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INEQUALITIES FOR RELATIVE ENTROPY IN TERMS OF TSALLIS' ENTROPY FOR OPERATORS SATISFYING A BOUNDEDNESS CONDITION (I)

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Abstract. In this paper we obtain some inequalities for relative operator entropy $S(A|B)$ in terms of Tsallis' relative entropy $T_{\pm t}(A|B)$, $t > 0$ in the case of positive invertible operators A, B that satisfy the boundedness condition $mA \leq B \leq MA$ with $0 < m < M$.

1. Introduction

Kamei and Fujii [10], [11] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B) := A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}, \tag{1.1}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [16].

In general, we can define for positive operators A, B

$$S(A|B) := s\text{-}\lim_{\varepsilon\to 0+} S(A+\varepsilon 1_H|B)$$

if it exists, here 1_H is the identity operator.

For the entropy function $\eta(t) = -t\ln t$, the *operator entropy* has the following expression:

$$\eta(A) = -A\ln A = S(A|1_H) \geq 0$$

for positive contraction A. This shows that the relative operator entropy (1.1) is a relative version of the operator entropy.

For results on the relative operator entropy see [12, p. 149-p. 155], [1], [8], [13], [14], [15] and [17].

In [19], A. Uhlmann has shown that the relative operator entropy $S(A|B)$ can be represented as the strong limit

$$S(A|B) = s\text{-}\lim_{t\to 0}\frac{A\sharp_t B - A}{t}, \tag{1.2}$$

where

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2},\ \nu\in[0,1]$$

is the *weighted geometric mean* of positive invertible operators A and B. For $\nu = \frac{1}{2}$ we denote $A\sharp B$.

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This definition of the weighted geometric mean can be extended for any real number ν with $\nu \neq 0$.

For $t \neq 0$ and the positive invertible operators A, B we define the *Tsallis' relative entropy* (see also [7]) by

$$T_t(A|B) := \frac{A\sharp_t B - A}{t}.$$

Consider the scalar function $T_t : (0,\infty) \to \mathbb{R}$ defined for $t \neq 0$ by

$$T_t(x) := \frac{x^t - 1}{t}. \tag{1.3}$$

We have

$$T_{-t}(x) := \frac{x^{-t} - 1}{-t} = \frac{1 - x^{-t}}{t} = \frac{x^t - 1}{tx^t} = T_t(x)\, x^{-t}. \tag{1.4}$$

For positive invertible operators A and B and $t > 0$ we then have

$$A^{1/2} T_t\left(A^{-1/2} B A^{-1/2}\right) A^{1/2} = A^{1/2} \frac{\left(A^{-1/2} B A^{-1/2}\right)^t - 1_H}{t} A^{1/2} = T_t(A|B).$$

Also by (1.4) we have

$$A^{1/2} T_{-t}\left(A^{-1/2} B A^{-1/2}\right) A^{1/2} = T_{-t}(A|B)$$

and

$$\begin{aligned} & A^{1/2} T_{-t}\left(A^{-1/2} B A^{-1/2}\right) A^{1/2} \\ & = A^{1/2} T_t\left(A^{-1/2} B A^{-1/2}\right)\left(A^{-1/2} B A^{-1/2}\right)^{-t} A^{1/2} \\ & = A^{1/2} T_t\left(A^{-1/2} B A^{-1/2}\right) A^{1/2} A^{-1/2}\left(A^{-1/2} B A^{-1/2}\right)^{-t} A^{-1/2} A \\ & = T_t(A|B)\left(A^{1/2}\left(A^{-1/2} B A^{-1/2}\right)^t A^{1/2}\right)^{-1} A \\ & = T_t(A|B)\left(A\sharp_t B\right)^{-1} A \end{aligned} \tag{1.5}$$

for any positive invertible operators A and B and $t > 0$.

The following result providing upper and lower bounds for relative operator entropy in terms of $T_t(\cdot|\cdot)$ has been obtained in [10] for $0 < t \leq 1$. However, it hods for any $t > 0$.

Theorem 1 (Fujii-Kamei, 1989, [10])**.** *Let A, B be two positive invertible operators, then for any $t > 0$ we have*

$$T_{-t}(A|B) \leq S(A|B) \leq T_t(A|B). \tag{1.6}$$

In particular, we have for $t = 1$ that

$$\left(1_H - AB^{-1}\right) A \leq S(A|B) \leq B - A, \ [10] \tag{1.7}$$

and for $t = 2$ that

$$\frac{1}{2}\left(1_H - \left(AB^{-1}\right)^2\right) A \leq S(A|B) \leq \frac{1}{2}\left(BA^{-1}B - A\right). \tag{1.8}$$

The case $t = \frac{1}{2}$ is of interest as well. Since in this case we have

$$T_{1/2}(A|B) := 2\left(A\sharp B - A\right)$$

and

$$T_{-1/2}(A|B) = T_{1/2}(A|B)\left(A\sharp_{1/2}B\right)^{-1}A = 2\left(1_H - A\left(A\sharp B\right)^{-1}\right)A,$$

hence by (1.6) we get

$$2\left(1_H - A\left(A\sharp B\right)^{-1}\right)A \le S(A|B) \le 2\left(A\sharp B - A\right) \le B - A. \tag{1.9}$$

Motivated by the Fujii-Kamei inequality (1.6) we establish in this paper some new results providing Taylor's like expansion bounds for the relative operator entropy $S(A|B)$ in the case of positive invertible operators A, B that satisfy the boundedness condition $mA \le B \le MA$ with $0 < m < M$ in terms of Tsallis' relative entropy $T_{\pm t}(A|B)$ with $t > 0$.

