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LOGARITHMIC INEQUALITIES FOR TWO POSITIVE NUMBERS VIA TAYLOR'S EXPANSION WITH INTEGRAL REMAINDER

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Abstract. In this paper we obtain several new logarithmic inequalities for two numbers a, b mainly in the case when $b \geq a > 0$ by the use of Taylor's expansion with integral remainder. An analysis of which bound is better is also performed.

1. Introduction

There are a number of inequalities for logarithm that are well know and widely used in literature, such as:

$$\frac{x-1}{x} \leq \ln x \leq x-1 \text{ for } x>0, \tag{1.1}$$

$$\frac{2x}{2+x} \leq \ln(1+x) \leq \frac{x}{\sqrt{x+1}} \text{ for } x \geq 0, \tag{1.2}$$

$$x \leq -\ln(1-x) \leq \frac{x}{1-x}, \text{ for } x<1,$$

$$\ln x \leq n\left(x^{1/n}-1\right) \text{ for } n>0 \text{ and } x>0,$$

$$\ln(1-|x|) \leq \ln(x+1) \leq -\ln(1-|x|) \text{ for } |x|<1,$$

and

$$-\frac{3}{2}x \leq \ln(1-x) \leq \frac{3}{2}x \text{ for } 0<x \leq 0.5838,$$

see for instance

`http://functions.wolfram.com/ElementaryFunctions/Log/29/`

and [4].

A simple proof of the first inequality in (1.2) may be found, for instance, in [5], see also [6] where the following rational bounds are provided as well:

$$\frac{x\left(1+\frac{5}{6}x\right)}{(1+x)\left(1+\frac{1}{3}x\right)} \leq \ln(1+x) \leq \frac{x\left(1+\frac{1}{6}x\right)}{1+\frac{2}{3}x} \text{ for } x \geq 0.$$

In the recent paper [1] we established the following inequalities for logarithm:

$$\frac{(b-a)^{2n}}{2n\max^{2n}\{a,b\}} \leq \sum_{k=1}^{2n-1}(-1)^{k-1}\frac{(b-a)^k}{ka^k} - \ln b + \ln a \leq \frac{(b-a)^{2n}}{2n\min^{2n}\{a,b\}} \tag{1.3}$$

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and

$$\frac{(b-a)^{2n}}{2n\max^{2n}\{a,b\}} \le \ln b - \ln a - \sum_{k=1}^{2n-1}\frac{(b-a)^k}{kb^k} \le \frac{(b-a)^{2n}}{2n\min^{2n}\{a,b\}} \tag{1.4}$$

for any a, $b>0$ and for $n\ge 1$. In particular, for $n=1$ we get

$$\frac{(b-a)^2}{2\max^2\{a,b\}} \le \frac{b-a}{a} - \ln b + \ln a \le \frac{(b-a)^2}{2\min^2\{a,b\}} \tag{1.5}$$

and

$$\frac{(b-a)^2}{2\max^2\{a,b\}} \le \ln b - \ln a - \frac{b-a}{b} \le \frac{(b-a)^2}{2\min^2\{a,b\}}. \tag{1.6}$$

We have the following upper bounds [1]:

$$(0\le)\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}(b-a)^k}{ka^k} - \ln b + \ln a \le \frac{(b-a)^{2n-1}\left(b^{2n-1}-a^{2n-1}\right)}{(2n-1)\,b^{2n-1}a^{2n-1}} \tag{1.7}$$

and

$$(0\le)\ln b - \ln a - \sum_{k=1}^{2n-1}\frac{(b-a)^k}{kb^k} \le \frac{(b-a)^{2n-1}\left(b^{2n-1}-a^{2n-1}\right)}{(2n-1)\,b^{2n-1}a^{2n-1}}$$

for any a, $b>0$ and for $n\ge 1$. For any a, $b>0$ we have the simpler inequalities

$$(0\le)\frac{b-a}{a} - \ln b + \ln a \le \frac{(b-a)^2}{ab} \tag{1.8}$$

and

$$(0\le)\ln b - \ln a - \frac{b-a}{b} \le \frac{(b-a)^2}{ab}. \tag{1.9}$$

We also have the Hölder's type upper bounds [2]:

$$\begin{aligned}(0\le)&\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}(b-a)^k}{ka^k} - \ln b + \ln a \\ &\le \frac{|b-a|^{2n-1+1/p}\left|b^{2nq-1}-a^{2nq-1}\right|^{1/q}}{[(2n-1)\,p+1]^{1/p}(2nq-1)^{1/q}(ba)^{2n-1/q}}\end{aligned} \tag{1.10}$$

and

$$\begin{aligned}(0\le)&\ln b - \ln a - \sum_{k=1}^{2n-1}\frac{(b-a)^k}{kb^k} \\ &\le \frac{|b-a|^{2n-1+1/p}\left|b^{2nq-1}-a^{2nq-1}\right|^{1/q}}{[(2n-1)\,p+1]^{1/p}(2nq-1)^{1/q}(ba)^{2n-1/q}},\end{aligned} \tag{1.11}$$

where p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and a, $b>0$ and $n\ge 1$. In particular, we get for $n=1$

$$(0\le)\frac{b-a}{a} - \ln b + \ln a \le \frac{|b-a|^{1+1/p}\left|b^{2q-1}-a^{2q-1}\right|^{1/q}}{[(p+1)]^{1/p}(2q-1)^{1/q}(ba)^{2-1/q}} \tag{1.12}$$

and

$$(0 \leq) \ln b - \ln a - \frac{b-a}{b} \leq \frac{|b-a|^{1+1/p} \left|b^{2q-1} - a^{2q-1}\right|^{1/q}}{[(p+1)]^{1/p} (2q-1)^{1/q} (ba)^{2-1/q}}. \tag{1.13}$$

