Received 19/08/16

INEQUALITIES FOR RELATIVE OPERATOR ENTROPY IN TERMS OF TSALLIS' ENTROPY

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Abstract. In this paper we obtain new inequalities for relative operator entropy $S(A|B)$ in terms of Tsallis' relative entropy $T_{\pm t}(A|B)$, $t>0$ in the case of positive invertible operators A, B. Further bounds for $B\geq A$ are also provided.

1. Introduction

Kamei and Fujii [10], [11] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B):=A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}, \tag{1.1}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [16].

In general, we can define for positive operators A, B

$$S(A|B):=s\text{-}\lim_{\varepsilon\to 0+}S(A+\varepsilon 1_H|B)$$

if it exists, here 1_H is the identity operator.

For the entropy function $\eta(t)=-t\ln t$, the *operator entropy* has the following expression:

$$\eta(A)=-A\ln A=S(A|1_H)\geq 0$$

for positive contraction A. This shows that the relative operator entropy (1.1) is a relative version of the operator entropy.

Following [12, p. 149-p. 155], we recall some important properties of relative operator entropy for A and B positive invertible operators:

(i) We have the equalities

$$S(A|B)=-A^{1/2}\left(\ln A^{1/2}B^{-1}A^{1/2}\right)A^{1/2}=B^{1/2}\eta\left(B^{-1/2}AB^{-1/2}\right)B^{1/2}; \tag{1.2}$$

(ii) We have the inequalities

$$S(A|B)\leq A(\ln\|B\|-\ln A)\ \text{and}\ S(A|B)\leq B-A; \tag{1.3}$$

(iii) For any C, D positive invertible operators we have that

$$S(A+B|C+D)\geq S(A|C)+S(B|D);$$

(iv) If $B\leq C$, then

$$S(A|B)\leq S(A|C);$$

1991 *Mathematics Subject Classification.* 47A63, 47A30,

Key words and phrases. Inequalities for Logarithm, Relative operator entropy, Operator entropy.

RGMIA Res. Rep. Coll. 19 (2016), Art. 133

(v) If $B_n \downarrow B$, then

$$S(A|B_n) \downarrow S(A|B);$$

(vi) For $\alpha > 0$ we have

$$S(\alpha A|\alpha B) = \alpha S(A|B);$$

(vii) For every operator T we have

$$T^* S(A|B) T \leq S(T^*AT|T^*BT).$$

The relative operator entropy is *jointly concave*, namely, for any positive invertible operators A, B, C, D we have

$$S(tA + (1-t)B|tC + (1-t)D) \geq tS(A|C) + (1-t)S(B|D)$$

for any $t \in [0,1]$.

For other results on the relative operator entropy see [1], [8], [13], [14], [15] and [17].

Observe that, if we replace in (1.2) B with A, then we get

$$\begin{aligned} S(B|A) &= A^{1/2}\eta\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} \\ &= A^{1/2}\left(-A^{-1/2}BA^{-1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}, \end{aligned}$$

therefore we have

$$A^{1/2}\left(A^{-1/2}BA^{-1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2} = -S(B|A) \tag{1.4}$$

for positive invertible operators A and B.

It is well know that, in general $S(A|B)$ is not equal to $S(B|A)$.

In [19], A. Uhlmann has shown that the relative operator entropy $S(A|B)$ can be represented as the strong limit

$$S(A|B) = s\text{-}\lim_{t\to 0}\frac{A\sharp_t B - A}{t}, \tag{1.5}$$

where

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2},\ \ \nu \in [0,1]$$

is the *weighted geometric mean* of positive invertible operators A and B. For $\nu = \frac{1}{2}$ we denote $A\sharp B$.

This definition of the weighted geometric mean can be extended for any real number ν with $\nu \neq 0$.

For $t \neq 0$ and the positive invertible operators A, B we define the *Tsallis' relative entropy* (see also [7]) by

$$T_t(A|B) := \frac{A\sharp_t B - A}{t}.$$

Consider the scalar function $T_t : (0,\infty) \to \mathbb{R}$ defined for $t \neq 0$ by

$$T_t(x) := \frac{x^t - 1}{t}.$$

We have

$$T_{-t}(x) := \frac{x^{-t}-1}{-t} = \frac{1-x^{-t}}{t} = \frac{x^t-1}{tx^t} = T_t(x)\,x^{-t}. \tag{1.6}$$

For positive invertible operators A and B and $t > 0$ we then have

$$A^{1/2}T_t\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} = A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^t - 1_H}{t}A^{1/2} = T_t(A|B).$$

Also by (1.6) we have

$$A^{1/2}T_{-t}\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}=T_{-t}\left(A|B\right)$$

and

$$\begin{aligned}(1.7)\qquad & A^{1/2}T_{-t}\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}\\ &=A^{1/2}T_{t}\left(A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2}\right)^{-t}A^{1/2}\\ &=A^{1/2}T_{t}\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}A^{-1/2}\left(A^{-1/2}BA^{-1/2}\right)^{-t}A^{-1/2}A\\ &=T_{t}\left(A|B\right)\left(A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{t}A^{1/2}\right)^{-1}A\\ &=T_{t}\left(A|B\right)\left(A\sharp_{t}B\right)^{-1}A\end{aligned}$$

for any positive invertible operators A and B and $t>0$.

The following result providing upper and lower bounds for relative operator entropy in terms of $T_t(\cdot|\cdot)$ has been obtained in [10] for $0<t\le 1$. However, it hods for any $t>0$.

Theorem 1 (Fujii-Kamei, 1989, [10])**.** *Let A, B be two positive invertible operators, then for any $t>0$ we have*

$$T_{-t}\left(A|B\right)\le S\left(A|B\right)\le T_{t}\left(A|B\right). \tag{1.8}$$

In particular, we have for $t=1$ that

$$\left(1_H-AB^{-1}\right)A\le S\left(A|B\right)\le B-A,\ [10] \tag{1.9}$$

and for $t=2$ that

$$\frac{1}{2}\left(1_H-\left(AB^{-1}\right)^2\right)A\le S\left(A|B\right)\le\frac{1}{2}\left(BA^{-1}B-A\right). \tag{1.10}$$

The case $t=\frac{1}{2}$ is of interest as well. Since in this case we have

$$T_{1/2}\left(A|B\right):=2\left(A\sharp B-A\right)$$

and

$$T_{-1/2}\left(A|B\right)=T_{1/2}\left(A|B\right)\left(A\sharp_{1/2}B\right)^{-1}A=2\left(1_H-A\left(A\sharp B\right)^{-1}\right)A,$$

hence by (1.8) we get

$$2\left(1_H-A\left(A\sharp B\right)^{-1}\right)A\le S\left(A|B\right)\le 2\left(A\sharp B-A\right)\le B-A. \tag{1.11}$$

Motivated by the Fujii-Kamei inequality (1.8) we establish in this paper some new results providing Taylor's like expansion bounds for the relative operator entropy $S(A|B)$ of positive invertible operators in terms of Tsallis' relative entropy $T_{\pm t}(A|B)$ with $t>0$. Further bounds in the case that $B\ge A$ are also provided.