2. Some Preliminary Results

We need the following result [3]

Lemma 1. *For any a, $b > 0$ we have for $n \ge 1$ that*

$$\frac{(b-a)^{2n}}{2n\max^{2n}\{a,b\}} \le \sum_{k=1}^{2n-1}(-1)^{k-1}\frac{(b-a)^k}{ka^k} - \ln b + \ln a \le \frac{(b-a)^{2n}}{2n\min^{2n}\{a,b\}} \tag{2.1}$$

and

$$\frac{(b-a)^{2n}}{2n\max^{2n}\{a,b\}} \le \ln b - \ln a - \sum_{k=1}^{2n-1}\frac{(b-a)^k}{kb^k} \le \frac{(b-a)^{2n}}{2n\min^{2n}\{a,b\}}. \tag{2.2}$$

Proof. For the sake of completeness, we give here a simple proof.

The following identity holds, see for instance [6] where further applications in Information Theory were provided

$$\ln b - \ln a + \sum_{k=1}^{m}\frac{(-1)^k(b-a)^k}{ka^k} = (-1)^m\int_a^b\frac{(b-t)^m}{t^{m+1}}dt. \tag{2.3}$$

for any a, $b > 0$ we have for $m \ge 1$

For $m = 2n-1$ with $n \ge 1$, then from (2.1) we have

$$\ln b - \ln a + \sum_{k=1}^{2n-1}\frac{(-1)^k(b-a)^k}{ka^k} = -\int_a^b\frac{(b-t)^{2n-1}}{t^{2n}}dt, \tag{2.4}$$

for any a, $b > 0$, giving that

$$\int_a^b\frac{(b-t)^{2n-1}}{t^{2n}}dt = \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}(b-a)^k}{ka^k} - \ln b + \ln a. \tag{2.5}$$

If $b > a > 0$, then

$$\int_a^b\frac{(b-t)^{2n-1}}{t^{2n}}dt \ge \frac{1}{b^{2n}}\int_a^b(b-t)^{2n-1}dt = \frac{(b-a)^{2n}}{2nb^{2n}} \tag{2.6}$$

and

$$\int_a^b\frac{(b-t)^{2n-1}}{t^{2n}}dt \le \frac{1}{a^{2n}}\int_a^b(b-t)^{2n-1}dt = \frac{(b-a)^{2n}}{2na^{2n}}. \tag{2.7}$$

If $a>b>0$, then

$$\int_a^b \frac{(b-t)^{2n-1}}{t^{2n}}dt = -\int_b^a \frac{(b-t)^{2n-1}}{t^{2n}}dt = \int_b^a \frac{(t-b)^{2n-1}}{t^{2n}}dt.$$

Therefore

$$\int_b^a \frac{(t-b)^{2n-1}}{t^{2n}}dt \geq \frac{1}{a^{2n}}\int_b^a (t-b)^{2n-1}\,dt = \frac{(a-b)^{2n}}{2na^{2n}} \tag{2.8}$$

and

$$\int_b^a \frac{(t-b)^{2n-1}}{t^{2n}}dt \leq \frac{1}{b^{2n}}\int_b^a (t-b)^{2n-1}\,dt = \frac{(a-b)^{2n}}{2nb^{2n}}. \tag{2.9}$$

By making use of (2.6)-(2.9), we deduce the desired result (2.1).

Now, if we replace a with b in (2.1) we get (2.2). □

Remark 1. *Since for $y>0$ we have:*

$$\frac{(y-1)^{2n}}{\max^{2n}\{1,y\}} = \left(1-\frac{\min\{1,y\}}{\max\{1,y\}}\right)^{2n},$$

$$\frac{(y-1)^{2n}}{\min^{2n}\{1,y\}} = \left(\frac{\max\{1,y\}}{\min\{1,y\}}-1\right)^{2n},$$

$$\frac{(y-1)^{2n}}{\max^{2n}\{1,y\}} = (\min\{1,y\})^{2n}\left(\frac{y-1}{y}\right)^{2n}$$

and

$$\frac{(y-1)^{2n}}{\min^{2n}\{1,y\}} = (\max\{1,y\})^{2n}\left(\frac{y-1}{y}\right)^{2n},$$

hence, by taking in Lemma 1 $b=y\in(0,\infty)$ and $a=1$, we get the inequalities

$$B_1(y,n) \leq \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}(y-1)^k - \ln y \leq B_2(y,n) \tag{2.10}$$

and

$$B_1(y,n) \leq \ln y - \sum_{k=1}^{2n-1}\frac{1}{k}\left(\frac{y-1}{y}\right)^k \leq B_2(y,n) \tag{2.11}$$

for $n\geq 1$, where the bounds

$$\begin{aligned} B_1(y,n) &:= \frac{(y-1)^{2n}}{2n\max^{2n}\{1,y\}} = \frac{1}{2n}(\min\{1,y\})^{2n}\left(\frac{y-1}{y}\right)^{2n} \\ &= \frac{1}{2n}\left(1-\frac{\min\{1,y\}}{\max\{1,y\}}\right)^{2n} \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} B_2(y,n) &:= \frac{(y-1)^{2n}}{2n\min^{2n}\{1,y\}} = \frac{1}{2n}(\max\{1,y\})^{2n}\left(\frac{y-1}{y}\right)^{2n} \\ &= \frac{1}{2n}\left(\frac{\max\{1,y\}}{\min\{1,y\}}-1\right)^{2n}. \end{aligned} \tag{2.13}$$