In [2] we obtained the following complementary results:

$$\frac{1}{2b \max\{a,b\}} (b-a)^2 \leq \ln b - \ln a - \frac{b-a}{b} \leq \frac{1}{2b \min\{a,b\}} (b-a)^2 \tag{1.14}$$

and

$$\frac{1}{2a \max\{a,b\}} (b-a)^2 \leq \frac{b-a}{a} - \ln b + \ln a \leq \frac{1}{2a \min\{a,b\}} (b-a)^2 \tag{1.15}$$

for any a, $b > 0$. If $n \geq 1$, then for any a, $b > 0$ we have that [2]

$$\begin{aligned} &\frac{(b-a)^{2n+2}}{(2n+1)(2n+2)\, b \max^{2n+1}\{a,b\}} \\ &\leq \ln b - \ln a - \frac{b-a}{b} + \frac{1}{b} \sum_{k=2}^{2n+1} \frac{(-1)^{k-1}}{k(k-1)} \frac{(b-a)^k}{a^{k-1}} \\ &\leq \frac{(b-a)^{2n+2}}{(2n+1)(2n+2)\, b \min^{2n+1}\{a,b\}} \end{aligned} \tag{1.16}$$

and

$$\begin{aligned} &\frac{(b-a)^{2n+2}}{(2n+1)(2n+2)\, a \max^{2n+1}\{a,b\}} \\ &\leq \frac{b-a}{a} - \frac{1}{a} \sum_{k=2}^{2n+1} \frac{1}{k(k-1)} \frac{(b-a)^k}{b^{k-1}} - \ln b + \ln a \\ &\leq \frac{(b-a)^{2n+2}}{(2n+1)(2n+2)\, a \min^{2n+1}\{a,b\}}. \end{aligned} \tag{1.17}$$

For other similar bounds see [2].

Motivated by the above results, we establish in this paper several bounds for the quantities

$$\ln b - \ln a - \sum_{k=1}^{2n} \frac{(-1)^{k-1} (b-a)^k}{k a^k},$$

$$\ln b - \ln a - \sum_{k=1}^{2n} \frac{(b-a)^k}{k b^k},$$

$$\frac{b-a}{b} + \frac{1}{b} \sum_{k=2}^{2n} \frac{(-1)^k}{k(k-1)} \frac{(b-a)^k}{a^{k-1}} - \ln b + \ln a$$

and

$$\frac{b-a}{a} - \frac{1}{a} \sum_{k=2}^{2n} \frac{1}{k(k-1)} \frac{(b-a)^k}{b^{k-1}} - \ln b + \ln a$$

where b, $a > 0$ and $n \geq 1$. The simpler cases when $n = 1$ are outlined and in this case some bounds are numerically compared to conclude that neither is always best.

2. Some Inequalities for Logarithm

The following result holds, see for instance [3] where various applications in Information Theory were provided:

Lemma 1. *For any a, $b>0$ we have for $m\geq 1$ that*

$$\ln b-\ln a+\sum_{k=1}^{m}\frac{(-1)^{k}(b-a)^{k}}{ka^{k}}=(-1)^{m}\int_{a}^{b}\frac{(b-t)^{m}}{t^{m+1}}dt. \tag{2.1}$$

For recent inequalities derived from this identity and a short proof, see [1].

We have the following result:

Theorem 1. *For any a, $b>0$ we have for $n\geq 1$ that*

$$\begin{aligned}\frac{1}{(2n+1)\,b^{2n+1}}(b-a)^{2n+1}&\leq \ln b-\ln a-\sum_{k=1}^{2n}\frac{(-1)^{k-1}(b-a)^{k}}{ka^{k}}\\&\leq\frac{1}{(2n+1)\,a^{2n+1}}(b-a)^{2n+1}\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\frac{1}{(2n+1)\,b^{2n+1}}(b-a)^{2n+1}&\leq \ln b-\ln a-\sum_{k=1}^{2n}\frac{(b-a)^{k}}{kb^{k}}\\&\leq\frac{1}{(2n+1)\,a^{2n+1}}(b-a)^{2n+1}.\end{aligned} \tag{2.3}$$

Proof. If we take $m=2n$ with $n\geq 1$ in (2.1), then we get for any a, $b>0$ that

$$\ln b-\ln a+\sum_{k=1}^{2n}\frac{(-1)^{k}(b-a)^{k}}{ka^{k}}=\int_{a}^{b}\frac{(b-t)^{2n}}{t^{2n+1}}dt. \tag{2.4}$$

If $b>a>0$, then we have

$$\frac{1}{b^{2n+1}}\int_{a}^{b}(b-t)^{2n}dt\leq\int_{a}^{b}\frac{(b-t)^{2n}}{t^{2n+1}}dt\leq\frac{1}{a^{2n+1}}\int_{a}^{b}(b-t)^{2n}dt$$

and since

$$\int_{a}^{b}(b-t)^{2n}dt=\frac{1}{2n+1}(b-a)^{2n+1}$$

we get, by (2.4) the desired inequality (2.2).