2. Some Operator Inequalities

We need the following result, see [5]:

Lemma 1. *For any* $a,\ b>0$ *we have for* $n\geq 1$ *that*

$$\frac{1}{(2n+1)\,b^{2n+1}}(b-a)^{2n+1}\leq \ln b-\ln a-\sum_{k=1}^{2n}\frac{(-1)^{k-1}(b-a)^{k}}{ka^{k}}\leq \frac{1}{(2n+1)\,a^{2n+1}}(b-a)^{2n+1} \tag{2.1}$$

and

$$\frac{1}{(2n+1)\,b^{2n+1}}(b-a)^{2n+1}\leq \ln b-\ln a-\sum_{k=1}^{2n}\frac{(b-a)^{k}}{kb^{k}}\leq \frac{1}{(2n+1)\,a^{2n+1}}(b-a)^{2n+1}. \tag{2.2}$$

Proof. For the sake of completeness, we give a short proof here. We use the following result, see for instance [6], where various applications in Information Theory were provided:

$$\ln b-\ln a+\sum_{k=1}^{m}\frac{(-1)^{k}(b-a)^{k}}{ka^{k}}=(-1)^{m}\int_{a}^{b}\frac{(b-t)^{m}}{t^{m+1}}dt, \tag{2.3}$$

for any $a,\ b>0$ and for $m\geq 1$. For recent inequalities derived from this identity and a short proof, see [3].

If we take $m=2n$ with $n\geq 1$ in (2.3), then we get for any $a,\ b>0$ that

$$\ln b-\ln a+\sum_{k=1}^{2n}\frac{(-1)^{k}(b-a)^{k}}{ka^{k}}=\int_{a}^{b}\frac{(b-t)^{2n}}{t^{2n+1}}dt. \tag{2.4}$$

If $b>a>0$, then we have

$$\frac{1}{b^{2n+1}}\int_{a}^{b}(b-t)^{2n}\,dt\leq \int_{a}^{b}\frac{(b-t)^{2n}}{t^{2n+1}}dt\leq \frac{1}{a^{2n+1}}\int_{a}^{b}(b-t)^{2n}\,dt$$

and since

$$\int_{a}^{b}(b-t)^{2n}\,dt=\frac{1}{2n+1}(b-a)^{2n+1}$$

we get, by (2.4) the desired inequality (2.1).

If $a>b>0$, then

$$\int_{a}^{b}\frac{(b-t)^{2n}}{t^{2n+1}}dt=-\int_{b}^{a}\frac{(b-t)^{2n}}{t^{2n+1}}dt. \tag{2.5}$$

We have

$$\frac{1}{a^{2n+1}}\int_{b}^{a}(b-t)^{2n}\,dt\leq \int_{b}^{a}\frac{(b-t)^{2n}}{t^{2n+1}}dt\leq \frac{1}{b^{2n+1}}\int_{b}^{a}(b-t)^{2n}\,dt. \tag{2.6}$$

Observe that

$$\int_{b}^{a}(b-t)^{2n}\,dt=\int_{b}^{a}(t-b)^{2n}\,dt=\frac{(a-b)^{2n+1}}{2n+1}=-\frac{(b-a)^{2n+1}}{2n+1}$$

and by (2.6) we then get

$$-\frac{(b-a)^{2n+1}}{(2n+1)\,a^{2n+1}}dt \le \int_b^a \frac{(b-t)^{2n}}{t^{2n+1}}dt \le -\frac{(b-a)^{2n+1}}{b^{2n+1}\,(2n+1)}$$

that is equivalent to

$$\frac{(b-a)^{2n+1}}{b^{2n+1}\,(2n+1)} \le -\int_b^a \frac{(b-t)^{2n}}{t^{2n+1}}dt \le \frac{(b-a)^{2n+1}}{(2n+1)\,a^{2n+1}}dt.$$

By using (2.4) and (2.5) we get (2.1) again.

Now, if we replace a with b in (2.1), then we get

$$\begin{aligned}\frac{1}{(2n+1)\,a^{2n+1}}(a-b)^{2n+1} &\le \ln a - \ln b + \sum_{k=1}^{2n}\frac{(-1)^k (a-b)^k}{kb^k}\\ &\le \frac{1}{(2n+1)\,b^{2n+1}}(a-b)^{2n+1},\end{aligned}$$

namely

$$\begin{aligned}-\frac{1}{(2n+1)\,a^{2n+1}}(b-a)^{2n+1} &\le \ln a - \ln b + \sum_{k=1}^{2n}\frac{(b-a)^k}{kb^k}\\ &\le -\frac{1}{(2n+1)\,b^{2n+1}}(b-a)^{2n+1}.\end{aligned}$$

If we multiply this inequality by -1 we get the desired inequality (2.2). □

Remark 1. *If we take $b = y \in (0,\infty)$ and $a = 1$ in (2.1) and (2.2), then we get*

$$\frac{1}{(2n+1)}\frac{(y-1)^{2n+1}}{y^{2n+1}} \le \ln y - \sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}(y-1)^k \le \frac{1}{(2n+1)}(y-1)^{2n+1} \tag{2.7}$$

and

$$\frac{1}{(2n+1)\,y^{2n+1}}(y-1)^{2n+1} \le \ln y - \sum_{k=1}^{2n}\frac{(y-1)^k}{ky^k} \le \frac{1}{(2n+1)}(y-1)^{2n+1} \tag{2.8}$$

for any $y \in (0,\infty)$ and $n \ge 1$.

The following operator inequality holds:

Theorem 2. *Let A, B be two positive invertible operators, then for any $t > 0$ we have*

$$\begin{aligned}&\frac{1}{(2n+1)}t^{2n}\left(T_{-t}(A|B)\,A^{-1}\right)^{2n+1}A\\ &\le S(A|B) - \sum_{k=1}^{2n}\frac{(-1)^{k-1}t^{k-1}}{k}\left(T_t(A|B)\,A^{-1}\right)^k A\\ &\le \frac{1}{(2n+1)}t^{2n}\left(T_t(A|B)\,A^{-1}\right)^{2n+1}A\end{aligned} \tag{2.9}$$

and

$$\frac{1}{(2n+1)}t^{2n}\left(T_{-t}(A|B)A^{-1}\right)^{2n+1}A \tag{2.10}$$
$$\leq S(A|B)-\sum_{k=1}^{2n}\frac{t^{k-1}}{k}\left(T_{-t}(A|B)A^{-1}\right)^{k}A$$
$$\leq \frac{1}{(2n+1)}t^{2n}\left(T_{t}(A|B)A^{-1}\right)^{2n+1}A$$

for any $n\geq 1$.