Assume that v, $V>0$ with $v<V$. If $y\in[v,V]\subset(0,\infty)$, then by analyzing all possible locations of the interval $[v,V]$ and 1 we have

$$\min\{1,v\}\le\min\{1,y\}\le\min\{1,V\}$$

and

$$\max\{1,v\}\le\max\{1,y\}\le\max\{1,V\}.$$

By using the inequalities (2.10) and (2.11) and the first two equalities in (2.12) and (2.13) we have *the local bounds:*

$$\frac{(y-1)^{2n}}{2n\max^{2n}\{1,V\}}\le\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}(y-1)^k-\ln y\le\frac{(y-1)^{2n}}{2n\min^{2n}\{1,v\}}, \tag{2.14}$$

$$\frac{(y-1)^{2n}}{2n\max^{2n}\{1,V\}}\le\ln y-\sum_{k=1}^{2n-1}\frac{1}{k}\left(\frac{y-1}{y}\right)^k\le\frac{(y-1)^{2n}}{2n\min^{2n}\{1,v\}}, \tag{2.15}$$

$$\begin{aligned}\frac{1}{2n}(\min\{1,v\})^{2n}\left(\frac{y-1}{y}\right)^{2n}&\le\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}(y-1)^k-\ln y\\&\le\frac{1}{2n}(\max\{1,V\})^{2n}\left(\frac{y-1}{y}\right)^{2n},\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}\frac{1}{2n}(\min\{1,v\})^{2n}\left(\frac{y-1}{y}\right)^{2n}&\le\ln y-\sum_{k=1}^{2n-1}\frac{1}{k}\left(\frac{y-1}{y}\right)^k\\&\le\frac{1}{2n}(\max\{1,V\})^{2n}\left(\frac{y-1}{y}\right)^{2n}\end{aligned} \tag{2.17}$$

for any $y\in[v,V]$ and $n\ge1$.

If $y\in[v,V]\subset(0,\infty)$, then by analyzing all possible locations of the interval $[v,V]$ and 1 we also have

$$0\le1-\frac{\min\{1,V\}}{\max\{1,v\}}\le1-\frac{\min\{1,y\}}{\max\{1,y\}}$$

and

$$0\le\frac{\max\{1,y\}}{\min\{1,y\}}-1\le\frac{\max\{1,V\}}{\min\{1,v\}}-1.$$

By using the inequalities (2.10) and (2.11) and the last equalities in (2.12) and (2.13) we have *the global bounds:*

$$\begin{aligned}\frac{1}{2n}\left(1-\frac{\min\{1,V\}}{\max\{1,v\}}\right)^{2n}&\le\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}(y-1)^k-\ln y\\&\le\frac{1}{2n}\left(\frac{\max\{1,V\}}{\min\{1,v\}}-1\right)^{2n},\end{aligned} \tag{2.18}$$

and

$$\frac{1}{2n}\left(1-\frac{\min\{1,V\}}{\max\{1,v\}}\right)^{2n} \le \ln y-\sum_{k=1}^{2n-1}\frac{1}{k}\left(\frac{y-1}{y}\right)^{k} \le \frac{1}{2n}\left(\frac{\max\{1,V\}}{\min\{1,v\}}-1\right)^{2n}, \tag{2.19}$$

for any $y\in[v,V]$ and $n\ge 1$.

3. Local Operator Inequalities

The following operator inequality holds:

Theorem 2. *Let A, B be two positive invertible operators satisfying the boundedness condition*

$$mA\le B\le MA \tag{3.1}$$

for some constants $m,M>0$ with $m<M$. Then for any $t>0$ and $n\ge 1$ we have

$$\begin{aligned}&\frac{t^{2n-1}}{2n\max^{2n}\{1,M^t\}}\left(T_t(A|B)A^{-1}\right)^{2n}A\\ &\le \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}t^{k-1}\left(T_t(A|B)A^{-1}\right)^{k}A-S(A|B)\\ &\le \frac{t^{2n-1}}{2n\min^{2n}\{1,m^t\}}\left(T_t(A|B)A^{-1}\right)^{2n}A\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}&\frac{t^{2n-1}}{2n\max^{2n}\{1,M^t\}}\left(T_t(A|B)A^{-1}\right)^{2n}A\\ &\le S(A|B)-\sum_{k=1}^{2n-1}\frac{1}{k}t^{k-1}\left(T_{-t}(A|B)A^{-1}\right)^{k}A\\ &\le \frac{t^{2n-1}}{2n\min^{2n}\{1,m^t\}}\left(T_t(A|B)A^{-1}\right)^{2n}A.\end{aligned} \tag{3.3}$$

Proof. Let $x\in[m,M]\subset(0,\infty)$ and for $t>0$ put $y=x^t\in[m^t,M^t]$. Then by (2.14) and (2.15) for $v=m^t$ and $V=M^t$ we have

$$\frac{(x^t-1)^{2n}}{2n\max^{2n}\{1,M^t\}}\le\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}\left(x^t-1\right)^k-\ln x^t\le\frac{(x^t-1)^{2n}}{2n\min^{2n}\{1,m^t\}},$$

and

$$\frac{(x^t-1)^{2n}}{2n\max^{2n}\{1,M^t\}}\le\ln y-\sum_{k=1}^{2n-1}\frac{1}{k}\left(\frac{x^t-1}{x^t}\right)^k\le\frac{(x^t-1)^{2n}}{2n\min^{2n}\{1,m^t\}},$$

for any $x\in[m,M]$, $t>0$ and $n\ge 1$.