If $a>b>0$, then

$$\int_{a}^{b}\frac{(b-t)^{2n}}{t^{2n+1}}dt=-\int_{b}^{a}\frac{(b-t)^{2n}}{t^{2n+1}}dt. \tag{2.5}$$

We have

$$\frac{1}{a^{2n+1}}\int_{b}^{a}(b-t)^{2n}dt\leq\int_{b}^{a}\frac{(b-t)^{2n}}{t^{2n+1}}dt\leq\frac{1}{b^{2n+1}}\int_{b}^{a}(b-t)^{2n}dt. \tag{2.6}$$

Observe that

$$\int_{b}^{a}(b-t)^{2n}dt=\int_{b}^{a}(t-b)^{2n}dt=\frac{(a-b)^{2n+1}}{2n+1}=-\frac{(b-a)^{2n+1}}{2n+1}$$

and by (2.6) we then get

$$-\frac{(b-a)^{2n+1}}{(2n+1)\,a^{2n+1}}dt \le \int_b^a \frac{(b-t)^{2n}}{t^{2n+1}}dt \le -\frac{(b-a)^{2n+1}}{b^{2n+1}\,(2n+1)}$$

that is equivalent to

$$\frac{(b-a)^{2n+1}}{b^{2n+1}\,(2n+1)} \le -\int_b^a \frac{(b-t)^{2n}}{t^{2n+1}}dt \le \frac{(b-a)^{2n+1}}{(2n+1)\,a^{2n+1}}dt.$$

By using (2.4) and (2.5) we get (2.2) again.

Now, if we replace a with b in (2.2) we get

$$\begin{aligned}\frac{1}{(2n+1)\,a^{2n+1}}(a-b)^{2n+1} &\le \ln a - \ln b + \sum_{k=1}^{2n}\frac{(-1)^k (a-b)^k}{kb^k}\\ &\le \frac{1}{(2n+1)\,b^{2n+1}}(a-b)^{2n+1}\end{aligned}$$

namely

$$\begin{aligned}-\frac{1}{(2n+1)\,a^{2n+1}}(b-a)^{2n+1} &\le \ln a - \ln b + \sum_{k=1}^{2n}\frac{(b-a)^k}{kb^k}\\ &\le -\frac{1}{(2n+1)\,b^{2n+1}}(b-a)^{2n+1}.\end{aligned}$$

If we multiply this inequality by -1 we get the desired inequality (2.3). □

For any a, $b>0$ we have by (2.2) and (2.3) for $n=1$ that

$$\frac{1}{3b^3}(b-a)^3 \le \ln b - \ln a - \frac{b-a}{a} + \frac{(b-a)^2}{2a^2} \le \frac{1}{3a^3}(b-a)^3 \tag{2.7}$$

and

$$\frac{1}{3b^3}(b-a)^3 \le \ln b - \ln a - \frac{b-a}{b} - \frac{(b-a)^2}{2b^2} \le \frac{1}{3a^3}(b-a)^3. \tag{2.8}$$

Corollary 1. *For any a, $b>0$ with $b\ge a>0$ we have for $n\ge 1$ that*

$$\sum_{k=1}^{2n}\frac{(-1)^{k-1}(b-a)^k}{ka^k} \le \ln b - \ln a \tag{2.9}$$

and

$$\sum_{k=1}^{2n}\frac{(b-a)^k}{kb^k} \le \ln b - \ln a. \tag{2.10}$$

For any a, $b>0$ with $b\ge a>0$ we have

$$\frac{b-a}{a} - \frac{(b-a)^2}{2a^2} \le \ln b - \ln a \left(\le \frac{b-a}{a}\right) \tag{2.11}$$

and

$$\left(\frac{b-a}{b} \le\right) \frac{b-a}{b} + \frac{(b-a)^2}{2b^2} \le \ln b - \ln a. \tag{2.12}$$

Remark 1. *If we take $b = x \in (0, \infty)$ and $a = 1$ in (2.2) and (2.3), then we get*

$$\frac{1}{(2n+1)} \frac{(x-1)^{2n+1}}{x^{2n+1}} \leq \ln x - \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} (x-1)^k \leq \frac{1}{(2n+1)} (x-1)^{2n+1} \tag{2.13}$$

and

$$\frac{1}{(2n+1) x^{2n+1}} (x-1)^{2n+1} \leq \ln x - \sum_{k=1}^{2n} \frac{(x-1)^k}{k x^k} \leq \frac{1}{(2n+1)} (x-1)^{2n+1} \tag{2.14}$$

for any $x \in (0, \infty)$ and $n \geq 1$.

In particular, for $n = 1$ we have

$$\frac{1}{3x^3} (x-1)^3 \leq \ln x - x + 1 + \frac{1}{2} (x-1)^2 \leq \frac{1}{3} (x-1)^3 \tag{2.15}$$

and

$$\frac{1}{3x^3} (x-1)^3 \leq \ln x - \frac{x-1}{x} - \frac{(x-1)^2}{2x^2} \leq \frac{1}{3} (x-1)^3 . \tag{2.16}$$

Now, if $x \geq 1$, then we get from (2.13) that

$$\sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} (x-1)^k \leq \ln x \tag{2.17}$$

and from (2.14) that

$$\sum_{k=1}^{2n} \frac{(x-1)^k}{k x^k} \leq \ln x. \tag{2.18}$$

We have:

Theorem 2. *For any a, $b > 0$ with $b \geq a > 0$ we have for $n \geq 1$ that*

$$(0 \leq) \ln b - \ln a - \sum_{k=1}^{2n} \frac{(-1)^{k-1} (b-a)^k}{k a^k} \leq \frac{1}{2n} \frac{(b-a)^{2n} \left(b^{2n} - a^{2n}\right)}{a^{2n} b^{2n}}. \tag{2.19}$$

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and a, $b > 0$ with $b \geq a > 0$ we have for $n \geq 1$ that

$$\begin{aligned} &(0 \leq) \ln b - \ln a - \sum_{k=1}^{2n} \frac{(-1)^{k-1} (b-a)^k}{k a^k} \\ &\leq \frac{1}{(2np+1)^{1/p} \left[(2n+1) q - 1\right]^{1/q}} \\ &\times \frac{(b-a)^{2n+1/p} \left(b^{(2n+1)q-1} - a^{(2n+1)q-1}\right)^{1/q}}{b^{2n+1/p} a^{2n+1/p}}. \end{aligned} \tag{2.20}$$

