and [17]). For other inequalities for isotonic functionals see [2], [5]-[15] and [18]-[20].

We note that common examples of such isotonic linear functionals A are given by

$$A(g) = \int_E g d\mu \text{ or } A(g) = \sum_{k \in E} p_k g_k,$$

where μ is a positive measure on E in the first case and E is a subset of the natural numbers $\mathbb{N}$, in the second ($p_k \geq 0$, $k \in E$).

We have the following inequality that provides a refinement and a reverse for the celebrated *Young's inequality*

$$\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max\{a,b\}} \leq (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \leq \frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min\{a,b\}} \tag{1.1}$$

for any $a, b > 0$ and $\nu \in [0,1]$.

This result was obtained in 1978 by Cartwright and Field [1] who established a more general result for n variables and gave an application for a probability measure supported on a finite interval.

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The functional version of *Callebaut's inequality* states that

$$A^2(fg) \le A\left(f^{2-\nu}g^{\nu}\right)A\left(f^{\gamma}g^{2-\nu}\right) \le A\left(f^2\right)A\left(g^2\right) \tag{1.2}$$

provided that f^2, g^2, $f^{2-\nu}g^{\nu}$, $f^{\nu}g^{2-\nu}$, $fg \in L$ for some $\nu \in [0,2]$. For the discrete and integral of one real variable versions see [4].

In this paper we obtain some inequalities for isotonic functionals via the reverse and refinement of Young's inequality (1.1) that are related to the second part of Callebaut's inequality (1.2). Applications for integrals and n-tuples of real numbers are also provided.

2. On Callebaut's Inequality

We have the following result that provides a refinement and reverse of Callebaut's second inequality:

Theorem 1. *Let A, $B : L \to \mathbb{R}$ be two normalised isotonic functionals. If f, $g : E \to \mathbb{R}$ are such that, f^2, g^2, $\frac{g^4}{f^2}$, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)} \in L$ for some $\nu \in [0,1]$, and*

$$0 < m \le \frac{f}{g} \le M < \infty \tag{2.1}$$

for real numbers $M > m > 0$, then

$$\begin{aligned} &\frac{1}{2}\nu(1-\nu)m^2\left(A\left(\frac{g^4}{f^2}\right)B\left(f^2\right)+A\left(f^2\right)B\left(\frac{g^4}{f^2}\right)-2\right) \\ &\le (1-\nu)A\left(f^2\right)B\left(g^2\right)+\nu A\left(g^2\right)B\left(f^2\right)-A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right) \\ &\le \frac{1}{2}\nu(1-\nu)M^2\left(A\left(\frac{g^4}{f^2}\right)B\left(f^2\right)+A\left(f^2\right)B\left(\frac{g^4}{f^2}\right)-2\right). \end{aligned} \tag{2.2}$$

Proof. Since $ab = \min\{a,b\}\max\{a,b\}$ for any $a, b > 0$, then from (1.1) we have

$$\begin{aligned} &\frac{1}{2}\nu(1-\nu)\min\{a,b\}\frac{(b-a)^2}{ab} \\ &\le (1-\nu)a+\nu b-a^{1-\nu}b^{\nu} \le \frac{1}{2}\nu(1-\nu)\max\{a,b\}\frac{(b-a)^2}{ab}, \end{aligned}$$

where $\nu \in [0,1]$. This can be written as

$$\begin{aligned} &\frac{1}{2}\nu(1-\nu)\min\{a,b\}\left(\frac{b}{a}+\frac{a}{b}-2\right) \\ &\le (1-\nu)a+\nu b-a^{1-\nu}b^{\nu} \le \frac{1}{2}\nu(1-\nu)\max\{a,b\}\left(\frac{b}{a}+\frac{a}{b}-2\right), \end{aligned} \tag{2.3}$$

for any a, $b > 0$.

Let x, $y \in E$ such that $g(x)$, $g(y) \neq 0$. If we use the inequalities (2.3) for

$$a = \frac{f^2(x)}{g^2(x)},\ b = \frac{f^2(y)}{g^2(y)} \in \left[m^2, M^2\right]$$

then we get

$$\begin{aligned}(2.4)\quad &\frac{1}{2}\nu(1-\nu)m^2\left(\frac{f^2(y)}{g^2(y)}\frac{g^2(x)}{f^2(x)}+\frac{f^2(x)}{g^2(x)}\frac{g^2(y)}{f^2(y)}-2\right)\\ &\le(1-\nu)\frac{f^2(x)}{g^2(x)}+\nu\frac{f^2(y)}{g^2(y)}-\left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu}\left(\frac{f^2(y)}{g^2(y)}\right)^{\nu}\\ &\le\frac{1}{2}\nu(1-\nu)M^2\left(\frac{f^2(y)}{g^2(y)}\frac{g^2(x)}{f^2(x)}+\frac{f^2(x)}{g^2(x)}\frac{g^2(y)}{f^2(y)}-2\right),\end{aligned}$$

where $\nu\in[0,1]$.