Proof. By (2.7) we have for $y=x^t$ with $x>0$ and $t>0$ that

$$\frac{1}{(2n+1)}\left(1-x^{-t}\right)^{2n+1}\leq \ln x^t-\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}\left(x^t-1\right)^k\leq \frac{1}{(2n+1)}\left(x^t-1\right)^{2n+1},$$

namely

$$\frac{1}{(2n+1)}t^{2n+1}\left(\frac{1-x^{-t}}{t}\right)^{2n+1} \leq t\ln x-\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}t^k\left(\frac{x^t-1}{t}\right)^k$$
$$\leq \frac{1}{(2n+1)}t^{2n+1}\left(\frac{x^t-1}{t}\right)^{2n+1}$$

and by division with $t>0$,

$$\frac{1}{(2n+1)}t^{2n}\left(\frac{1-x^{-t}}{t}\right)^{2n+1} \leq \ln x-\sum_{k=1}^{2n}\frac{(-1)^{k-1}t^{k-1}}{k}\left(\frac{x^t-1}{t}\right)^k$$
$$\leq \frac{1}{(2n+1)}t^{2n}\left(\frac{x^t-1}{t}\right)^{2n+1}$$

for any x, $t>0$ and $n\geq 1$.

This inequality can be written in terms of T_t as

$$\frac{1}{(2n+1)}t^{2n}T_{-t}^{2n+1}(x)\leq \ln x-\sum_{k=1}^{2n}\frac{(-1)^{k-1}t^{k-1}}{k}T_t^k(x) \tag{2.11}$$
$$\leq \frac{1}{(2n+1)}t^{2n}T_t^{2n+1}(x)$$

for any x, $t>0$ and $n\geq 1$.

Using the continuous functional calculus for the positive invertible operator X we have

$$\frac{1}{(2n+1)}t^{2n}\left(T_{-t}(X)\right)^{2n+1}\leq \ln X-\sum_{k=1}^{2n}\frac{(-1)^{k-1}t^{k-1}}{k}\left(T_t(X)\right)^k$$
$$\leq \frac{1}{(2n+1)}t^{2n}\left(T_t(X)\right)^{2n+1}.$$

Now, if we take in this inequality $X = A^{-1/2}BA^{-1/2}$, then we get

$$\begin{aligned}
&\frac{1}{(2n+1)} t^{2n} \left(T_{-t}\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+1} \\
&\leq \ln\left(A^{-1/2}BA^{-1/2}\right) - \sum_{k=1}^{2n} \frac{(-1)^{k-1} t^{k-1}}{k} \left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^k \\
&\leq \frac{1}{(2n+1)} t^{2n} \left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+1}.
\end{aligned}$$

If we multiply this inequality in both sides by $A^{1/2}$, then we get

$$\begin{aligned}
&\frac{1}{(2n+1)} t^{2n} A^{1/2} \left(T_{-t}\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+1} A^{1/2} \\
&\leq A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right) A^{1/2} \\
&- \sum_{k=1}^{2n} \frac{(-1)^{k-1} t^{k-1}}{k} A^{1/2} \left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^k A^{1/2} \\
&\leq \frac{1}{(2n+1)} t^{2n} A^{1/2} \left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+1} A^{1/2}.
\end{aligned} \tag{2.12}$$

Observe that for $k = 1$ we have

$$A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^k A^{1/2} = A^{1/2} T_t\left(A^{-1/2}BA^{-1/2}\right) A^{1/2} = T_t(A|B).$$

For $k \geq 2$ we have

$$\begin{aligned}
&A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^k A^{1/2} \\
&= A^{1/2}\left(A^{-1/2}A^{1/2}T_t\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}A^{-1/2}\right)^k A^{1/2} \\
&= A^{1/2}\left(A^{-1/2}T_t(A|B)A^{-1/2}\right)^k A^{1/2} \\
&= A^{1/2}A^{-1/2}T_t(A|B)A^{-1/2}...A^{-1/2}T_t(A|B)A^{-1/2}A^{1/2} \\
&= T_t(A|B)A^{-1}...T_t(A|B)A^{-1/2}A^{1/2} \\
&= T_t(A|B)A^{-1}...T_t(A|B)A^{-1}A = \left(T_t(A|B)A^{-1}\right)^k A.
\end{aligned}$$

We observe that, this formula also holds for $k = 1$, therefore for any $k \geq 1$ we have

$$A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^k A^{1/2} = \left(T_t(A|B)A^{-1}\right)^k A.$$

Similarly,

$$A^{1/2}\left(T_{-t}\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+1} A^{1/2} = \left(T_{-t}(A|B)A^{-1}\right)^{2n+1} A$$

and by (2.12) we get the desired result (2.9).

From (2.8) we have for $y = x^t$ with $x > 0$ and $t > 0$ that

$$\frac{1}{(2n+1)} t^{2n} T_{-t}^{2n+1}(x) \leq \ln x - \sum_{k=1}^{2n} \frac{1}{k} t^{k-1} T_{-t}^k(x) \leq \frac{1}{(2n+1)} t^{2n} T_t^{2n+1}(x).$$

On making use of a similar argument to the one outlined above we get the desired result (2.10) and the details are omitted. □

Corollary 1. *Let A, B be two positive invertible operators such that $B \geq A$ then for any $t > 0$ we have the lower bounds for the relative operator entropy*

$$\sum_{k=1}^{2n} \frac{(-1)^{k-1} t^{k-1}}{k} \left(T_t (A|B) A^{-1}\right)^k A \leq S(A|B) \tag{2.13}$$

and

$$\sum_{k=1}^{2n} \frac{t^{k-1}}{k} \left(T_{-t} (A|B) A^{-1}\right)^k A \leq S(A|B) \tag{2.14}$$

for any $n \geq 1$.