These inequalities are equivalent to

$$\begin{aligned}\frac{t^{2n-1}}{2n\max^{2n}\{1,M^t\}}\left(\frac{x^t-1}{t}\right)^{2n} &\le \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}t^{k-1}\left(\frac{x^t-1}{t}\right)^k-\ln x\\ &\le \frac{t^{2n-1}}{2n\min^{2n}\{1,m^t\}}\left(\frac{x^t-1}{t}\right)^{2n},\end{aligned}$$

and

$$\begin{aligned}\frac{t^{2n-1}}{2n\max^{2n}\{1,M^t\}}\left(\frac{x^t-1}{t}\right)^{2n} &\le \ln x-\sum_{k=1}^{2n-1}\frac{1}{k}t^{k-1}\left(\frac{x^t-1}{tx^t}\right)^k\\ &\le \frac{t^{2n-1}}{2n\min^{2n}\{1,m^t\}}\left(\frac{x^t-1}{t}\right)^{2n},\end{aligned}$$

for any $x\in[m,M]$, $t>0$ and $n\ge 1$.

These inequalities can the be written as

$$\begin{aligned}\frac{t^{2n-1}}{2n\max^{2n}\{1,M^t\}}(T_t(x))^{2n} &\le \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}t^{k-1}(T_t(x))^k-\ln x\\ &\le \frac{t^{2n-1}}{2n\min^{2n}\{1,m^t\}}(T_t(x))^{2n},\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}\frac{t^{2n-1}}{2n\max^{2n}\{1,M^t\}}(T_t(x))^{2n} &\le \ln x-\sum_{k=1}^{2n-1}\frac{1}{k}t^{k-1}(T_{-t}(x))^k\\ &\le \frac{t^{2n-1}}{2n\min^{2n}\{1,m^t\}}(T_t(x))^{2n},\end{aligned}\tag{3.5}$$

for any $x\in[m,M]$, $t>0$ and $n\ge 1$.

Using the continuous functional calculus for the positive invertible operator X with spectrum in $[m,M]$ we have by (3.4) that

$$\begin{aligned}\frac{t^{2n-1}}{2n\max^{2n}\{1,M^t\}}(T_t(X))^{2n} &\le \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}t^{k-1}(T_t(X))^k-\ln X\\ &\le \frac{t^{2n-1}}{2n\min^{2n}\{1,m^t\}}(T_t(X))^{2n},\end{aligned}\tag{3.6}$$

for any $t>0$ and $n\ge 1$.

If in the inequality (3.1) we multiply both sides by $A^{-1/2}$ we get $m1_H\le A^{-1/2}BA^{-1/2}\le M1_H$ and by (3.6) we obtain

$$\begin{aligned}&\frac{t^{2n-1}}{2n\max^{2n}\{1,M^t\}}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n}\\ &\le \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}t^{k-1}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^k-\ln\left(A^{-1/2}BA^{-1/2}\right)\\ &\le \frac{t^{2n-1}}{2n\min^{2n}\{1,m^t\}}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n},\end{aligned}\tag{3.7}$$

for any $t>0$ and $n\ge 1$.

If we multiply this inequality in both sides by $A^{1/2}$, then we get

$$\begin{aligned} &\frac{t^{2n-1}}{2n\max^{2n}\{1,M^t\}}A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n}A^{1/2} \\ &\leq \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}t^{k-1}A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{k}A^{1/2} \\ &-A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2} \\ &\leq \frac{t^{2n-1}}{2n\min^{2n}\{1,m^t\}}A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n}A^{1/2}, \end{aligned} \tag{3.8}$$

for any $t>0$ and $n\geq 1$.

Observe that for $k=1$ we have

$$A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{k}A^{1/2} = A^{1/2}T_t\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} = T_t(A|B).$$

For $k\geq 2$ we have

$$\begin{aligned} &A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{k}A^{1/2} \\ &= A^{1/2}\left(A^{-1/2}A^{1/2}T_t\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}A^{-1/2}\right)^{k}A^{1/2} \\ &= A^{1/2}\left(A^{-1/2}T_t(A|B)\,A^{-1/2}\right)^{k}A^{1/2} \\ &= A^{1/2}A^{-1/2}T_t(A|B)\,A^{-1/2}...A^{-1/2}T_t(A|B)\,A^{-1/2}A^{1/2} \\ &= T_t(A|B)\,A^{-1}...T_t(A|B)\,A^{-1/2}A^{1/2} \\ &= T_t(A|B)\,A^{-1}...T_t(A|B)\,A^{-1}A = \left(T_t(A|B)\,A^{-1}\right)^{k}A. \end{aligned}$$

We observe that, this formula also holds for $k=1$, therefore for any $k\geq 1$ we have

$$A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{k}A^{1/2} = \left(T_t(A|B)\,A^{-1}\right)^{k}A.$$

By using the inequality (3.8) we deduce the desired result (3.2).