In particular, for $p = q = 2$ we have

$$\begin{aligned} &(0 \leq) \ln b - \ln a - \sum_{k=1}^{2n} \frac{(-1)^{k-1} (b-a)^k}{k a^k} \\ &\leq \frac{1}{(4n+1)} \frac{(b-a)^{2n+1/2} \left(b^{4n+1} - a^{4n+1}\right)^{1/2}}{b^{2n+1/2} a^{2n+1/2}}. \end{aligned} \tag{2.21}$$

Proof. By (2.4) we have for $b \geq a > 0$ that

$$\ln b - \ln a - \sum_{k=1}^{2n} \frac{(-1)^{k-1}(b-a)^k}{ka^k} = \int_a^b \frac{(b-t)^{2n}}{t^{2n+1}} dt \tag{2.22}$$
$$\leq (b-a)^{2n} \int_a^b \frac{1}{t^{2n+1}} dt$$
$$= (b-a)^{2n} \frac{b^{2n} - a^{2n}}{2na^{2n}b^{2n}}$$

that proves the inequality (2.19).

Using Hölder's integral inequality we have for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\int_a^b \frac{(b-t)^{2n}}{t^{2n+1}} dt \leq \left(\int_a^b (b-t)^{2np} dt \right)^{1/p} \left(\int_a^b t^{-(2n+1)q} dt \right)^{1/q}.$$

Observe that

$$\left(\int_a^b (b-t)^{2np} dt \right)^{1/p} = \left(\frac{(b-a)^{2np+1}}{2np+1} \right)^{1/p} = \frac{(b-a)^{2n+1/p}}{(2np+1)^{1/p}}$$

and

$$\left(\int_a^b t^{-(2n+1)q} dt \right)^{1/q} = \left(\frac{b^{-(2n+1)q+1} - a^{-(2n+1)q+1}}{-(2n+1)q+1} \right)^{1/q}$$
$$= \left(\frac{b^{(2n+1)q-1} - a^{(2n+1)q-1}}{[(2n+1)q-1] b^{(2n+1)q-1} a^{(2n+1)q-1}} \right)^{1/q}$$
$$= \frac{\left(b^{(2n+1)q-1} - a^{(2n+1)q-1} \right)^{1/q}}{[(2n+1)q-1]^{1/q} b^{2n+1-1/q} a^{2n+1-1/q}}$$
$$= \frac{\left(b^{(2n+1)q-1} - a^{(2n+1)q-1} \right)^{1/q}}{[(2n+1)q-1]^{1/q} b^{2n+1/p} a^{2n+1/p}}.$$

By utilising the equality in (2.22) we deduce the desired result (2.20). □

For any a, $b > 0$ with $b \geq a > 0$ we have by (2.19), (2.20) and (2.21) for $n = 1$ that

$$(0 \leq) \ln b - \ln a - \frac{b-a}{a} + \frac{(b-a)^2}{2a^2} \leq \frac{1}{2} \frac{(b-a)^2 (b^2 - a^2)}{a^2 b^2}, \tag{2.23}$$

$$(0 \leq) \ln b - \ln a - \frac{b-a}{a} + \frac{(b-a)^2}{2a^2} \tag{2.24}$$
$$\leq \frac{1}{(2p+1)^{1/p} (3q-1)^{1/q}} \frac{(b-a)^{2+1/p} \left(b^{3q-1} - a^{3q-1} \right)^{1/q}}{b^{2+1/p} a^{2+1/p}}$$

with p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and

$$(0 \leq) \ln b - \ln a - \frac{b-a}{a} + \frac{(b-a)^2}{2a^2} \leq \frac{1}{5} \frac{(b-a)^{5/2} \left(b^5 - a^5 \right)^{1/2}}{b^{5/2} a^{5/2}}. \tag{2.25}$$

If $x \geq 1$, then we have for $n \geq 1$ that

$$(0 \leq) \ln x - \sum_{k=1}^{2n} \frac{(-1)^{k-1} (x-1)^k}{k} \leq \frac{1}{2n} \frac{(x-1)^{2n} \left(x^{2n}-1\right)}{x^{2n}}. \tag{2.26}$$

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $x \geq 1$, then we have for $n \geq 1$ that

$$\begin{aligned}(0 \leq) \ln x - \sum_{k=1}^{2n} \frac{(-1)^{k-1} (x-1)^k}{k} &\leq \frac{1}{(2np+1)^{1/p} \left[(2n+1) q - 1\right]^{1/q}} \\ &\quad \times \frac{(x-1)^{2n+1/p} \left(x^{(2n+1)q-1} - 1\right)^{1/q}}{x^{2n+1/p}}.\end{aligned} \tag{2.27}$$

In particular, for $p = q = 2$ we have

$$(0 \leq) \ln x - \sum_{k=1}^{2n} \frac{(-1)^{k-1} (x-1)^k}{k} \leq \frac{1}{(4n+1)} \frac{(x-1)^{2n+1/2} \left(x^{4n+1}-1\right)^{1/2}}{x^{2n+1/2}} \tag{2.28}$$

for any $x \geq 1$.