If we multiply (2.4) by $g^2(x)\,g^2(y)$, then we get

$$\begin{aligned}(2.5)\quad &\frac{1}{2}\nu(1-\nu)m^2\left(f^2(y)\frac{g^4(x)}{f^2(x)}+f^2(x)\frac{g^4(y)}{f^2(y)}-2\right)\\ &\le(1-\nu)g^2(y)f^2(x)+\nu f^2(y)g^2(x)\\ &-f^{2\nu}(y)g^{2(1-\nu)}(y)f^{2(1-\nu)}(x)g^{2\nu}(x)\\ &\le\frac{1}{2}\nu(1-\nu)M^2\left(f^2(y)\frac{g^4(x)}{f^2(x)}+f^2(x)\frac{g^4(y)}{f^2(y)}-2\right),\end{aligned}$$

which holds for any $x,\,y\in E$.

Fix $y\in E$. Then by (2.5) we have in the order of L that

$$\begin{aligned}(2.6)\quad &\frac{1}{2}\nu(1-\nu)m^2\left(f^2(y)\frac{g^4}{f^2}+\frac{g^4(y)}{f^2(y)}f^2-2\right)\\ &\le(1-\nu)g^2(y)f^2+\nu f^2(y)g^2-f^{2\nu}(y)g^{2(1-\nu)}(y)f^{2(1-\nu)}g^{2\nu}\\ &\le\frac{1}{2}\nu(1-\nu)M^2\left(f^2(y)\frac{g^4}{f^2}+\frac{g^4(y)}{f^2(y)}f^2-2\right).\end{aligned}$$

If we take the functional A in (2.6), then we get

$$\begin{aligned}&\frac{1}{2}\nu(1-\nu)m^2\left(f^2(y)A\left(\frac{g^4}{f^2}\right)+\frac{g^4(y)}{f^2(y)}A\left(f^2\right)-2\right)\\ &\le(1-\nu)g^2(y)A\left(f^2\right)+\nu f^2(y)A\left(g^2\right)-f^{2\nu}(y)g^{2(1-\nu)}(y)A\left(f^{2(1-\nu)}g^{2\nu}\right)\\ &\le\frac{1}{2}\nu(1-\nu)M^2\left(f^2(y)A\left(\frac{g^4}{f^2}\right)+\frac{g^4(y)}{f^2(y)}A\left(f^2\right)-2\right),\end{aligned}$$

for any $y\in E$.

If we write this inequality in the order of L, then we have

$$\begin{aligned}&\frac{1}{2}\nu(1-\nu)m^2\left(A\left(\frac{g^4}{f^2}\right)f^2+A\left(f^2\right)\frac{g^4}{f^2}-2\right)\\ &\le(1-\nu)A\left(f^2\right)g^2+\nu A\left(g^2\right)f^2-A\left(f^{2(1-\nu)}g^{2\nu}\right)f^{2\nu}g^{2(1-\nu)}\\ &\le\frac{1}{2}\nu(1-\nu)M^2\left(A\left(\frac{g^4}{f^2}\right)f^2+A\left(f^2\right)\frac{g^4}{f^2}-2\right),\end{aligned}$$

and by taking the functional B we deduce the desired result (2.2). □

Corollary 1. *Let $A:L\to\mathbb{R}$ be a normalised isotonic functional. If $f,\,g:E\to\mathbb{R}$ are such that $f^2,\ g^2,\ \frac{g^4}{f^2},\ f^{2(1-\nu)}g^{2\nu},\ f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$ and the* *condition (2.1) holds, then*

$$\nu(1-\nu)m^2\left(A\left(\frac{g^4}{f^2}\right)A\left(f^2\right)-1\right) \tag{2.7}$$
$$\leq A\left(f^2\right)A\left(g^2\right)-A\left(f^{2(1-\nu)}g^{2\nu}\right)A\left(f^{2\nu}g^{2(1-\nu)}\right)$$
$$\leq \nu(1-\nu)M^2\left(A\left(\frac{g^4}{f^2}\right)A\left(f^2\right)-1\right).$$

In particular, if f^2, g^2, $\frac{g^4}{f^2}$, $fg\in L$ and the condition (2.1) holds, then

$$\frac{1}{4}m^2\left(A\left(\frac{g^4}{f^2}\right)A\left(f^2\right)-1\right)\leq A\left(f^2\right)A\left(g^2\right)-A^2(fg) \tag{2.8}$$
$$\leq \frac{1}{4}M^2\left(A\left(\frac{g^4}{f^2}\right)A\left(f^2\right)-1\right).$$

The following result also holds:

Theorem 2. *Let A, $B : L\to\mathbb{R}$ be two normalised isotonic functionals. If f, $g : E\to\mathbb{R}$ are such that $f\geq 0$, $g>0$, f^2, g^2, $\frac{f^4}{g^2}$, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$ and the condition (2.1) holds, then*

$$\frac{1}{2M^2}\nu(1-\nu)\left(A\left(g^2\right)B\left(\frac{f^4}{g^2}\right)-2A\left(f^2\right)B\left(f^2\right)+A\left(\frac{f^4}{g^2}\right)B\left(g^2\right)\right) \tag{2.9}$$
$$\leq (1-\nu)A\left(f^2\right)B\left(g^2\right)+\nu A\left(g^2\right)B\left(f^2\right)$$
$$-A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right)$$
$$\leq \frac{1}{2m^2}\nu(1-\nu)\left(A\left(g^2\right)B\left(\frac{f^4}{g^2}\right)-2A\left(f^2\right)B\left(f^2\right)+A\left(\frac{f^4}{g^2}\right)B\left(g^2\right)\right).$$

Proof. For any x, $y\in E$ we have

$$m^2\leq\frac{f^2(x)}{g^2(x)},\frac{f^2(y)}{g^2(y)}\leq M^2.$$

If we use the inequality (1.1) for

$$a=\frac{f^2(x)}{g^2(x)},\ b=\frac{f^2(y)}{g^2(y)},$$

then we get

$$\frac{1}{2M^2}\nu(1-\nu)\left(\frac{f^2(y)}{g^2(y)}-\frac{f^2(x)}{g^2(x)}\right)^2$$
$$\leq(1-\nu)\frac{f^2(x)}{g^2(x)}+\nu\frac{f^2(y)}{g^2(y)}-\left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu}\left(\frac{f^2(y)}{g^2(y)}\right)^{\nu}$$
$$\leq\frac{1}{2m^2}\nu(1-\nu)\left(\frac{f^2(y)}{g^2(y)}-\frac{f^2(x)}{g^2(x)}\right)^2$$

for any x, $y\in E$.