Similarly, for any $k\geq 1$ we have

$$A^{1/2}\left(T_{-t}\left(A^{-1/2}BA^{-1/2}\right)\right)^{k}A^{1/2} = \left(T_{-t}(A|B)\,A^{-1}\right)^{k}A.$$

Making use of the inequality (3.5) and a similar argument we deduce the second result (3.3). □

If we take $n=1$ in (3.2) and (3.3), then we get for any $t>0$

$$\begin{aligned} \frac{t}{2\max^2\{1,M^t\}}\left(T_t(A|B)\,A^{-1}\right)^2A &\leq T_t(A|B)-S(A|B) \\ &\leq \frac{t}{2\min^2\{1,m^t\}}\left(T_t(A|B)\,A^{-1}\right)^2A \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \frac{t}{2\max^2\{1,M^t\}}\left(T_t(A|B)\,A^{-1}\right)^2A &\leq S(A|B)-T_{-t}(A|B) \\ &\leq \frac{t}{2\min^2\{1,m^t\}}\left(T_t(A|B)\,A^{-1}\right)^2A, \end{aligned} \tag{3.10}$$

where A, B are two positive invertible operators satisfying the condition (3.1).

From Fujii-Kamei's inequalities we know that the operators

$$R_t(A|B) := T_t(A|B) - S(A|B)$$

and

$$L_t(A|B) := S(A|B) - T_{-t}(A|B)$$

are positive for A, B two positive invertible operators and $t > 0$. Moreover, if A, B satisfy the condition (3.1), then we have by (3.9) and (3.10) some positive lower bounds and upper bounds for $R_t(A|B)$ and $L_t(A|B)$.

If we take $n = 2$ in (3.2) and (3.3), then we get for any $t > 0$

$$\begin{aligned} &\frac{t^3}{4\max^4\{1, M^t\}}\left(T_t(A|B)A^{-1}\right)^4 A \\ &\le T_t(A|B) - \frac{1}{2}t\left(T_t(A|B)A^{-1}\right)^2 A - S(A|B) \\ &\le \frac{t^3}{4\min^4\{1, M^t\}}\left(T_t(A|B)A^{-1}\right)^4 A \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &\frac{t^3}{4\max^4\{1, M^t\}}\left(T_t(A|B)A^{-1}\right)^4 A \\ &\le S(A|B) - T_{-t}(A|B) - \frac{1}{2}t\left(T_{-t}(A|B)A^{-1}\right)^2 A \\ &\le \frac{t^3}{4\min^4\{1, M^t\}}\left(T_t(A|B)A^{-1}\right)^4 A. \end{aligned} \tag{3.12}$$

where A, B are two positive invertible operators satisfying the condition (3.1).

Similar comments apply by utilising the inequalities (3.11) and (3.12).

Remark 2. *If we take in the inequalities (3.2) and (3.3) $t = 1$ and since*

$$T_{-1}(A|B)A^{-1} = 1_H - AB^{-1} \text{ and } T_1(A|B)A^{-1} = BA^{-1} - 1_H,$$

then we get

$$\begin{aligned} &\frac{1}{2n\max^{2n}\{1, M\}}\left(BA^{-1} - 1_H\right)^{2n} A \\ &\le \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}\left(BA^{-1} - 1_H\right)^k A - S(A|B) \\ &\le \frac{1}{2n\min^{2n}\{1, m\}}\left(BA^{-1} - 1_H\right)^{2n} A \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \frac{1}{2n\max^{2n}\{1, M\}}\left(BA^{-1} - 1_H\right)^{2n} A &\le S(A|B) - \sum_{k=1}^{2n-1}\frac{1}{k}\left(1_H - AB^{-1}\right)^k A \\ &\le \frac{1}{2n\min^{2n}\{1, m\}}\left(BA^{-1} - 1_H\right)^{2n} A, \end{aligned} \tag{3.14}$$

where A, B are two positive invertible operators satisfying the condition (3.1).

If we take in the inequalities (3.2) and (3.3) $t=2$ and since

$$T_{-2}(A|B)A^{-1}=\frac{1}{2}\left(1_H-\left(AB^{-1}\right)^2\right)$$

and

$$T_2(A|B)A^{-1}=\frac{1}{2}\left(\left(BA^{-1}\right)^2-1_H\right),$$

then we get

$$\begin{aligned}\frac{1}{4n\max^{2n}\{1,M^2\}}&\left(\left(BA^{-1}\right)^2-1_H\right)^{2n}A \\ &\leq \frac{1}{2}\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}\left(\left(BA^{-1}\right)^2-1_H\right)^{k}A-S(A|B) \\ &\leq \frac{1}{4n\min^{2n}\{1,M^2\}}\left(\left(BA^{-1}\right)^2-1_H\right)^{2n}A\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}\frac{1}{4n\max^{2n}\{1,M^2\}}&\left(\left(BA^{-1}\right)^2-1_H\right)^{2n}A \\ &\leq S(A|B)-\frac{1}{2}\sum_{k=1}^{2n-1}\frac{1}{k}\left(1_H-\left(AB^{-1}\right)^2\right)^{k}A \\ &\leq \frac{1}{4n\min^{2n}\{1,m^2\}}\left(\left(BA^{-1}\right)^2-1_H\right)^{2n}A\end{aligned}\tag{3.16}$$

where A, B are two positive invertible operators satisfying the condition (3.1).