3. Further Inequalities for Logarithm

We have the following representation result [2]:

Lemma 2. *For any $m \geq 2$ and any a, $b > 0$ we have*

$$\ln b - \ln a - \frac{b-a}{b} + \frac{1}{b} \sum_{k=2}^{m} \frac{(-1)^{k-1}}{k(k-1)} \frac{(b-a)^k}{a^{k-1}} = \frac{(-1)^{m-1}}{mb} \int_a^b \frac{(b-t)^m}{t^m} dt. \tag{3.1}$$

We have:

Theorem 3. *For any a, $b > 0$ we have for $n \geq 1$ that*

$$\begin{aligned}\frac{1}{2n(2n+1)} \frac{(b-a)^{2n+1}}{b^{2n+1}} &\leq \frac{b-a}{b} + \frac{1}{b} \sum_{k=2}^{2n} \frac{(-1)^k}{k(k-1)} \frac{(b-a)^k}{a^{k-1}} - \ln b + \ln a \\ &\leq \frac{1}{2n(2n+1)} \frac{(b-a)^{2n+1}}{a^{2n} b}\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\frac{1}{2n(2n+1)} \frac{(b-a)^{2n+1}}{b^{2n} a} &\leq \frac{b-a}{a} - \frac{1}{a} \sum_{k=2}^{2n} \frac{1}{k(k-1)} \frac{(b-a)^k}{b^{k-1}} - \ln b + \ln a \\ &\leq \frac{1}{2n(2n+1)} \frac{(b-a)^{2n+1}}{a^{2n+1}}.\end{aligned} \tag{3.3}$$

Proof. If we take $m = 2n$ with $n \geq 1$ in (3.1), then we get

$$\ln b - \ln a - \frac{b-a}{b} + \frac{1}{b} \sum_{k=2}^{2n} \frac{(-1)^{k-1}}{k(k-1)} \frac{(b-a)^k}{a^{k-1}} = -\frac{1}{2nb} \int_a^b \frac{(b-t)^{2n}}{t^{2n}} dt$$

that is equivalent to

$$\frac{1}{b} \sum_{k=2}^{2n} \frac{(-1)^k}{k(k-1)} \frac{(b-a)^k}{a^{k-1}} - \ln b + \ln a + \frac{b-a}{b} = \frac{1}{2nb} \int_a^b \frac{(b-t)^{2n}}{t^{2n}} dt \tag{3.4}$$

for any a, $b > 0$.

If $b > a > 0$, then we have

$$\frac{1}{b^{2n}}\int_a^b (b-t)^{2n}\,dt \le \int_a^b \frac{(b-t)^{2n}}{t^{2n}}dt \le \frac{1}{a^{2n}}\int_a^b (b-t)^{2n}\,dt$$

namely

$$\frac{1}{(2n+1)\,b^{2n}}(b-a)^{2n+1} \le \int_a^b \frac{(b-t)^{2n}}{t^{2n}}dt \le \frac{1}{(2n+1)\,a^{2n}}(b-a)^{2n+1}. \tag{3.5}$$

If $a > b > 0$, then

$$\int_a^b \frac{(b-t)^{2n}}{t^{2n}}dt = -\int_b^a \frac{(b-t)^{2n}}{t^{2n}}dt.$$

Observe that

$$\int_b^a (b-t)^{2n}\,dt = \int_b^a (t-b)^{2n}\,dt = \frac{(a-b)^{2n+1}}{2n+1} = -\frac{(b-a)^{2n+1}}{2n+1}.$$

We have

$$\frac{1}{a^{2n}}\int_b^a (t-b)^{2n}\,dt \le \int_b^a \frac{(b-t)^{2n}}{t^{2n}}dt \le \frac{1}{b^{2n}}\int_b^a (t-b)^{2n}\,dt$$

namely

$$-\frac{1}{a^{2n}}\frac{(b-a)^{2n+1}}{2n+1} \le \int_b^a \frac{(b-t)^{2n}}{t^{2n}}dt \le -\frac{1}{b^{2n}}\frac{(b-a)^{2n+1}}{2n+1},$$

which, by multiplying with -1 gives

$$\frac{1}{b^{2n}}\frac{(b-a)^{2n+1}}{2n+1} \le \int_a^b \frac{(b-t)^{2n}}{t^{2n}}dt \le \frac{1}{a^{2n}}\frac{(b-a)^{2n+1}}{2n+1} \tag{3.6}$$

for $a \ge b > 0$.

Using the representation (3.4) and the inequalities (3.5) and (3.6) we get (3.2).

If we replace a with b in (3.2) then we get

$$\begin{aligned}\frac{1}{2n\,(2n+1)}\frac{(a-b)^{2n+1}}{a^{2n+1}} &\le \frac{1}{a}\sum_{k=2}^{2n}\frac{(-1)^k}{k\,(k-1)}\frac{(a-b)^k}{b^{k-1}} - \ln a + \ln b + \frac{a-b}{a}\\ &\le \frac{1}{2n\,(2n+1)}\frac{(a-b)^{2n+1}}{b^{2n}a}\end{aligned}$$

namely

$$\begin{aligned}-\frac{1}{2n\,(2n+1)}\frac{(b-a)^{2n+1}}{a^{2n+1}} &\le \frac{1}{a}\sum_{k=2}^{2n}\frac{1}{k\,(k-1)}\frac{(b-a)^k}{b^{k-1}} - \ln a + \ln b + \frac{a-b}{a}\\ &\le -\frac{1}{2n\,(2n+1)}\frac{(b-a)^{2n+1}}{b^{2n}a}.\end{aligned} \tag{3.7}$$

If we multiply (3.7) by -1, then we get (3.3). □

For any a, $b > 0$ we have by (3.2) and (3.3) for $n = 1$ that

$$\frac{1}{6}\frac{(b-a)^3}{b^3} \le \frac{b-a}{b} + \frac{1}{2}\frac{(b-a)^2}{ab} - \ln b + \ln a \le \frac{1}{6}\frac{(b-a)^3}{a^2b} \tag{3.8}$$

and

$$\frac{1}{6}\frac{(b-a)^3}{b^2a} \le \frac{b-a}{a} - \frac{1}{2}\frac{(b-a)^2}{ab} - \ln b + \ln a \le \frac{1}{6}\frac{(b-a)^3}{a^3}. \tag{3.9}$$