This can be written as

$$\begin{aligned}(2.10)\qquad & \frac{1}{2M^2}\nu(1-\nu)\left(\frac{f^4(y)}{g^4(y)}-2\frac{f^2(y)}{g^2(y)}\frac{f^2(x)}{g^2(x)}+\frac{f^4(x)}{g^4(x)}\right)\\ &\le(1-\nu)\frac{f^2(x)}{g^2(x)}+\nu\frac{f^2(y)}{g^2(y)}-\left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu}\left(\frac{f^2(y)}{g^2(y)}\right)^{\nu}\\ &\le\frac{1}{2m^2}\nu(1-\nu)\left(\frac{f^4(y)}{g^4(y)}-2\frac{f^2(y)}{g^2(y)}\frac{f^2(x)}{g^2(x)}+\frac{f^4(x)}{g^4(x)}\right).\end{aligned}$$

Now, if we multiply (2.10) by $g^2(x)\,g^2(y)>0$ then we get

$$\begin{aligned}(2.11)\qquad & \frac{1}{2M^2}\nu(1-\nu)\left(\frac{f^4(y)}{g^2(y)}g^2(x)-2f^2(y)\,f^2(x)+\frac{f^4(x)}{g^2(x)}g^2(y)\right)\\ &\le(1-\nu)\,g^2(y)\,f^2(x)+\nu f^2(y)\,g^2(x)\\ &-f^{2\nu}(y)\,g^{2(1-\nu)}(y)\,f^{2(1-\nu)}(x)\,g^{2\nu}(x)\\ &\le\frac{1}{2m^2}\nu(1-\nu)\left(\frac{f^4(y)}{g^2(y)}g^2(x)-2f^2(y)\,f^2(x)+\frac{f^4(x)}{g^2(x)}g^2(y)\right)\end{aligned}$$

for any $x,\,y\in E$.

Fix $y\in E$. Then by (2.11) we have in the order of L that

$$\begin{aligned}(2.12)\qquad & \frac{1}{2M^2}\nu(1-\nu)\left(\frac{f^4(y)}{g^2(y)}g^2-2f^2(y)\,f^2+g^2(y)\,\frac{f^4}{g^2}\right)\\ &\le(1-\nu)\,g^2(y)\,f^2+\nu f^2(y)\,g^2-f^{2\nu}(y)\,g^{2(1-\nu)}(y)\,f^{2(1-\nu)}g^{2\nu}\\ &\le\frac{1}{2m^2}\nu(1-\nu)\left(\frac{f^4(y)}{g^2(y)}g^2-2f^2(y)\,f^2+g^2(y)\,\frac{f^4}{g^2}\right).\end{aligned}$$

If we take the functional A in (2.12), then we get

$$\begin{aligned}(2.13)\qquad & \frac{1}{2M^2}\nu(1-\nu)\left(\frac{f^4(y)}{g^2(y)}A\left(g^2\right)-2f^2(y)\,A\left(f^2\right)+g^2(y)\,A\left(\frac{f^4}{g^2}\right)\right)\\ &\le(1-\nu)\,g^2(y)\,A\left(f^2\right)+\nu f^2(y)\,A\left(g^2\right)\\ &-f^{2\nu}(y)\,g^{2(1-\nu)}(y)\,A\left(f^{2(1-\nu)}g^{2\nu}\right)\\ &\le\frac{1}{2m^2}\nu(1-\nu)\left(\frac{f^4(y)}{g^2(y)}A\left(g^2\right)-2f^2(y)\,A\left(f^2\right)+g^2(y)\,A\left(\frac{f^4}{g^2}\right)\right)\end{aligned}$$

for any $y\in E$.

This inequality can be written in the order of L as

$$\begin{aligned}(2.14)\qquad & \frac{1}{2M^2}\nu(1-\nu)\left(A\left(g^2\right)\frac{f^4}{g^2}-2A\left(f^2\right)f^2+A\left(\frac{f^4}{g^2}\right)g^2\right)\\ &\le(1-\nu)\,A\left(f^2\right)g^2+\nu A\left(g^2\right)f^2-A\left(f^{2(1-\nu)}g^{2\nu}\right)f^{2\nu}g^{2(1-\nu)}\\ &\le\frac{1}{2m^2}\nu(1-\nu)\left(A\left(g^2\right)\frac{f^4}{g^2}-2A\left(f^2\right)f^2+A\left(\frac{f^4}{g^2}\right)g^2\right).\end{aligned}$$

Now, if we take the functional B in (2.14), then we get the desired result (2.9). □