Since for $t=1/2$ we have

$$T_{-1/2}(A|B)A^{-1}=2\left(1_H-A\left(A\sharp B\right)^{-1}\right)$$

and

$$T_{1/2}(A|B)A^{-1}=2\left(\left(A\sharp B\right)A^{-1}-1_H\right),$$

then by the inequalities (3.2) and (3.3) we get

$$\begin{aligned}\frac{1}{n\max^{2n}\left\{1,\sqrt{M}\right\}}&\left(\left(A\sharp B\right)A^{-1}-1_H\right)^{2n}A \\ &\leq 2\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}\left(\left(A\sharp B\right)A^{-1}-1_H\right)^{k}A-S(A|B) \\ &\leq \frac{1}{n\min^{2n}\{1,\sqrt{m}\}}\left(\left(A\sharp B\right)A^{-1}-1_H\right)^{2n}A\end{aligned}\tag{3.17}$$

and

$$\begin{aligned}\frac{1}{n\max^{2n}\left\{1,\sqrt{M}\right\}}&\left(\left(A\sharp B\right)A^{-1}-1_H\right)^{2n}A \\ &\leq S(A|B)-2\sum_{k=1}^{2n-1}\frac{1}{k}\left(1_H-A\left(A\sharp B\right)^{-1}\right)^{k}A \\ &\leq \frac{1}{n\min^{2n}\{1,\sqrt{m}\}}\left(\left(A\sharp B\right)A^{-1}-1_H\right)^{2n}A,\end{aligned}\tag{3.18}$$

where A, B are two positive invertible operators satisfying the condition (3.1).

Theorem 3. *Let A, B be two positive invertible operators satisfying the boundedness condition (3.1) for some constants m, $M > 0$ with $m < M$. Then for any $t > 0$ and $n \geq 1$ we have*

$$\begin{aligned}
&\frac{1}{2n}\left(\min\left\{1, m^t\right\}\right)^{2n} t^{2n-1}\left(T_{-t}\left(A|B\right)A^{-1}\right)^{2n} A \\
&\leq \sum_{k=1}^{2n-1}\frac{\left(-1\right)^{k-1}}{k} t^{k-1}\left(T_t\left(A|B\right)A^{-1}\right)^{k} A - S\left(A|B\right) \\
&\leq \frac{1}{2n}\left(\max\left\{1, M^t\right\}\right)^{2n} t^{2n-1}\left(T_{-t}\left(A|B\right)A^{-1}\right)^{2n} A
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
&\frac{1}{2n}\left(\min\left\{1, m^t\right\}\right)^{2n} t^{2n-1}\left(T_{-t}\left(A|B\right)A^{-1}\right)^{2n} A \\
&\leq S\left(A|B\right) - \sum_{k=1}^{2n-1}\frac{1}{k} t^{k-1}\left(T_{-t}\left(A|B\right)A^{-1}\right)^{k} A \\
&\leq \frac{1}{2n}\left(\max\left\{1, M^t\right\}\right)^{2n} t^{2n-1}\left(T_{-t}\left(A|B\right)A^{-1}\right)^{2n} A.
\end{aligned} \tag{3.20}$$

Proof. Let $x \in [m, M] \subset (0, \infty)$ and for $t > 0$ put $y = x^t \in [m^t, M^t]$. Then by (2.16) and (2.17) for $v = m^t$ and $V = M^t$ we have

$$\begin{aligned}
\frac{1}{2n}\left(\min\left\{1, m^t\right\}\right)^{2n}\left(\frac{x^t-1}{x^t}\right)^{2n} &\leq \sum_{k=1}^{2n-1}\frac{\left(-1\right)^{k-1}}{k}\left(x^t-1\right)^k - \ln x^t \\
&\leq \frac{1}{2n}\left(\max\left\{1, M^t\right\}\right)^{2n}\left(\frac{x^t-1}{x^t}\right)^{2n},
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2n}\left(\min\left\{1, m^t\right\}\right)^{2n}\left(\frac{x^t-1}{x^t}\right)^{2n} &\leq \ln x^t - \sum_{k=1}^{2n-1}\frac{1}{k}\left(\frac{x^t-1}{x^t}\right)^k \\
&\leq \frac{1}{2n}\left(\max\left\{1, M^t\right\}\right)^{2n}\left(\frac{x^t-1}{x^t}\right)^{2n}.
\end{aligned}$$

These inequalities can be written as

$$\begin{aligned}
&\frac{1}{2n}\left(\min\left\{1, m^t\right\}\right)^{2n} t^{2n-1}\left(T_{-t}\left(x\right)\right)^{2n} \\
&\leq \sum_{k=1}^{2n-1}\frac{\left(-1\right)^{k-1}}{k} t^{k-1}\left(T_t\left(x\right)\right)^{k} - \ln x \\
&\leq \frac{1}{2n}\left(\max\left\{1, M^t\right\}\right)^{2n} t^{2n-1}\left(T_{-t}\left(x\right)\right)^{2n},
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned} &\frac{1}{2n}\left(\min\left\{1,m^t\right\}\right)^{2n} t^{2n-1}\left(T_{-t}(x)\right)^{2n} \\ &\leq \ln x-\sum_{k=1}^{2n-1}\frac{1}{k}t^{k-1}\left(T_{-t}(x)\right)^k \\ &\leq \frac{1}{2n}\left(\max\left\{1,M^t\right\}\right)^{2n} t^{2n-1}\left(T_{-t}(x)\right)^{2n}, \end{aligned} \tag{3.22}$$

for any $x\in[m,M]\subset(0,\infty)$, $n\geq 1$ and for $t>0$.