Proof. If $B \geq A$, then by multiplying both sides by $A^{-1/2}$ we get

$$A^{-1/2} B A^{-1/2} \geq 1_H.$$

If $x \geq 1$ then for $t > 0$ we have

$$T_{-t}(x) = \frac{x^t - 1}{t x^t} \geq 0,$$

which implies for the operator $X \geq 1_H$ that $T_{-t}(X) \geq 0$ and for $X = A^{-1/2} B A^{-1/2}$ that $T_{-t}\left(A^{-1/2} B A^{-1/2}\right) \geq 0$. By multiplying both sides with $A^{1/2}$ we get

$$A^{1/2} T_{-t} \left(A^{-1/2} B A^{-1/2}\right) A^{1/2} \geq 0.$$

Therefore

$$\left(T_{-t}(A|B) A^{-1}\right)^{2n+1} A = A^{1/2} \left(T_{-t}\left(A^{-1/2} B A^{-1/2}\right)\right)^{2n+1} A^{1/2} \geq 0$$

for $n \geq 1$ and by (2.9) and (2.10) we get the desired results. □

If we take $n = 1$ in (2.9) and (2.10), then we get

$$\begin{aligned} \frac{1}{3} t^2 \left(T_{-t}(A|B) A^{-1}\right)^3 A &\leq S(A|B) - T_t(A|B) + \frac{1}{2} t \left(T_t(A|B) A^{-1}\right)^2 A \\ &\leq \frac{1}{3} t^2 \left(T_t(A|B) A^{-1}\right)^3 A \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} \frac{1}{3} t^2 \left(T_{-t}(A|B) A^{-1}\right)^3 A &\leq S(A|B) - T_{-t}(A|B) - \frac{1}{2} t \left(T_{-t}(A|B) A^{-1}\right)^2 A \\ &\leq \frac{1}{3} t^2 \left(T_t(A|B) A^{-1}\right)^3 A, \end{aligned} \tag{2.16}$$

for any A, B two positive invertible operators and any $t > 0$.

If A, B are two positive invertible operators with $B \geq A$ and $t > 0$, then

$$T_t(A|B) - \frac{1}{2} t \left(T_t(A|B) A^{-1}\right)^2 A \leq S(A|B) \quad \left(\leq T_t(A|B) \text{ from (1.8)}\right) \tag{2.17}$$

and

$$T_{-t}(A|B) \leq T_{-t}(A|B) + \frac{1}{2} t \left(T_{-t}(A|B) A^{-1}\right)^2 A \leq S(A|B). \tag{2.18}$$

The inequality between the first and last term in (2.18) holds for any positive invertible operators A, B, as shown in the first part of (1.8). Therefore, (2.18) can be seen as a refinement of that inequality.

Remark 2. *If we take in the inequalities (2.9) and (2.10) $t=1$ and since*

$$T_{-1}(A|B)A^{-1}=1_H-AB^{-1} \text{ and } T_1(A|B)A^{-1}=BA^{-1}-1_H,$$

then we get

$$\frac{1}{(2n+1)}\left(1_H-AB^{-1}\right)^{2n+1}A\le S(A|B)-\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}\left(BA^{-1}-1_H\right)^kA \le \frac{1}{(2n+1)}\left(BA^{-1}-1_H\right)^{2n+1}A \tag{2.19}$$

and

$$\frac{1}{(2n+1)}\left(1_H-AB^{-1}\right)^{2n+1}A\le S(A|B)-\sum_{k=1}^{2n}\frac{1}{k}\left(1_H-AB^{-1}\right)^kA \le \frac{1}{(2n+1)}\left(BA^{-1}-1_H\right)^{2n+1}A \tag{2.20}$$

for any positive invertible operators A, B and $n\ge 1$.

If $B\ge A$, then

$$\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}\left(BA^{-1}-1_H\right)^kA\le S(A|B)$$

and

$$\sum_{k=1}^{2n}\frac{1}{k}\left(1_H-AB^{-1}\right)^kA\le S(A|B)$$

for any $n\ge 1$.

If we take in the inequalities (2.9) and (2.10) $t=2$ and since

$$T_{-2}(A|B)A^{-1}=\frac{1}{2}\left(1_H-\left(AB^{-1}\right)^2\right)$$

and

$$T_2(A|B)A^{-1}=\frac{1}{2}\left(\left(BA^{-1}\right)^2-1_H\right),$$

then we get

$$\begin{aligned}&\frac{1}{2(2n+1)}\left(1_H-\left(AB^{-1}\right)^2\right)^{2n+1}A\\ &\le S(A|B)-\frac{1}{2}\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}\left(\left(BA^{-1}\right)^2-1_H\right)^kA\\ &\le \frac{1}{2(2n+1)}\left(\left(BA^{-1}\right)^2-1_H\right)^{2n+1}A\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}&\frac{1}{2(2n+1)}\left(1_H-\left(AB^{-1}\right)^2\right)^{2n+1}A\\ &\le S(A|B)-\sum_{k=1}^{2n}\frac{1}{2k}\left(1_H-\left(AB^{-1}\right)^2\right)^kA\\ &\le \frac{1}{2(2n+1)}\left(\left(BA^{-1}\right)^2-1_H\right)^{2n+1}A\end{aligned} \tag{2.22}$$

for any positive invertible operators A, B and $n \geq 1$.

If $B \geq A$, then

$$\frac{1}{2}\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}\left(\left(BA^{-1}\right)^{2}-1_{H}\right)^{k}A\leq S\left(A|B\right) \tag{2.23}$$

and

$$\sum_{k=1}^{2n}\frac{1}{2k}\left(1_{H}-\left(AB^{-1}\right)^{2}\right)^{k}A\leq S\left(A|B\right) \tag{2.24}$$

for any $n \geq 1$.

Since for $t = 1/2$ we have

$$T_{-1/2}\left(A|B\right)A^{-1}=2\left(1_{H}-A\left(A\sharp B\right)^{-1}\right)$$

and

$$T_{1/2}\left(A|B\right)A^{-1}=2\left(\left(A\sharp B\right)A^{-1}-1_{H}\right),$$

then by the inequalities (2.9) and (2.10) we get

$$\begin{aligned}
&\frac{2}{(2n+1)}\left(1_{H}-A\left(A\sharp B\right)^{-1}\right)^{2n+1}A \\
&\leq S\left(A|B\right)-2\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}\left(\left(A\sharp B\right)A^{-1}-1_{H}\right)^{k}A \\
&\leq \frac{2}{(2n+1)}\left(\left(A\sharp B\right)A^{-1}-1_{H}\right)^{2n+1}A
\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}
&\frac{2}{(2n+1)}\left(1_{H}-A\left(A\sharp B\right)^{-1}\right)^{2n+1}A \\
&\leq S\left(A|B\right)-2\sum_{k=1}^{2n}\frac{1}{k}\left(1_{H}-A\left(A\sharp B\right)^{-1}\right)^{k}A \\
&\leq \frac{2}{(2n+1)}\left(\left(A\sharp B\right)A^{-1}-1_{H}\right)^{2n+1}A
\end{aligned} \tag{2.26}$$

for any $n \geq 1$.

If $B \geq A$, then

$$2\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}\left(\left(A\sharp B\right)A^{-1}-1_{H}\right)^{k}A\leq S\left(A|B\right) \tag{2.27}$$

and

$$2\sum_{k=1}^{2n}\frac{1}{k}\left(1_{H}-A\left(A\sharp B\right)^{-1}\right)^{k}A\leq S\left(A|B\right) \tag{2.28}$$

for any $n \geq 1$.