Corollary 2. *Let $A : L \to \mathbb{R}$ be a normalised isotonic functional. If $f,\ g : E \to \mathbb{R}$ are such that $f \geq 0$, $g > 0$, f^2, g^2, $\frac{f^4}{g^2}$, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)} \in L$ for some $\nu \in [0,1]$ and the condition (2.1) is valid, then*

$$\begin{aligned}(2.15)\qquad & \frac{1}{M^2}\nu(1-\nu)\left(A\left(g^2\right)A\left(\frac{f^4}{g^2}\right) - A^2\left(f^2\right)\right) \\ & \leq A\left(f^2\right)A\left(g^2\right) - A\left(f^{2(1-\nu)}g^{2\nu}\right)A\left(f^{2\nu}g^{2(1-\nu)}\right) \\ & \leq \frac{1}{m^2}\nu(1-\nu)\left(A\left(g^2\right)A\left(\frac{f^4}{g^2}\right) - A^2\left(f^2\right)\right).\end{aligned}$$

In particular, if f^2, g^2, $\frac{f^4}{g^2}$, $fg \in L$ and the condition (2.1) is valid, then we have

$$\begin{aligned}(2.16)\qquad \frac{1}{4M^2}\left(A\left(g^2\right)A\left(\frac{f^4}{g^2}\right) - A^2\left(f^2\right)\right) & \leq A\left(f^2\right)A\left(g^2\right) - A^2(fg) \\ & \leq \frac{1}{4m^2}\left(A\left(g^2\right)A\left(\frac{f^4}{g^2}\right) - A^2\left(f^2\right)\right).\end{aligned}$$

3. Other Related Results

If we write the inequality (1.1) for $a = 1$ and $b = x$ we get

$$(3.1)\qquad \frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\max\{x,1\}} \leq 1 - \nu + \nu x - x^\nu \leq \frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\min\{x,1\}}$$

for any $x > 0$ and for any $\nu \in [0,1]$.

If $x \in [t,T] \subset (0,\infty)$, then $\max\{x,1\} \leq \max\{T,1\}$ and $\min\{t,1\} \leq \min\{x,1\}$ and by (3.1) we get

$$\begin{aligned}(3.2)\qquad \frac{1}{2}\nu(1-\nu)\frac{\min_{x\in[t,T]}(x-1)^2}{\max\{T,1\}} & \leq \frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\max\{T,1\}} \\ & \leq 1-\nu+\nu x - x^\nu \\ & \leq \frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\min\{t,1\}} \\ & \leq \frac{1}{2}\nu(1-\nu)\frac{\max_{x\in[t,T]}(x-1)^2}{\min\{t,1\}}\end{aligned}$$

for any $x \in [t,T]$ and for any $\nu \in [0,1]$.

Observe that

$$\min_{x\in[t,T]}(x-1)^2 = \begin{cases} (T-1)^2 \text{ if } T < 1, \\ 0 \text{ if } t \leq 1 \leq T, \\ (t-1)^2 \text{ if } 1 < t \end{cases}$$

and

$$\max_{x\in[t,T]}(x-1)^2 = \begin{cases} (t-1)^2 \text{ if } T < 1, \\ \max\left\{(t-1)^2, (T-1)^2\right\} \text{ if } t \leq 1 \leq T, \\ (T-1)^2 \text{ if } 1 < t. \end{cases}$$

Then

$$(3.3)\qquad c(t,T) := \frac{\min_{x\in[t,T]}(x-1)^2}{\max\{T,1\}} = \begin{cases} (T-1)^2 \text{ if } T < 1, \\ 0 \text{ if } t \leq 1 \leq T, \\ \frac{(t-1)^2}{T} \text{ if } 1 < t \end{cases}$$

and

$$C(t,T) := \frac{\max_{x\in[t,T]}(x-1)^2}{\min\{t,1\}} = \begin{cases} \frac{(t-1)^2}{t} \text{ if } T<1, \\ \frac{1}{t}\max\left\{(t-1)^2,(T-1)^2\right\} \text{ if } t\le 1\le T, \\ (T-1)^2 \text{ if } 1<t. \end{cases} \tag{3.4}$$

Using the inequality (3.2) we have

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)c(t,T) &\le \frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\max\{T,1\}} \\ &\le 1-\nu+\nu x - x^{\nu} \\ &\le \frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\min\{t,1\}} \le \frac{1}{2}\nu(1-\nu)C(t,T)\end{aligned} \tag{3.5}$$

for any $x\in[t,T]$ and for any $\nu\in[0,1]$.

Now, if a, $b>0$ and assume that $\frac{b}{a}\in[t,T]$, then by (3.5) we get

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)c(t,T)a &\le \frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max\{T,1\}a} \\ &\le (1-\nu)a+\nu b - b^{\nu}a^{1-\nu} \\ &\le \frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min\{t,1\}a} \le \frac{1}{2}\nu(1-\nu)C(t,T)a\end{aligned} \tag{3.6}$$

for any $\nu\in[0,1]$, where $c(t,T)$ and $C(t,T)$ are defined by (3.3) and (3.4), respectively.