On making use of a similar argument to the one in the proof of Theorem (2) we obtain from (3.21) and (3.22) the desired results (3.19) and (3.20). □

For $n=1$ we get from (3.19) and (3.20) for $t>0$ that

$$\begin{aligned} &\frac{1}{2}\left(\min\left\{1,m^t\right\}\right)^2 t\left(T_{-t}(A|B)A^{-1}\right)^2 A \\ &\leq T_t(A|B)-S(A|B) \\ &\leq \frac{1}{2}\left(\max\left\{1,M^t\right\}\right)^2 t\left(T_{-t}(A|B)A^{-1}\right)^2 A \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} &\frac{1}{2}\left(\min\left\{1,m^t\right\}\right)^2 t\left(T_{-t}(A|B)A^{-1}\right)^2 A \\ &\leq S(A|B)-T_{-t}(A|B) \\ &\leq \frac{1}{2}\left(\max\left\{1,M^t\right\}\right)^2 t\left(T_{-t}(A|B)A^{-1}\right)^2 A, \end{aligned} \tag{3.24}$$

where A, B are two positive invertible operators satisfying the condition (3.1).

We notice that the inequalities (3.23) and (3.24) provide other lower and upper bounds for the positive operators $R_t(A|B)$ and $L_t(A|B)$ defined above.

If we take in (3.19) and (3.20) for $t=1$, then we get

$$\begin{aligned} &\frac{1}{2n}\left(\min\{1,m\}\right)^{2n}\left(1_H-AB^{-1}\right)^{2n} A \\ &\leq \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}\left(BA^{-1}-1_H\right)^k A-S(A|B) \\ &\leq \frac{1}{2n}\left(\max\{1,M\}\right)^{2n}\left(1_H-AB^{-1}\right)^{2n} A \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} &\frac{1}{2n}\left(\min\{1,m\}\right)^{2n}\left(1_H-AB^{-1}\right)^{2n} A \\ &\leq S(A|B)-\sum_{k=1}^{2n-1}\frac{1}{k}\left(1_H-AB^{-1}\right)^k A \\ &\leq \frac{1}{2n}\left(\max\{1,M\}\right)^{2n}\left(1_H-AB^{-1}\right)^{2n} A \end{aligned} \tag{3.26}$$

for any $n\geq 1$, where A, B are two positive invertible operators satisfying the condition (3.1).

4. Global Operator Inequalities

We have:

Theorem 4. *Let A, B be two positive invertible operators satisfying the boundedness condition (3.1) for some constants m, $M>0$ with $m<M$. Then for any $t>0$ and $n\geq 1$ we have*

$$\begin{aligned}(4.1)\qquad & \frac{1}{2n}t^{2n-1}T_t^{2n}\left(\frac{\min\{1,M\}}{\max\{1,m\}}\right)A \\ & \leq \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}t^{k-1}\left(T_t(A|B)A^{-1}\right)^k A-S(A|B) \\ & \leq \frac{1}{2n}t^{2n-1}T_t^{2n}\left(\frac{\max\{1,M\}}{\min\{1,m\}}\right)A\end{aligned}$$

and

$$\begin{aligned}(4.2)\quad \frac{1}{2n}t^{2n-1}T_t^{2n}\left(\frac{\min\{1,M\}}{\max\{1,m\}}\right)A & \leq S(A|B)-\sum_{k=1}^{2n-1}\frac{1}{k}t^{k-1}\left(T_{-t}(A|B)A^{-1}\right)^k A \\ & \leq \frac{1}{2n}t^{2n-1}T_t^{2n}\left(\frac{\max\{1,M\}}{\min\{1,m\}}\right)A,\end{aligned}$$

where T_t is defined by (1.3).

Proof. Let $x\in[m,M]\subset(0,\infty)$ and for $t>0$ put $y=x^t\in[m^t,M^t]$. Then by (2.18) and (2.19) for $v=m^t$ and $V=M^t$ we have

$$\begin{aligned}\frac{1}{2n}\left(1-\frac{\min\{1,M^t\}}{\max\{1,m^t\}}\right)^{2n} & \leq \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}\left(x^t-1\right)^k-\ln x^t \\ & \leq \frac{1}{2n}\left(\frac{\max\{1,M^t\}}{\min\{1,m^t\}}-1\right)^{2n},\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2n}\left(1-\frac{\min\{1,M^t\}}{\max\{1,m^t\}}\right)^{2n} & \leq \ln x^t-\sum_{k=1}^{2n-1}\frac{1}{k}\left(\frac{x^t-1}{x^t}\right)^k \\ & \leq \frac{1}{2n}\left(\frac{\max\{1,M^t\}}{\min\{1,m^t\}}-1\right)^{2n},\end{aligned}$$

for any $n\geq 1$.

These inequalities may be written as

$$\begin{aligned}\frac{1}{2n}\left(1-\left(\frac{\min\{1,M\}}{\max\{1,m\}}\right)^t\right)^{2n} & \leq \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}\left(x^t-1\right)^k-\ln x^t \\ & \leq \frac{1}{2n}\left(\left(\frac{\max\{1,M\}}{\min\{1,m\}}\right)^t-1\right)^{2n},\end{aligned}$$

and

$$\frac{1}{2n}\left(1-\left(\frac{\min\{1,M\}}{\max\{1,m\}}\right)^{t}\right)^{2n} \le \ln x^{t}-\sum_{k=1}^{2n-1}\frac{1}{k}\left(\frac{x^{t}-1}{x^{t}}\right)^{k} \le \frac{1}{2n}\left(\left(\frac{\max\{1,M\}}{\min\{1,m\}}\right)^{t}-1\right)^{2n}$$