3. Further Operator Inequalities

We have the following inequalities for the logarithm [5]:

Lemma 2. *For any $a, b > 0$ we have for $n \geq 1$ that*

$$\begin{aligned}
\frac{1}{2n(2n+1)}\frac{(b-a)^{2n+1}}{b^{2n+1}} &\leq \frac{b-a}{b} + \frac{1}{b}\sum_{k=2}^{2n}\frac{(-1)^k}{k(k-1)}\frac{(b-a)^k}{a^{k-1}} - \ln b + \ln a \\
&\leq \frac{1}{2n(2n+1)}\frac{(b-a)^{2n+1}}{a^{2n}b}
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
\frac{1}{2n(2n+1)}\frac{(b-a)^{2n+1}}{b^{2n}a} &\leq \frac{b-a}{a} - \frac{1}{a}\sum_{k=2}^{2n}\frac{1}{k(k-1)}\frac{(b-a)^k}{b^{k-1}} - \ln b + \ln a \\
&\leq \frac{1}{2n(2n+1)}\frac{(b-a)^{2n+1}}{a^{2n+1}}.
\end{aligned} \tag{3.2}$$

Proof. For the sake of completeness, we give a short proof here. We have the following representation result [4]:

$$\ln b - \ln a - \frac{b-a}{b} + \frac{1}{b}\sum_{k=2}^{m}\frac{(-1)^{k-1}}{k(k-1)}\frac{(b-a)^k}{a^{k-1}} = \frac{(-1)^{m-1}}{mb}\int_a^b \frac{(b-t)^m}{t^m}dt \tag{3.3}$$

for any $m \geq 2$ and any $a, b > 0$.

If we take $m = 2n$ with $n \geq 1$ in (3.3), then we get

$$\ln b - \ln a - \frac{b-a}{b} + \frac{1}{b}\sum_{k=2}^{2n}\frac{(-1)^{k-1}}{k(k-1)}\frac{(b-a)^k}{a^{k-1}} = -\frac{1}{2nb}\int_a^b \frac{(b-t)^{2n}}{t^{2n}}dt$$

that is equivalent to

$$\frac{1}{b}\sum_{k=2}^{2n}\frac{(-1)^k}{k(k-1)}\frac{(b-a)^k}{a^{k-1}} - \ln b + \ln a + \frac{b-a}{b} = \frac{1}{2nb}\int_a^b \frac{(b-t)^{2n}}{t^{2n}}dt \tag{3.4}$$

for any $a, b > 0$.

If $b > a > 0$, then we have

$$\frac{1}{b^{2n}}\int_a^b (b-t)^{2n}\,dt \leq \int_a^b \frac{(b-t)^{2n}}{t^{2n}}dt \leq \frac{1}{a^{2n}}\int_a^b (b-t)^{2n}\,dt$$

namely

$$\frac{1}{(2n+1)b^{2n}}(b-a)^{2n+1} \leq \int_a^b \frac{(b-t)^{2n}}{t^{2n}}dt \leq \frac{1}{(2n+1)a^{2n}}(b-a)^{2n+1}. \tag{3.5}$$

If $a > b > 0$, then

$$\int_a^b \frac{(b-t)^{2n}}{t^{2n}}dt = -\int_b^a \frac{(b-t)^{2n}}{t^{2n}}dt.$$

Observe that

$$\int_b^a (b-t)^{2n}\,dt = \int_b^a (t-b)^{2n}\,dt = \frac{(a-b)^{2n+1}}{2n+1} = -\frac{(b-a)^{2n+1}}{2n+1}.$$

We have

$$\frac{1}{a^{2n}}\int_b^a (t-b)^{2n}\,dt \le \int_b^a \frac{(b-t)^{2n}}{t^{2n}}dt \le \frac{1}{b^{2n}}\int_b^a (t-b)^{2n}\,dt$$

namely

$$-\frac{1}{a^{2n}}\frac{(b-a)^{2n+1}}{2n+1} \le \int_b^a \frac{(b-t)^{2n}}{t^{2n}}dt \le -\frac{1}{b^{2n}}\frac{(b-a)^{2n+1}}{2n+1},$$

which, by multiplying with -1 gives

$$\frac{1}{b^{2n}}\frac{(b-a)^{2n+1}}{2n+1} \le \int_a^b \frac{(b-t)^{2n}}{t^{2n}}dt \le \frac{1}{a^{2n}}\frac{(b-a)^{2n+1}}{2n+1} \tag{3.6}$$

for $a \ge b > 0$.

Using the representation (3.4) and the inequalities (3.5) and (3.6) we get (3.1).

If we replace a with b in (3.1), then we get

$$\begin{aligned}\frac{1}{2n(2n+1)}\frac{(a-b)^{2n+1}}{a^{2n+1}} &\le \frac{1}{a}\sum_{k=2}^{2n}\frac{(-1)^k}{k(k-1)}\frac{(a-b)^k}{b^{k-1}} - \ln a + \ln b + \frac{a-b}{a}\\ &\le \frac{1}{2n(2n+1)}\frac{(a-b)^{2n+1}}{b^{2n}a},\end{aligned}$$

namely

$$\begin{aligned}-\frac{1}{2n(2n+1)}\frac{(b-a)^{2n+1}}{a^{2n+1}} &\le \frac{1}{a}\sum_{k=2}^{2n}\frac{1}{k(k-1)}\frac{(b-a)^k}{b^{k-1}} - \ln a + \ln b + \frac{a-b}{a}\\ &\le -\frac{1}{2n(2n+1)}\frac{(b-a)^{2n+1}}{b^{2n}a}.\end{aligned} \tag{3.7}$$

If we multiply (3.7) by -1, then we get (3.2). □

Remark 3. *If we take $b = y \in (0,\infty)$ and $a = 1$ in (3.1) and (3.2), then we get*

$$\begin{aligned}\frac{1}{2n(2n+1)}\frac{(y-1)^{2n+1}}{y^{2n+1}} &\le \frac{y-1}{y} + \frac{1}{y}\sum_{k=2}^{2n}\frac{(-1)^k}{k(k-1)}(y-1)^k - \ln y\\ &\le \frac{1}{2n(2n+1)}\frac{(y-1)^{2n+1}}{y}\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\frac{1}{2n(2n+1)}\frac{(y-1)^{2n+1}}{y^{2n}} &\le y-1-\sum_{k=2}^{2n}\frac{1}{k(k-1)}\frac{(y-1)^k}{y^{k-1}} - \ln y\\ &\le \frac{1}{2n(2n+1)}(y-1)^{2n+1}.\end{aligned} \tag{3.9}$$