Theorem 3. *Let A, $B : L\to\mathbb{R}$ be two normalised isotonic functionals. If f, $g : E\to\mathbb{R}$ are such that, f^2, g^2, $\frac{g^4}{f^2}$, $\frac{f^4}{g^2}$, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$ and the condition (2.1) holds, then*

$$\begin{aligned}0 &\le \frac{1}{2}\nu(1-\nu)\frac{m^2}{M^2}\left(A\left(\frac{g^4}{f^2}\right)B\left(\frac{f^4}{g^2}\right)-2A\left(g^2\right)B\left(f^2\right)+A\left(f^2\right)B\left(g^2\right)\right) \\ &\le (1-\nu)A\left(f^2\right)B\left(g^2\right)+\nu A\left(g^2\right)B\left(f^2\right) \\ &\quad - A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right) \\ &\le \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\left(A\left(\frac{g^4}{f^2}\right)B\left(\frac{f^4}{g^2}\right)-2A\left(g^2\right)B\left(f^2\right)+A\left(f^2\right)B\left(g^2\right)\right) \\ &\le \frac{1}{2}\nu(1-\nu)\left(\frac{M^2}{m^2}-1\right)^2 A\left(f^2\right)B\left(g^2\right).\end{aligned} \tag{3.7}$$

Proof. For any x, $y\in E$ we have

$$m^2\le\frac{f^2(x)}{g^2(x)},\frac{f^2(y)}{g^2(y)}\le M^2.$$

Consider

$$a=\frac{f^2(x)}{g^2(x)},\ b=\frac{f^2(y)}{g^2(y)},$$

then $\frac{b}{a} \in \left[\frac{m^2}{M^2}, \frac{M^2}{m^2}\right]$ and by (3.6) we get

$$\begin{aligned}
0 &\leq \frac{1}{2}\nu(1-\nu)\frac{\left(\frac{f^2(y)}{g^2(y)} - \frac{f^2(x)}{g^2(x)}\right)^2}{\frac{M^2}{m^2}\frac{f^2(x)}{g^2(x)}} \\
&\leq (1-\nu)\frac{f^2(x)}{g^2(x)} + \nu\frac{f^2(y)}{g^2(y)} - \left(\frac{f^2(y)}{g^2(y)}\right)^{\nu}\left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu} \\
&\leq \frac{1}{2}\nu(1-\nu)\frac{\left(\frac{f^2(y)}{g^2(y)} - \frac{f^2(x)}{g^2(x)}\right)^2}{\frac{m^2}{M^2}\frac{f^2(x)}{g^2(x)}} \\
&\leq \frac{1}{2}\nu(1-\nu)\max\left\{\left(\frac{m^2}{M^2} - 1\right)^2, \left(\frac{M^2}{m^2} - 1\right)^2\right\}\frac{f^2(x)}{g^2(x)} \\
&= \frac{1}{2}\nu(1-\nu)\left(\frac{M^2}{m^2} - 1\right)^2\frac{f^2(x)}{g^2(x)}
\end{aligned}$$

for any x, $y \in E$ and $\nu \in [0,1]$.

This inequality is equivalent to

$$\begin{aligned}
0 &\leq \frac{1}{2}\nu(1-\nu)\frac{m^2}{M^2}\frac{\left(\frac{f^2(y)}{g^2(y)} - \frac{f^2(x)}{g^2(x)}\right)^2 g^2(x)}{f^2(x)} \\
&\leq (1-\nu)\frac{f^2(x)}{g^2(x)} + \nu\frac{f^2(y)}{g^2(y)} - \left(\frac{f^2(y)}{g^2(y)}\right)^{\nu}\left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu} \\
&\leq \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\frac{\left(\frac{f^2(y)}{g^2(y)} - \frac{f^2(x)}{g^2(x)}\right)^2 g^2(x)}{f^2(x)} \\
&\leq \frac{1}{2}\nu(1-\nu)\left(\frac{M^2}{m^2} - 1\right)^2\frac{f^2(x)}{g^2(x)}
\end{aligned} \tag{3.8}$$

for any $x, y \in E$ and $\nu \in [0,1]$.

Now, if we multiply (3.8) by $g^2(x)\,g^2(y) > 0$ then we get

$$\begin{aligned}
0 &\leq \frac{1}{2}\nu(1-\nu)\frac{m^2}{M^2}\frac{\left(\frac{f^2(y)}{g^2(y)} - \frac{f^2(x)}{g^2(x)}\right)^2 g^4(x)\,g^2(y)}{f^2(x)} \\
&\leq (1-\nu)\,g^2(y)\,f^2(x) + \nu f^2(y)\,g^2(x) \\
&\quad - f^{2\nu}(y)\,g^{2(1-\nu)}(y)\,f^{2(1-\nu)}(x)\,g^{2\nu}(x) \\
&\leq \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\frac{\left(\frac{f^2(y)}{g^2(y)} - \frac{f^2(x)}{g^2(x)}\right)^2 g^4(x)\,g^2(y)}{f^2(x)} \\
&\leq \frac{1}{2}\nu(1-\nu)\left(\frac{M^2}{m^2} - 1\right)^2 f^2(x)\,g^2(y)
\end{aligned} \tag{3.9}$$

for any x, $y \in E$ and $\nu \in [0,1]$.