or, in terms of $T_{\pm t}$ as

$$\frac{1}{2n}t^{2n-1}T_t^{2n}\left(\frac{\min\{1,M\}}{\max\{1,m\}}\right) \le \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}t^{k-1}\left(T_t(x)\right)^{k}-\ln x \le \frac{1}{2n}t^{2n-1}T_t^{2n}\left(\frac{\max\{1,M\}}{\min\{1,m\}}\right), \tag{4.3}$$

and

$$\frac{1}{2n}t^{2n-1}T_t^{2n}\left(\frac{\min\{1,M\}}{\max\{1,m\}}\right) \le \ln x-\sum_{k=1}^{2n-1}\frac{1}{k}t^{k-1}\left(T_{-t}(x)\right)^{k} \le \frac{1}{2n}t^{2n-1}T_t^{2n}\left(\frac{\max\{1,M\}}{\min\{1,m\}}\right) \tag{4.4}$$

for $x\in[m,M]\subset(0,\infty)$, $n\ge 1$ and for $t>0$.

On making use of a similar argument to the one in the proof of Theorem (2) we obtain from (4.3) and (4.4) the desired results (4.1) and (4.2). □

For $n=1$, we get from (4.1) and (4.2) that

$$\frac{1}{2}tT_t^{2}\left(\frac{\min\{1,M\}}{\max\{1,m\}}\right)A \le T_t(A|B)-S(A|B) \le \frac{1}{2}tT_t^{2}\left(\frac{\max\{1,M\}}{\min\{1,m\}}\right)A \tag{4.5}$$

and

$$\frac{1}{2}tT_t^{2}\left(\frac{\min\{1,M\}}{\max\{1,m\}}\right)A \le S(A|B)-T_{-t}(A|B) \le \frac{1}{2}tT_t^{2}\left(\frac{\max\{1,M\}}{\min\{1,m\}}\right)A, \tag{4.6}$$

for any $t>0$, where A, B are two positive invertible operators satisfying the condition (3.1).

If we take in (4.1) and (4.2) $t=1$, then we get

$$\frac{1}{2n}\left(1-\frac{\min\{1,M\}}{\max\{1,m\}}\right)^{2n}A \le \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}\left(BA^{-1}-1_H\right)^{k}A-S(A|B) \le \frac{1}{2n}\left(\frac{\max\{1,M\}}{\min\{1,m\}}-1\right)^{2n}A \tag{4.7}$$

and

$$\frac{1}{2n}\left(1-\frac{\min\{1,M\}}{\max\{1,m\}}\right)^{2n}A \le S(A|B)-\sum_{k=1}^{2n-1}\frac{1}{k}\left(1_H-AB^{-1}\right)^{k}A \le \frac{1}{2n}\left(\frac{\max\{1,M\}}{\min\{1,m\}}-1\right)^{2n}A, \tag{4.8}$$

for any $n\ge 1$, where A, B are two positive invertible operators satisfying the condition (3.1).

Finally, we can prove some inequalities that do not require the assumption (3.1) in order to get upper and lower bounds for the relative operator entropy:

Theorem 5. *For any A, B two positive invertible operators and for any $t>0$ and $n \geq 1$ we have*

$$\begin{aligned}\sum_{k=1}^{2n-1} \frac{1}{k} t^{k-1} \left(T_{-t}(A|B) A^{-1}\right)^k A &\leq S(A|B) \\ &\leq \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} t^{k-1} \left(T_t(A|B) A^{-1}\right)^k A.\end{aligned} \tag{4.9}$$

Proof. If we use the first inequalities in (2.1) and (2.2) we have

$$0 \leq \frac{(b-a)^{2n}}{2n \max^{2n}\{a,b\}} \leq \sum_{k=1}^{2n-1} (-1)^{k-1} \frac{(b-a)^k}{ka^k} - \ln b + \ln a$$

and

$$0 \leq \frac{(b-a)^{2n}}{2n \max^{2n}\{a,b\}} \leq \ln b - \ln a - \sum_{k=1}^{2n-1} \frac{(b-a)^k}{kb^k}$$

for any a, $b>0$.

These inequalities imply for $b=x$ and $a=1$ that

$$\sum_{k=1}^{2n-1} \frac{(x-1)^k}{kx^k} \leq \ln x \leq \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} (x-1)^k, \tag{4.10}$$

for any $x>0$ and $n \geq 1$.

Now, by using the functional calculus for invertible positive operators, we obtain the desired result (4.9).

The details are omitted. □

For $n=1$, we recapture from (4.9) the Fujii-Kamei's result (1.6).

For $n=2$, we get from (4.9) that

$$\begin{aligned}(T_{-t}(A|B) \leq) T_{-t}(A|B) &+ \frac{1}{2} t \left(T_{-t}(A|B) A^{-1}\right)^2 A \\ &\leq S(A|B) \\ &\leq T_t(A|B) - \frac{1}{2} t \left(T_t(A|B) A^{-1}\right)^2 A \left(\leq T_t(A|B)\right),\end{aligned} \tag{4.11}$$

for any A, B two positive invertible operators and for any $t>0$, which provides a refinement for each of the inequalities in (1.6).

For $t=1$ in (4.9) we get

$$\sum_{k=1}^{2n-1} \frac{1}{k} \left(1_H - AB^{-1}\right)^k A \leq S(A|B) \leq \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} \left(BA^{-1} - 1_H\right)^k A \tag{4.12}$$

for any A, B two positive invertible operators.

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