We have:

Corollary 2. *For any a, $b>0$ with $b\geq a>0$ we have for $n\geq 1$ that*

$$\left(\frac{b-a}{b}\leq\right)\ \ln b-\ln a\leq\frac{b-a}{b}+\frac{1}{b}\sum_{k=2}^{2n}\frac{(-1)^{k}}{k(k-1)}\frac{(b-a)^{k}}{a^{k-1}} \tag{3.10}$$

and

$$\ln b-\ln a\leq\frac{b-a}{a}-\frac{1}{a}\sum_{k=2}^{2n}\frac{1}{k(k-1)}\frac{(b-a)^{k}}{b^{k-1}}\ \left(\leq\frac{b-a}{a}\right). \tag{3.11}$$

For any a, $b>0$ with $b\geq a>0$ we have

$$\left(\frac{b-a}{b}\leq\right)\ \ln b-\ln a\leq\frac{b-a}{b}+\frac{1}{2}\frac{(b-a)^{2}}{ab} \tag{3.12}$$

and

$$\ln b-\ln a\leq\frac{b-a}{a}-\frac{1}{2}\frac{(b-a)^{2}}{ab}\ \left(\leq\frac{b-a}{a}\right). \tag{3.13}$$

Remark 2. *If we take $b=x\in(0,\infty)$ and $a=1$ in (3.2) and (3.3), then we get*

$$\begin{aligned}\frac{1}{2n(2n+1)}\frac{(x-1)^{2n+1}}{x^{2n+1}}&\leq\frac{x-1}{x}+\frac{1}{x}\sum_{k=2}^{2n}\frac{(-1)^{k}}{k(k-1)}(x-1)^{k}-\ln x\\&\leq\frac{1}{2n(2n+1)}\frac{(x-1)^{2n+1}}{x}\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\frac{1}{2n(2n+1)}\frac{(x-1)^{2n+1}}{x^{2n}}&\leq x-1-\sum_{k=2}^{2n}\frac{1}{k(k-1)}\frac{(x-1)^{k}}{x^{k-1}}-\ln x\\&\leq\frac{1}{2n(2n+1)}(x-1)^{2n+1}.\end{aligned} \tag{3.15}$$

In particular, for $n=1$ we have

$$\frac{1}{6}\frac{(x-1)^{3}}{x^{3}}\leq\frac{x-1}{x}+\frac{1}{2}\frac{(x-1)^{2}}{x}-\ln x\leq\frac{1}{6}\frac{(x-1)^{3}}{x} \tag{3.16}$$

and

$$\frac{1}{6}\frac{(x-1)^{3}}{x^{2}}\leq x-1-\frac{1}{2}\frac{(x-1)^{2}}{x}-\ln x\leq\frac{1}{6}(x-1)^{3} \tag{3.17}$$

for any $x>0$.

For $x\geq 1$ we also have

$$\left(\frac{x-1}{x}\leq\right)\ \ln x\leq\frac{x-1}{x}+\frac{1}{x}\sum_{k=2}^{2n}\frac{(-1)^{k}}{k(k-1)}(x-1)^{k} \tag{3.18}$$

and

$$\ln x\leq x-1-\sum_{k=2}^{2n}\frac{1}{k(k-1)}\frac{(x-1)^{k}}{x^{k-1}}\ \ (\leq x-1) \tag{3.19}$$

for any $n\geq 1$.

In particular, we have

$$\left(\frac{x-1}{x}\leq\right)\ \ln x\leq\frac{x-1}{x}+\frac{1}{2}\frac{(x-1)^{2}}{x} \tag{3.20}$$

and

$$\ln x\leq x-1-\frac{1}{2}\frac{(x-1)^{2}}{x}\ \ (\leq x-1) \tag{3.21}$$

for any $x\geq 1$.

We have:

Theorem 4. *For any* $a,\ b>0$ *with* $b\geq a>0$ *we have for* $n\geq 1$ *that*

$$\begin{aligned}(0\leq)\ &\frac{b-a}{b}+\frac{1}{b}\sum_{k=2}^{2n}\frac{(-1)^{k}}{k(k-1)}\frac{(b-a)^{k}}{a^{k-1}}-\ln b+\ln a\\ &\leq\frac{1}{2n(2n-1)}\frac{(b-a)^{2n}\left(b^{2n-1}-a^{2n-1}\right)}{b^{2n}a^{2n-1}}.\end{aligned} \tag{3.22}$$

If $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $a,\ b>0$ *with* $b\geq a>0$ *we have for* $n\geq 1$ *that*

$$\begin{aligned}(0\leq)\ &\frac{b-a}{b}+\frac{1}{b}\sum_{k=2}^{2n}\frac{(-1)^{k}}{k(k-1)}\frac{(b-a)^{k}}{a^{k-1}}-\ln b+\ln a\\ &\leq\frac{1}{2n(2np+1)^{1/p}(2nq-1)^{1/q}}\frac{(b-a)^{2n+1/p}\left(b^{2nq-1}-a^{2nq-1}\right)^{1/q}}{b^{2n+1-1/q}a^{2n-1/q}}.\end{aligned} \tag{3.23}$$