Observe that

$$
\begin{aligned}
&\frac{\left(\frac{f^2(y)}{g^2(y)}-\frac{f^2(x)}{g^2(x)}\right)^2 g^4(x)\,g^2(y)}{f^2(x)} \\
&=\frac{\left(\frac{f^4(y)}{g^4(y)}-2\frac{f^2(y)}{g^2(y)}\frac{f^2(x)}{g^2(x)}+\frac{f^4(x)}{g^4(x)}\right) g^4(x)\,g^2(y)}{f^2(x)} \\
&=\frac{\frac{f^4(y)g^4(x)}{g^2(y)}-2f^2(y)\,f^2(x)\,g^2(x)+f^4(x)\,g^2(y)}{f^2(x)} \\
&=\frac{f^4(y)\,g^4(x)}{g^2(y)\,f^2(x)}-2f^2(y)\,g^2(x)+f^2(x)\,g^2(y)
\end{aligned}
$$

and by (3.9) we get

$$
\begin{aligned}
0 &\le \frac{1}{2}\nu(1-\nu)\frac{m^2}{M^2}\left(\frac{f^4(y)\,g^4(x)}{g^2(y)\,f^2(x)}-2f^2(y)\,g^2(x)+f^2(x)\,g^2(y)\right) \\
&\le (1-\nu)\,g^2(y)\,f^2(x)+\nu f^2(y)\,g^2(x)-f^{2\nu}(y)\,g^{2(1-\nu)}(y)\,f^{2(1-\nu)}(x)\,g^{2\nu}(x) \\
&\le \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\left(\frac{f^4(y)\,g^4(x)}{g^2(y)\,f^2(x)}-2f^2(y)\,g^2(x)+f^2(x)\,g^2(y)\right) \\
&\le \frac{1}{2}\nu(1-\nu)\left(\frac{M^2}{m^2}-1\right)^2 f^2(x)\,g^2(y)
\end{aligned}
$$

for any $x,\ y\in E$ and $\nu\in[0,1]$.

Now, if we use a similar argument to the one from the proof of Theorem 1 we deduce the desired result (3.7). □

Corollary 3. *Let $A:L\to\mathbb{R}$ be a normalised isotonic functional. If $f,\ g:E\to\mathbb{R}$ are such that, f^2, g^2, $\frac{g^4}{f^2}$, $\frac{f^4}{g^2}$, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$ and the condition (2.1) holds, then*

$$
\begin{aligned}
0 &\le \frac{1}{2}\nu(1-\nu)\frac{m^2}{M^2}\left(A\left(\frac{g^4}{f^2}\right)A\left(\frac{f^4}{g^2}\right)-A\left(g^2\right)A\left(f^2\right)\right) \\
&\le A\left(f^2\right)A\left(g^2\right)-A\left(f^{2(1-\nu)}g^{2\nu}\right)A\left(f^{2\nu}g^{2(1-\nu)}\right) \\
&\le \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\left(A\left(\frac{g^4}{f^2}\right)A\left(\frac{f^4}{g^2}\right)-A\left(g^2\right)A\left(f^2\right)\right) \\
&\le \frac{1}{2}\nu(1-\nu)\left(\frac{M^2}{m^2}-1\right)^2 A\left(f^2\right)A\left(g^2\right).
\end{aligned} \tag{3.10}
$$

In particular, if f^2, g^2, $\frac{g^4}{f^2}$, $\frac{f^4}{g^2}$, $fg \in L$, *then we have*

$$\begin{aligned} 0 &\leq \frac{1}{8}\frac{m^2}{M^2}\left(A\left(\frac{g^4}{f^2}\right)A\left(\frac{f^4}{g^2}\right)-A\left(g^2\right)A\left(f^2\right)\right) \\ &\leq A\left(f^2\right)A\left(g^2\right)-A^2\left(fg\right) \\ &\leq \frac{1}{8}\frac{M^2}{m^2}\left(A\left(\frac{g^4}{f^2}\right)A\left(\frac{f^4}{g^2}\right)-A\left(g^2\right)A\left(f^2\right)\right) \\ &\leq \frac{1}{8}\left(\frac{M^2}{m^2}-1\right)^2 A\left(f^2\right)A\left(g^2\right). \end{aligned} \tag{3.11}$$

We observe that the inequality (3.11) can be written as

$$\begin{aligned} 0 &\leq \frac{1}{8}\frac{m^2}{M^2}\left(\frac{A\left(\frac{g^4}{f^2}\right)A\left(\frac{f^4}{g^2}\right)}{A\left(g^2\right)A\left(f^2\right)}-1\right) \leq 1-\frac{A^2\left(fg\right)}{A\left(f^2\right)A\left(g^2\right)} \\ &\leq \frac{1}{8}\frac{M^2}{m^2}\left(\frac{A\left(\frac{g^4}{f^2}\right)A\left(\frac{f^4}{g^2}\right)}{A\left(g^2\right)A\left(f^2\right)}-1\right) \leq \frac{1}{8}\left(\frac{M^2}{m^2}-1\right)^2. \end{aligned} \tag{3.12}$$

4. Applications for Integrals

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ -a.e. (almost every) $x \in \Omega$ and $p \geq 1$ consider the Lebesgue space

$$L_w^p(\Omega,\mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)|^p w(x)\,d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$. The same for other integrals involved below. We assume that $\int_\Omega w d\mu = 1$.