The following inequalities for the relative operator entropy may be stated as well:

Theorem 3. *Let A, B be two positive invertible operators, then for any $t>0$ we have*

$$\begin{aligned}(3.10)\quad & \frac{1}{2n(2n+1)}t^{2n}\left(T_{-t}(A|B)A^{-1}\right)^{2n+1}A \\ & \leq T_{-t}(A|B)+\sum_{k=2}^{2n}\frac{(-1)^k}{k(k-1)}t^{k-1}A(A\sharp_t B)^{-1}\left(T_t(A|B)A^{-1}\right)^k A \\ & -S(A|B) \\ & \leq \frac{1}{2n(2n+1)}t^{2n}A(A\sharp_t B)^{-1}\left(T_t(A|B)A^{-1}\right)^{2n+1}A\end{aligned}$$

and

$$\begin{aligned}(3.11)\quad & \frac{1}{2n(2n+1)}t^{2n}(A\sharp_t B)A^{-1}\left(T_{-t}(A|B)A^{-1}\right)^{2n+1}A \\ & \leq T_t(A|B)-\sum_{k=2}^{2n}\frac{1}{k(k-1)}t^{k-1}(A\sharp_t B)A^{-1}\left(T_{-t}(A|B)A^{-1}\right)^k A \\ & -S(A|B) \\ & \leq \frac{1}{2n(2n+1)}t^{2n}\left(T_t(A|B)A^{-1}\right)^{2n+1}A\end{aligned}$$

for any $n\geq 1$.

Proof. By (3.8) we have for $y=x^t$ with $x>0$ and $t>0$ that

$$\begin{aligned}\frac{1}{2n(2n+1)}\left(\frac{x^t-1}{x^t}\right)^{2n+1} & \leq \frac{x^t-1}{x^t}+\frac{1}{x^t}\sum_{k=2}^{2n}\frac{(-1)^k}{k(k-1)}\left(x^t-1\right)^k-\ln x^t \\ & \leq \frac{1}{2n(2n+1)}\frac{\left(x^t-1\right)^{2n+1}}{x^t}\end{aligned}$$

that is equivalent to

$$\begin{aligned}& \frac{1}{2n(2n+1)}t^{2n}\left(\frac{x^t-1}{tx^t}\right)^{2n+1} \\ & \leq \frac{x^t-1}{tx^t}+\frac{1}{x^t}\sum_{k=2}^{2n}\frac{(-1)^k}{k(k-1)}t^{k-1}\left(\frac{x^t-1}{t}\right)^k-\ln x \\ & \leq \frac{1}{2n(2n+1)}\frac{t^{2n}}{x^t}\left(\frac{x^t-1}{t}\right)^{2n+1},\end{aligned}$$

which can be written as

$$\begin{aligned}& \frac{1}{2n(2n+1)}t^{2n}\left(T_{-t}(x)\right)^{2n+1} \\ & \leq T_{-t}(x)+\frac{1}{x^t}\sum_{k=2}^{2n}\frac{(-1)^k}{k(k-1)}t^{k-1}\left(T_t(x)\right)^k-\ln x \\ & \leq \frac{1}{2n(2n+1)}\frac{t^{2n}}{x^t}\left(T_t(x)\right)^{2n+1},\end{aligned}$$

for any $x>0$, $n\geq 1$ and $t>0$.

Using the continuous functional calculus we have, as in the proof of Theorem 2 that

$$\begin{aligned}
&\frac{1}{2n(2n+1)}t^{2n}\left(T_{-t}\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+1}\\
&\leq T_{-t}\left(A^{-1/2}BA^{-1/2}\right)\\
&+\sum_{k=2}^{2n}\frac{(-1)^k}{k(k-1)}t^{k-1}\left(A^{-1/2}BA^{-1/2}\right)^{-t}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^k\\
&-\ln\left(A^{-1/2}BA^{-1/2}\right)\\
&\leq\frac{1}{2n(2n+1)}t^{2n}\left(A^{-1/2}BA^{-1/2}\right)^{-t}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+1}
\end{aligned}$$

and by multiplying both sides with $A^{1/2}$ we get

$$\begin{aligned}
&\frac{1}{2n(2n+1)}t^{2n}A^{1/2}\left(T_{-t}\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+1}A^{1/2}\\
&\leq A^{1/2}T_{-t}\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}\\
&+\sum_{k=2}^{2n}\frac{(-1)^k}{k(k-1)}t^{k-1}A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{-t}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^kA^{1/2}\\
&-A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}\\
&\leq\frac{1}{2n(2n+1)}t^{2n}A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{-t}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+1}A^{1/2}
\end{aligned}\tag{3.12}$$

for any A, B positive invertible operators for any $t>0$ and $n\geq 1$.

As above

$$A^{1/2}\left(T_{-t}\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+1}A^{1/2}=\left(T_{-t}(A|B)A^{-1}\right)^{2n+1}A,$$

$$A^{1/2}T_{-t}\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}=T_{-t}(A|B)$$

and for $k\geq 2$

$$\begin{aligned}
&A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{-t}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^kA^{1/2}\\
&=AA^{-1/2}\left(A^{-1/2}BA^{-1/2}\right)^{-t}A^{-1/2}A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^kA^{1/2}\\
&=A\left(A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{t}A^{1/2}\right)^{-1}A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^kA^{1/2}\\
&=A\left(A\sharp_tB\right)^{-1}\left(T_t(A|B)A^{-1}\right)^kA,
\end{aligned}$$

then by (2.10) we get the desired result (3.10).

By (3.9) we have

$$\begin{aligned}
\frac{1}{2n(2n+1)}y\left(\frac{y-1}{y}\right)^{2n+1}&\leq y-1-\sum_{k=2}^{2n}\frac{1}{k(k-1)}y\left(\frac{y-1}{y}\right)^k-\ln y\\
&\leq\frac{1}{2n(2n+1)}(y-1)^{2n+1}
\end{aligned}$$

that by taking $y = x^t$ with $x > 0$ and $t > 0$ gives

$$\begin{aligned}&\frac{1}{2n(2n+1)}x^t\left(\frac{x^t-1}{x^t}\right)^{2n+1}\\&\le x^t-1-\sum_{k=2}^{2n}\frac{1}{k(k-1)}x^t\left(\frac{x^t-1}{x^t}\right)^k-\ln x^t\\&\le\frac{1}{2n(2n+1)}\left(x^t-1\right)^{2n+1},\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\frac{1}{2n(2n+1)}t^{2n}x^t\left(\frac{x^t-1}{tx^t}\right)^{2n+1}\\&\le\frac{x^t-1}{t}-\sum_{k=2}^{2n}\frac{1}{k(k-1)}t^{k-1}x^t\left(\frac{x^t-1}{tx^t}\right)^k-\ln x\\&\le\frac{1}{2n(2n+1)}t^{2n}\left(\frac{x^t-1}{t}\right)^{2n+1},\end{aligned}$$

for any $x > 0$, $t > 0$ and $n \ge 1$.