In particular, for $p=q=2$ *we have*

$$\begin{aligned}(0\leq)\ &\frac{b-a}{b}+\frac{1}{b}\sum_{k=2}^{2n}\frac{(-1)^{k}}{k(k-1)}\frac{(b-a)^{k}}{a^{k-1}}-\ln b+\ln a\\ &\leq\frac{1}{2n(4n+1)^{1/2}(4n-1)^{1/2}}\frac{(b-a)^{2n+1/2}\left(b^{4n-1}-a^{4n-1}\right)^{1/2}}{b^{2n+1/2}a^{2n-1/2}}.\end{aligned} \tag{3.24}$$

Proof. For any $a,\ b>0$ with $b\geq a>0$ we have for $n\geq 1$ that

$$\begin{aligned}&\frac{1}{b}\sum_{k=2}^{2n}\frac{(-1)^{k}}{k(k-1)}\frac{(b-a)^{k}}{a^{k-1}}-\ln b+\ln a+\frac{b-a}{b}\\ &=\frac{1}{2nb}\int_{a}^{b}\frac{(b-t)^{2n}}{t^{2n}}dt\\ &\leq\frac{1}{2nb}(b-a)^{2n}\int_{a}^{b}t^{-2n}dt\\ &=\frac{1}{2nb}(b-a)^{2n}\left(\frac{b^{-2n+1}}{-2n+1}-\frac{a^{-2n+1}}{-2n+1}\right)\\ &=\frac{1}{2n(2n-1)}\frac{(b-a)^{2n}\left(b^{2n-1}-a^{2n-1}\right)}{b^{2n}a^{2n-1}},\end{aligned} \tag{3.25}$$

which proves (3.22).

Using Hölder's integral inequality we have for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\begin{aligned}
\int_a^b \frac{(b-t)^{2n}}{t^{2n}} dt &\le \left(\int_a^b (b-t)^{2np} \, dt\right)^{1/p} \left(\int_a^b t^{-2nq} dt\right)^{1/q} \\
&= \left(\frac{(b-a)^{2np+1}}{2np+1}\right)^{1/p} \left(\frac{b^{-2nq+1}}{-2nq+1} - \frac{b^{-2nq+1}}{-2nq+1}\right)^{1/q} \\
&= \frac{(b-a)^{2n+1/p}}{(2np+1)^{1/p}} \left(\frac{b^{2nq-1} - a^{2nq-1}}{(2nq-1) \, b^{2nq-1} a^{2nq-1}}\right)^{1/q} \\
&= \frac{(b-a)^{2n+1/p}}{(2np+1)^{1/p}} \frac{\left(b^{2nq-1} - a^{2nq-1}\right)^{1/q}}{(2nq-1)^{1/q} \, b^{2n-1/q} a^{2n-1/q}}
\end{aligned}$$

for any a, $b > 0$ with $b \ge a > 0$ and $n \ge 1$.

Using the first identity in (3.25) we get (3.23). □

For any a, $b > 0$ with $b \ge a > 0$ we have by (3.22)-(3.24) for $n = 1$ that

$$(0 \le) \, \frac{b-a}{b} + \frac{1}{2} \frac{(b-a)^2}{ab} - \ln b + \ln a \le \frac{1}{2} \frac{(b-a)^3}{b^2 a}, \tag{3.26}$$

$$\begin{aligned}
&(0 \le) \, \frac{b-a}{b} + \frac{1}{2} \frac{(b-a)^2}{ab} - \ln b + \ln a \\
&\le \frac{1}{2 (2p+1)^{1/p} (2q-1)^{1/q}} \frac{(b-a)^{2+1/p} \left(b^{2q-1} - a^{2q-1}\right)^{1/q}}{b^{3-1/q} a^{2-1/q}}
\end{aligned} \tag{3.27}$$

and

$$(0 \le) \, \frac{b-a}{b} + \frac{1}{2} \frac{(b-a)^2}{ab} - \ln b + \ln a \le \frac{1}{2\sqrt{15}} \frac{(b-a)^3}{b^2 a} \sqrt{\frac{b^2 + ba + a^2}{ba}}. \tag{3.28}$$

Remark 3. *For $x \ge 1$ we also have*

$$\begin{aligned}
&(0 \le) \, \frac{x-1}{x} + \frac{1}{x} \sum_{k=2}^{2n} \frac{(-1)^k}{k (k-1)} (x-1)^k - \ln x \\
&\le \frac{1}{2n (2n-1)} \frac{(x-1)^{2n} \left(x^{2n-1} - 1\right)}{x^{2n}},
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
&(0 \le) \, \frac{x-1}{x} + \frac{1}{x} \sum_{k=2}^{2n} \frac{(-1)^k}{k (k-1)} (x-1)^k - \ln x \\
&\le \frac{1}{2n (2np+1)^{1/p} (2nq-1)^{1/q}} \frac{(x-1)^{2n+1/p} \left(x^{2nq-1} - 1\right)^{1/q}}{x^{2n+1-1/q}}
\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}
&(0 \le) \, \frac{x-1}{x} + \frac{1}{x} \sum_{k=2}^{2n} \frac{(-1)^k}{k (k-1)} (x-1)^k - \ln x \\
&\le \frac{1}{2n (4n+1)^{1/2} (4n-1)^{1/2}} \frac{(x-1)^{2n+1/2} \left(x^{4n-1} - 1\right)^{1/2}}{x^{2n+1/2}}.
\end{aligned} \tag{3.31}$$

In particular, for $n=1$, we have

$$(0\leq)\ \frac{x-1}{x}+\frac{1}{2}\frac{(x-1)^2}{x}-\ln x\leq\frac{1}{2}\frac{(x-1)^3}{x^2}, \tag{3.32}$$

$$\begin{aligned}(0\leq)\ &\frac{x-1}{x}+\frac{1}{2}\frac{(x-1)^2}{x}-\ln x\\ &\leq\frac{1}{2(2p+1)^{1/p}(2q-1)^{1/q}}\frac{(x-1)^{2+1/p}\left(x^{2q-1}-1\right)^{1/q}}{x^{3-1/q}}\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}(0\leq)\ &\frac{x-1}{x}+\frac{1}{2}\frac{(x-1)^2}{x}-\ln x\\ &\leq\frac{1}{2\sqrt{15}}\frac{(x-1)^3}{x^2}\sqrt{\frac{x^2+x+1}{x}}\end{aligned} \tag{3.34}$$

for any $x\geq 1$.