Let f, g be μ-measurable functions with the property that there exists the constants M, $m > 0$ such that

$$0 < m \leq \frac{f}{g} \leq M < \infty \ \mu\text{-almost everywhere (a.e.) on } \Omega. \tag{4.1}$$

If $f^2, g^2, \frac{g^4}{f^2}, f^{2(1-\nu)}g^{2\nu}, f^{2\nu}g^{2(1-\nu)} \in L_w(\Omega,\mu)$ for some $\nu \in [0,1]$ and the condition (4.1) holds, then by (2.7) we have

$$\begin{aligned} &\nu(1-\nu)\,m^2\left(\int_\Omega w\frac{g^4}{f^2}d\mu\int_\Omega wf^2d\mu-1\right) \\ &\leq \int_\Omega wf^2d\mu\int_\Omega wg^2d\mu-\int_\Omega wf^{2(1-\nu)}g^{2\nu}d\mu\int_\Omega wf^{2\nu}g^{2(1-\nu)}d\mu \\ &\leq \nu(1-\nu)\,M^2\left(\int_\Omega w\frac{g^4}{f^2}d\mu\int_\Omega wf^2d\mu-1\right). \end{aligned} \tag{4.2}$$

In particular, if f^2, g^2, $\frac{g^4}{f^2}$, $fg \in L_w(\Omega,\mu)$ and the condition (4.1) holds, then

$$\begin{aligned}(4.3)\quad \frac{1}{4}m^2\left(\int_\Omega w\frac{g^4}{f^2}d\mu\int_\Omega wf^2d\mu-1\right) &\le \int_\Omega wf^2d\mu\int_\Omega wg^2d\mu-\left(\int_\Omega wfgd\mu\right)^2\\ &\le \frac{1}{4}M^2\left(\int_\Omega w\frac{g^4}{f^2}d\mu\int_\Omega wf^2d\mu-1\right).\end{aligned}$$

If f^2, g^2, $\frac{f^4}{g^2}$, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)} \in L_w(\Omega,\mu)$ for some $\nu\in[0,1]$ and the condition (4.1) holds, then by (2.15) we have

$$\begin{aligned}(4.4)\quad &\frac{1}{M^2}\nu(1-\nu)\left(\int_\Omega wg^2d\mu\int_\Omega w\frac{f^4}{g^2}d\mu-\left(\int_\Omega wf^2d\mu\right)^2\right)\\ &\le \int_\Omega wf^2d\mu\int_\Omega wg^2d\mu-\int_\Omega wf^{2(1-\nu)}g^{2\nu}d\mu\int_\Omega wf^{2\nu}g^{2(1-\nu)}d\mu\\ &\le \frac{1}{m^2}\nu(1-\nu)\left(\int_\Omega wg^2d\mu\int_\Omega w\frac{f^4}{g^2}d\mu-\left(\int_\Omega wf^2d\mu\right)^2\right).\end{aligned}$$

In particular, if f^2, g^2, $\frac{f^4}{g^2}$, $fg \in L_w(\Omega,\mu)$ and the condition (4.1) is valid, then we have

$$\begin{aligned}(4.5)\quad &\frac{1}{4M^2}\left(\int_\Omega wg^2d\mu\int_\Omega w\frac{f^4}{g^2}d\mu-\left(\int_\Omega wf^2d\mu\right)^2\right)\\ &\le \int_\Omega wf^2d\mu\int_\Omega wg^2d\mu-\left(\int_\Omega wfgd\mu\right)^2\\ &\le \frac{1}{4m^2}\left(\int_\Omega wg^2d\mu\int_\Omega w\frac{f^4}{g^2}d\mu-\left(\int_\Omega wf^2d\mu\right)^2\right).\end{aligned}$$

If f^2, g^2, $\frac{g^4}{f^2}$, $\frac{f^4}{g^2}$, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)} \in L_w(\Omega,\mu)$ for some $\nu\in[0,1]$, and the condition (4.1) holds, then

$$\begin{aligned}(4.6)\quad 0 &\le \frac{1}{2}\nu(1-\nu)\frac{m^2}{M^2}\left(\int_\Omega w\frac{g^4}{f^2}d\mu\int_\Omega w\frac{f^4}{g^2}d\mu-\int_\Omega wg^2d\mu\int_\Omega wf^2d\mu\right)\\ &\le \int_\Omega wf^2d\mu\int_\Omega wg^2d\mu-\int_\Omega wf^{2(1-\nu)}g^{2\nu}d\mu\int_\Omega wf^{2\nu}g^{2(1-\nu)}d\mu\\ &\le \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\left(\int_\Omega w\frac{g^4}{f^2}d\mu\int_\Omega w\frac{f^4}{g^2}d\mu-\int_\Omega wg^2d\mu\int_\Omega wf^2d\mu\right)\\ &\le \frac{1}{2}\nu(1-\nu)\left(\frac{M^2}{m^2}-1\right)^2\int_\Omega wf^2d\mu\int_\Omega wg^2d\mu.\end{aligned}$$

In particular, if f^2, g^2, $\frac{g^4}{f^2}$, $\frac{f^4}{g^2}$, $fg \in L_w(\Omega, \mu)$, then we have

$$0 \le \frac{1}{8}\frac{m^2}{M^2}\left(\int_\Omega w\frac{g^4}{f^2}d\mu \int_\Omega w\frac{f^4}{g^2}d\mu - \int_\Omega wg^2 d\mu \int_\Omega wf^2 d\mu\right) \tag{4.7}$$
$$\le \int_\Omega wf^2 d\mu \int_\Omega wg^2 d\mu - \left(\int_\Omega wfg d\mu\right)^2$$
$$\le \frac{1}{8}\frac{M^2}{m^2}\left(\int_\Omega w\frac{g^4}{f^2}d\mu \int_\Omega w\frac{f^4}{g^2}d\mu - \int_\Omega wg^2 d\mu \int_\Omega wf^2 d\mu\right)$$
$$\le \frac{1}{8}\left(\frac{M^2}{m^2} - 1\right)^2 \int_\Omega wf^2 d\mu \int_\Omega wg^2 d\mu.$$