This inequality can be written as

$$\begin{aligned}\frac{1}{2n(2n+1)}t^{2n}x^tT_{-t}^{2n+1}(x)&\le T_t(x)-\sum_{k=2}^{2n}\frac{1}{k(k-1)}t^{k-1}x^tT_{-t}^k(x)-\ln x\\&\le\frac{1}{2n(2n+1)}t^{2n}T_t^{2n+1}(x),\end{aligned}$$

for any $x > 0$, $t > 0$ and $n \ge 1$.

Using the continuous functional calculus we have,

$$\begin{aligned}&\frac{1}{2n(2n+1)}t^{2n}\left(A^{-1/2}BA^{-1/2}\right)^t\left(T_{-t}\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+1}\\&\le T_t\left(A^{-1/2}BA^{-1/2}\right)\\&-\sum_{k=2}^{2n}\frac{1}{k(k-1)}t^{k-1}\left(A^{-1/2}BA^{-1/2}\right)^t\left(T_{-t}\left(A^{-1/2}BA^{-1/2}\right)\right)^k\\&-\ln\left(A^{-1/2}BA^{-1/2}\right)\\&\le\frac{1}{2n(2n+1)}t^{2n}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+1},\end{aligned}$$

for any A, B positive invertible operators for any $t > 0$ and $n \ge 1$.

If we multiply both sides of this inequality by $A^{1/2}$ then we get

$$\begin{aligned}(3.13)\quad & \frac{1}{2n(2n+1)}t^{2n}A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{t}\left(T_{-t}\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+1}A^{1/2} \\ & \leq A^{1/2}T_{t}\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} \\ & -\sum_{k=2}^{2n}\frac{1}{k(k-1)}t^{k-1}A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{t}\left(T_{-t}\left(A^{-1/2}BA^{-1/2}\right)\right)^{k}A^{1/2} \\ & -A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2} \\ & \leq\frac{1}{2n(2n+1)}t^{2n}A^{1/2}\left(T_{t}\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+1}A^{1/2}.\end{aligned}$$

Observe that for $k\geq 2$

$$A^{1/2}\left(T_{-t}\left(A^{-1/2}BA^{-1/2}\right)\right)^{k}A^{1/2}=\left(T_{-t}(A|B)A^{-1}\right)^{k}A,$$

$$\begin{aligned}& A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{t}\left(T_{-t}\left(A^{-1/2}BA^{-1/2}\right)\right)^{k}A^{1/2} \\ & =A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{t}A^{1/2}A^{-1}A^{1/2}\left(T_{-t}\left(A^{-1/2}BA^{-1/2}\right)\right)^{k}A^{1/2} \\ & =(A\sharp_{t}B)A^{-1}\left(T_{-t}(A|B)A^{-1}\right)^{k}A\end{aligned}$$

and

$$A^{1/2}\left(T_{t}\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+1}A^{1/2}=\left(T_{t}(A|B)A^{-1}\right)^{2n+1}A$$

and by (3.13) we get the desired result (3.11). □

Corollary 2. *Let A, B be two positive invertible operators such that $B\geq A$, then for any $t>0$ we have the upper bounds for the relative operator entropy*

$$(3.14)\quad S(A|B)\leq T_{-t}(A|B)+\sum_{k=2}^{2n}\frac{(-1)^{k}}{k(k-1)}t^{k-1}A(A\sharp_{t}B)^{-1}\left(T_{t}(A|B)A^{-1}\right)^{k}A$$

and

$$(3.15)\quad S(A|B)\leq T_{t}(A|B)-\sum_{k=2}^{2n}\frac{1}{k(k-1)}t^{k-1}(A\sharp_{t}B)A^{-1}\left(T_{-t}(A|B)A^{-1}\right)^{k}A$$

for any $n\geq 1$.

If we take $n=1$ in (3.10) and (3.11), then we get

$$\begin{aligned}(3.16)\quad & \frac{1}{6}t^{2}\left(T_{-t}(A|B)A^{-1}\right)^{3}A \\ & \leq T_{-t}(A|B)+\frac{1}{2}tA(A\sharp_{t}B)^{-1}\left(T_{t}(A|B)A^{-1}\right)^{2}A-S(A|B) \\ & \leq\frac{1}{6}t^{2}A(A\sharp_{t}B)^{-1}\left(T_{t}(A|B)A^{-1}\right)^{3}A\end{aligned}$$

and

$$\frac{1}{6}t^2\left(A\sharp_t B\right)A^{-1}\left(T_{-t}\left(A|B\right)A^{-1}\right)^3 A \tag{3.17}$$
$$\leq T_t\left(A|B\right)-\frac{1}{2}t\left(A\sharp_t B\right)A^{-1}\left(T_{-t}\left(A|B\right)A^{-1}\right)^2 A-S\left(A|B\right)$$
$$\leq \frac{1}{6}t^2\left(T_t\left(A|B\right)A^{-1}\right)^3 A.$$

for any A, B positive invertible operators and $t>0$.

If A, B are two positive invertible operators with $B\geq A$ and $t>0$, then

$$\left(T_{-t}\left(A|B\right)\leq\right)\ S\left(A|B\right)\leq T_{-t}\left(A|B\right)+\frac{1}{2}tA\left(A\sharp_t B\right)^{-1}\left(T_t\left(A|B\right)A^{-1}\right)^2 A \tag{3.18}$$

and

$$S\left(A|B\right)\leq T_t\left(A|B\right)-\frac{1}{2}t\left(A\sharp_t B\right)A^{-1}\left(T_{-t}\left(A|B\right)A^{-1}\right)^2 A\ \ \left(\leq T_t\left(A|B\right)\right). \tag{3.19}$$

The inequality (3.18) provides a reverse for the first inequality in (1.8) while the inequality (3.19) provides a refinement for the second inequality in (1.8).