4. Comparison

Using the inequalities (2.7), (2.23) and (2.25) we have for any $b\geq a>0$ the following upper bounds for the positive quantity

$$\ln b-\ln a-\frac{b-a}{a}+\frac{(b-a)^2}{2a^2}\leq B_1(a,b),\ B_2(a,b)\ \text{ and } B_3(a,b) \tag{4.1}$$

where

$$B_1(a,b):=\frac{1}{3a^3}(b-a)^3,\ B_2(a,b)=\frac{1}{2}\frac{(b-a)^3(b+a)}{a^2b^2} \tag{4.2}$$

and

$$B_3(a,b):=\frac{1}{5}\frac{(b-a)^3}{b^2a^2}\sqrt{\frac{b^4+b^3a+b^2a^2+ba^3+a^4}{ba}}. \tag{4.3}$$

Consider the simpler quantities

$$C_1(a,b):=\frac{1}{3a},C_2(a,b)=\frac{1}{2}\frac{b+a}{b^2}$$

and

$$C_3(a,b):=\frac{1}{5}\frac{1}{b^2}\sqrt{\frac{b^4+b^3a+b^2a^2+ba^3+a^4}{ba}}.$$

If we plot the difference $D_1(a,b):=C_1(a,b)-C_2(a,b)$ on the domain $1\leq a\leq b\leq 10$ we have the Figure 1, the plot of the difference $D_2(a,b):=C_2(a,b)-C_3(a,b)$ on the domain $0\leq a\leq b\leq 2$ is depicted in Figure 2 while the plot of the difference $D_3(a,b):=C_1(a,b)-C_3(a,b)$ on the triangle $1\leq a\leq b\leq 10$ is incorporated in Figure 3

The plots in Figure 1-3 show that neither of the upper bounds $B_1(a,b)$, $B_2(a,b)$ and $B_3(a,b)$ for the positive quantity

$$\ln b-\ln a-\frac{b-a}{a}+\frac{(b-a)^2}{2a^2},0<a\leq b$$

is always best.

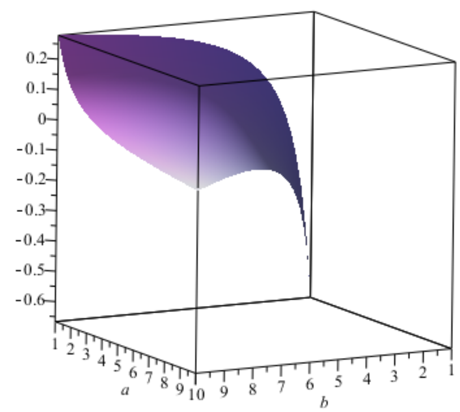

FIGURE 1. Plot of $D_1(a,b)$ for $1 \le a \le b \le 10$

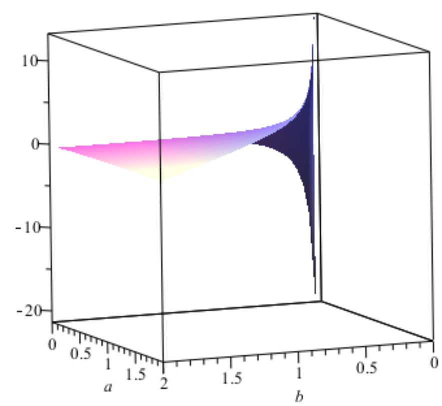

FIGURE 2. Plot of $D_2(a,b)$ for $0 \le a \le b \le 2$

From the inequalities (3.8), (3.26) and (3.28) we have for any $b \ge a > 0$ the following upper bounds for the positive quantity

$$\frac{b-a}{b} + \frac{1}{2}\frac{(b-a)^2}{ab} - \ln b + \ln a \le K_1(a,b)\,,\ K_2(a,b) \text{ and } K_3(a,b) \tag{4.4}$$

where

$$K_1(a,b) := \frac{1}{6}\frac{(b-a)^3}{a^2 b},\ K_2(a,b) := \frac{1}{2}\frac{(b-a)^3}{b^2 a} \tag{4.5}$$

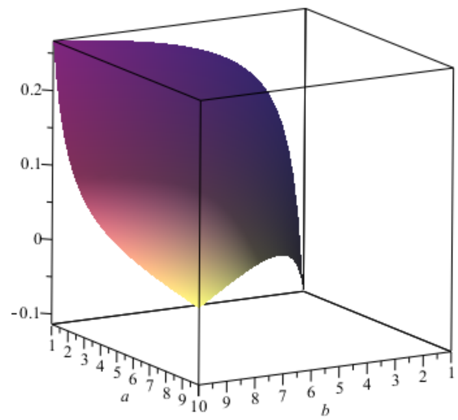

FIGURE 3. Plot of $D_3(a,b)$ for $1 \leq a \leq b \leq 10$

and

$$K_3(a,b) := \frac{1}{2\sqrt{15}} \frac{(b-a)^3}{b^2 a} \sqrt{\frac{b^2+ba+a^2}{ba}}. \tag{4.6}$$

The interested reader may perform a similar analysis for these bounds. However the details are not provided here.

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