5. Applications for Real Numbers

We consider the n-tuples of positive numbers $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ and the probability distribution $p = (p_1, ..., p_n)$, i.e. $p_i \ge 0$ for any $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$. If there exist the constants m, $M > 0$ such that

$$0 < m \le \frac{a_i}{b_i} \le M < \infty \text{ for any } i \in \{1, ..., n\}, \tag{5.1}$$

then by (4.2) and (4.3) for the counting discrete measure, we have

$$\nu(1-\nu)m^2\left(\sum_{i=1}^n p_i \frac{b_i^4}{a_i^2}\sum_{i=1}^n p_i a_i^2 - 1\right) \tag{5.2}$$
$$\le \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 - \sum_{i=1}^n p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{i=1}^n p_i a_i^{2\nu} b_i^{2(1-\nu)}$$
$$\le \nu(1-\nu)M^2\left(\sum_{i=1}^n p_i \frac{b_i^4}{a_i^2}\sum_{i=1}^n p_i a_i^2 - 1\right)$$

for any $\nu \in [0, 1]$ and

$$\frac{1}{4}m^2\left(\sum_{i=1}^n p_i \frac{b_i^4}{a_i^2}\sum_{i=1}^n p_i a_i^2 - 1\right) \le \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 - \left(\sum_{i=1}^n p_i a_i b_i\right)^2 \tag{5.3}$$
$$\le \frac{1}{4}M^2\left(\sum_{i=1}^n p_i \frac{b_i^4}{a_i^2}\sum_{i=1}^n p_i a_i^2 - 1\right).$$

If $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ satisfy (5.1), then by (4.4) and (4.5) for the counting discrete measure, we have

$$\frac{1}{M^2}\nu(1-\nu)\left(\sum_{i=1}^n p_i b_i^2 \sum_{i=1}^n p_i \frac{a_i^4}{b_i^2} - \left(\sum_{i=1}^n p_i a_i^2\right)^2\right) \tag{5.4}$$
$$\le \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 - \sum_{i=1}^n p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{i=1}^n p_i a_i^{2\nu} b_i^{2(1-\nu)}$$
$$\le \frac{1}{m^2}\nu(1-\nu)\left(\sum_{i=1}^n p_i b_i^2 \sum_{i=1}^n p_i \frac{a_i^4}{b_i^2} - \left(\sum_{i=1}^n p_i a_i^2\right)^2\right)$$

for any $\nu \in [0,1]$ and

$$\begin{aligned}(5.5)\quad \frac{1}{4M^2}\left(\sum_{i=1}^n p_i b_i^2 \sum_{i=1}^n p_i \frac{a_i^4}{b_i^2} - \left(\sum_{i=1}^n p_i a_i^2\right)^2\right) &\\ \le \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 - \left(\sum_{i=1}^n p_i a_i b_i\right)^2 &\\ \le \frac{1}{4m^2}\left(\sum_{i=1}^n p_i b_i^2 \sum_{i=1}^n p_i \frac{a_i^4}{b_i^2} - \left(\sum_{i=1}^n p_i a_i^2\right)^2\right).\end{aligned}$$

If $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ satisfy (5.1), then by (4.6) and (4.7) for the counting discrete measure, we have

$$\begin{aligned}(5.6)\quad 0 &\le \frac{1}{2}\nu(1-\nu)\frac{m^2}{M^2}\left(\sum_{i=1}^n p_i \frac{b_i^4}{a_i^2}\sum_{i=1}^n p_i \frac{a_i^4}{b_i^2} - \sum_{i=1}^n p_i b_i^2 \sum_{i=1}^n p_i a_i^2\right)\\ &\le \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 - \sum_{i=1}^n p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{i=1}^n p_i a_i^{2\nu} b_i^{2(1-\nu)}\\ &\le \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\left(\sum_{i=1}^n p_i \frac{b_i^4}{a_i^2}\sum_{i=1}^n p_i \frac{a_i^4}{b_i^2} - \sum_{i=1}^n p_i b_i^2 \sum_{i=1}^n p_i a_i^2\right)\\ &\le \frac{1}{2}\nu(1-\nu)\left(\frac{M^2}{m^2}-1\right)^2 \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2\end{aligned}$$

for any $\nu \in [0,1]$ and

$$\begin{aligned}(5.7)\quad 0 &\le \frac{1}{8}\frac{m^2}{M^2}\left(\sum_{i=1}^n p_i \frac{b_i^4}{a_i^2}\sum_{i=1}^n p_i \frac{a_i^4}{b_i^2} - \sum_{i=1}^n p_i b_i^2 \sum_{i=1}^n p_i a_i^2\right)\\ &\le \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 - \left(\sum_{i=1}^n p_i a_i b_i\right)^2\\ &\le \frac{1}{8}\frac{M^2}{m^2}\left(\sum_{i=1}^n p_i \frac{b_i^4}{a_i^2}\sum_{i=1}^n p_i \frac{a_i^4}{b_i^2} - \sum_{i=1}^n p_i b_i^2 \sum_{i=1}^n p_i a_i^2\right)\\ &\le \frac{1}{8}\left(\frac{M^2}{m^2}-1\right)^2 \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2.\end{aligned}$$

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