Remark 4. *If we take in the inequalities (3.10) and (3.11) $t=1$ and since $A\sharp_1 B=B$*

$$T_{-1}\left(A|B\right)A^{-1}=1_H-AB^{-1}\ \text{and}\ T_1\left(A|B\right)A^{-1}=BA^{-1}-1_H,$$

then we get

$$\frac{1}{2n\left(2n+1\right)}\left(1_H-AB^{-1}\right)^{2n+1}A \tag{3.20}$$
$$\leq 1_H-AB^{-1}+\sum_{k=2}^{2n}\frac{\left(-1\right)^k}{k\left(k-1\right)}AB^{-1}\left(BA^{-1}-1_H\right)^k A-S\left(A|B\right)$$
$$\leq\frac{1}{2n\left(2n+1\right)}AB^{-1}\left(BA^{-1}-1_H\right)^{2n+1}A$$

and

$$\frac{1}{2n\left(2n+1\right)}BA^{-1}\left(1_H-AB^{-1}\right)^{2n+1}A \tag{3.21}$$
$$\leq BA^{-1}-1_H-\sum_{k=2}^{2n}\frac{1}{k\left(k-1\right)}BA^{-1}\left(1_H-AB^{-1}\right)^k A-S\left(A|B\right)$$
$$\leq\frac{1}{2n\left(2n+1\right)}\left(BA^{-1}-1_H\right)^{2n+1}A$$

for any positive invertible operators A, B and $n\geq 1$.

If $B\geq A$, then

$$S\left(A|B\right)\leq 1_H-AB^{-1}+\sum_{k=2}^{2n}\frac{\left(-1\right)^k}{k\left(k-1\right)}AB^{-1}\left(BA^{-1}-1_H\right)^k A \tag{3.22}$$

and

$$S\left(A|B\right)\leq BA^{-1}-1_H-\sum_{k=2}^{2n}\frac{1}{k\left(k-1\right)}BA^{-1}\left(1_H-AB^{-1}\right)^k A \tag{3.23}$$

for any $n\geq 1$.

If we take in the inequalities (3.10) and (3.11) $t=2$ *and since* $A\sharp_2 B = BA^{-1}B$,

$$T_{-2}(A|B)A^{-1} = \frac{1}{2}\left(1_H - \left(AB^{-1}\right)^2\right)$$

and

$$T_2(A|B)A^{-1} = \frac{1}{2}\left(\left(BA^{-1}\right)^2 - 1_H\right),$$

then

$$\begin{aligned}(3.24)\quad & \frac{1}{4n(2n+1)}\left(1_H - \left(AB^{-1}\right)^2\right)^{2n+1} A \\ & \le \frac{1}{2}\left(1_H - \left(AB^{-1}\right)^2\right) + \frac{1}{2}\sum_{k=2}^{2n}\frac{(-1)^k}{k(k-1)}\left(AB^{-1}\right)^2\left(\left(BA^{-1}\right)^2 - 1_H\right)^k A \\ & - S(A|B) \\ & \le \frac{1}{4n(2n+1)}\left(AB^{-1}\right)^2\left(\left(BA^{-1}\right)^2 - 1_H\right)^{2n+1} A\end{aligned}$$

and

$$\begin{aligned}(3.25)\quad & \frac{1}{4n(2n+1)}\left(BA^{-1}\right)^2\left(1_H - \left(AB^{-1}\right)^2\right)^{2n+1} A \\ & \le \frac{1}{2}\left(\left(BA^{-1}\right)^2 - 1_H\right) - \frac{1}{2}\sum_{k=2}^{2n}\frac{1}{k(k-1)}\left(BA^{-1}\right)^2\left(1_H - \left(AB^{-1}\right)^2\right)^k A \\ & - S(A|B) \\ & \le \frac{1}{4n(2n+1)}\left(\left(BA^{-1}\right)^2 - 1_H\right)^{2n+1} A\end{aligned}$$

for any positive invertible operators A *and* B *and* $n \ge 1$.

If $B \ge A$, *then*

$$\begin{aligned}(3.26)\quad S(A|B) \le & \frac{1}{2}\left(1_H - \left(AB^{-1}\right)^2\right) \\ & + \frac{1}{2}\sum_{k=2}^{2n}\frac{(-1)^k}{k(k-1)}\left(AB^{-1}\right)^2\left(\left(BA^{-1}\right)^2 - 1_H\right)^k A\end{aligned}$$

and

$$\begin{aligned}(3.27)\quad S(A|B) \le & \frac{1}{2}\left(\left(BA^{-1}\right)^2 - 1_H\right) \\ & - \frac{1}{2}\sum_{k=2}^{2n}\frac{1}{k(k-1)}\left(BA^{-1}\right)^2\left(1_H - \left(AB^{-1}\right)^2\right)^k A\end{aligned}$$

for any $n \ge 1$.

If we take in the inequalities (3.10) and (3.11) $t = 1/2$ *and since*

$$T_{-1/2}(A|B)A^{-1} = 2\left(1_H - A(A\sharp B)^{-1}\right)$$

and

$$T_{1/2}(A|B)A^{-1} = 2\left((A\sharp B)A^{-1} - 1_H\right),$$

then we get

$$\begin{aligned}(3.28)\quad &\frac{1}{n(2n+1)}\left(1_H - A(A\sharp B)^{-1}\right)^{2n+1} A \\ &\leq 2\left(1_H - A(A\sharp B)^{-1}\right) + 2\sum_{k=2}^{2n}\frac{(-1)^k}{k(k-1)} A(A\sharp B)^{-1}\left((A\sharp B)A^{-1} - 1_H\right)^k A \\ &- S(A|B) \\ &\leq \frac{1}{n(2n+1)} A(A\sharp B)^{-1}\left(1_H - A(A\sharp B)^{-1}\right)^{2n+1} A\end{aligned}$$

and

$$\begin{aligned}(3.29)\quad &\frac{1}{n(2n+1)}(A\sharp B)A^{-1}\left(1_H - A(A\sharp B)^{-1}\right)^{2n+1} A \\ &\leq 2\left((A\sharp B)A^{-1} - 1_H\right) \\ &- 2\sum_{k=2}^{2n}\frac{1}{k(k-1)}(A\sharp B)A^{-1}\left(1_H - A(A\sharp B)^{-1}\right)^k A - S(A|B) \\ &\leq \frac{1}{n(2n+1)}\left((A\sharp B)A^{-1} - 1_H\right)^{2n+1} A\end{aligned}$$

for any $n \geq 1$.

If $B \geq A$, *then*

$$\begin{aligned}(3.30)\quad S(A|B) &\leq 2\left(1_H - A(A\sharp B)^{-1}\right) \\ &+ 2\sum_{k=2}^{2n}\frac{(-1)^k}{k(k-1)} A(A\sharp B)^{-1}\left((A\sharp B)A^{-1} - 1_H\right)^k A\end{aligned}$$

and

$$\begin{aligned}(3.31)\quad S(A|B) &\leq 2\left((A\sharp B)A^{-1} - 1_H\right) \\ &- 2\sum_{k=2}^{2n}\frac{1}{k(k-1)}(A\sharp B)A^{-1}\left(1_H - A(A\sharp B)^{-1}\right)^k A\end{aligned}$$

for any $n \geq 1$.

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