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INEQUALITIES FOR RELATIVE ENTROPY IN TERMS OF TSALLIS' ENTROPY FOR OPERATORS SATISFYING A BOUNDEDNESS CONDITION (II)

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Abstract. In this paper, by building on the previous work with (I) in the title above, we obtain some new inequalities for relative operator entropy $S(A|B)$ in terms of Tsallis' relative entropy $T_{\pm t}(A|B)$, $t>0$ in the case of positive invertible operators A, B that satisfy the boundedness condition $mA \leq B \leq MA$ with $0<m<M$.

1. Introduction

Kamei and Fujii [12], [13] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B) := A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}, \tag{1.1}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [18].

In general, we can define for positive operators A, B

$$S(A|B) := s\text{-}\lim_{\varepsilon\to 0+} S(A+\varepsilon 1_H|B)$$

if it exists, here 1_H is the identity operator.

For the entropy function $\eta(t) = -t\ln t$, the *operator entropy* has the following expression:

$$\eta(A) = -A\ln A = S(A|1_H) \geq 0$$

for positive contraction A. This shows that the relative operator entropy (1.1) is a relative version of the operator entropy.

For results on the relative operator entropy see [14, p. 149-p. 155], [1], [10], [15], [16], [17] and [19].

In [21], A. Uhlmann has shown that the relative operator entropy $S(A|B)$ can be represented as the strong limit

$$S(A|B) = s\text{-}\lim_{t\to 0}\frac{A\sharp_t B - A}{t}, \tag{1.2}$$

where

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu}A^{1/2},\ \nu\in[0,1]$$

is the *weighted geometric mean* of positive invertible operators A and B. For $\nu=\frac{1}{2}$ we denote $A\sharp B$.

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This definition of the weighted geometric mean can be extended for any real number ν with $\nu \neq 0$.

For $t \neq 0$ and the positive invertible operators A, B we define the *Tsallis' relative entropy* (see also [9]) by

$$T_t(A|B) := \frac{A\sharp_t B - A}{t}.$$

Consider the scalar function $T_t : (0, \infty) \to \mathbb{R}$ defined for $t \neq 0$ by

$$T_t(x) := \frac{x^t - 1}{t}. \tag{1.3}$$

We have

$$T_{-t}(x) = \frac{x^t - 1}{tx^t} = T_t(x)\, x^{-t}. \tag{1.4}$$

For positive invertible operators A and B and $t > 0$ we then have

$$A^{1/2} T_t\left(A^{-1/2} B A^{-1/2}\right) A^{1/2} = T_t(A|B).$$

Also by (1.4) we have

$$A^{1/2} T_{-t}\left(A^{-1/2} B A^{-1/2}\right) A^{1/2} = T_{-t}(A|B)$$

and

$$A^{1/2} T_{-t}\left(A^{-1/2} B A^{-1/2}\right) A^{1/2} = T_t(A|B)\left(A\sharp_t B\right)^{-1} A \tag{1.5}$$

for any positive invertible operators A and B and $t > 0$.

The following result providing upper and lower bounds for relative operator entropy in terms of $T_t(\cdot|\cdot)$ has been obtained in [12] for $0 < t \leq 1$. However, it hods for any $t > 0$.

Theorem 1 (Fujii-Kamei, 1989, [12])**.** *Let A, B be two positive invertible operators, then for any $t > 0$ we have*

$$T_{-t}(A|B) \leq S(A|B) \leq T_t(A|B). \tag{1.6}$$

In particular, we have for $t = 1$ that

$$\left(1_H - AB^{-1}\right) A \leq S(A|B) \leq B - A, \ [12]. \tag{1.7}$$

In the previous paper [7], we obtained among others the following result:

Theorem 2 (Dragomir, 2016, [7])**.** *Let A, B be two positive invertible operators satisfying the boundedness condition*

$$mA \leq B \leq MA \tag{1.8}$$

for some constants m, $M > 0$ with $m < M$. Then for any $t > 0$ and $n \geq 1$ we have

$$\begin{aligned} &\frac{t^{2n-1}}{2n \max^{2n}\{1, M^t\}} \left(T_t(A|B) A^{-1}\right)^{2n} A \\ &\leq \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} t^{k-1} \left(T_t(A|B) A^{-1}\right)^k A - S(A|B) \\ &\leq \frac{t^{2n-1}}{2n \min^{2n}\{1, m^t\}} \left(T_t(A|B) A^{-1}\right)^{2n} A \end{aligned} \tag{1.9}$$

and

$$\begin{aligned}\tag{1.10} &\frac{t^{2n-1}}{2n\max^{2n}\{1,M^t\}}\left(T_t(A|B)A^{-1}\right)^{2n}A \\ &\leq S(A|B)-\sum_{k=1}^{2n-1}\frac{1}{k}t^{k-1}\left(T_{-t}(A|B)A^{-1}\right)^kA \\ &\leq \frac{t^{2n-1}}{2n\min^{2n}\{1,m^t\}}\left(T_t(A|B)A^{-1}\right)^{2n}A.\end{aligned}$$

If we take in the inequalities (1.9) and (1.10) $t=1$ and since

$$T_{-1}(A|B)A^{-1}=1_H-AB^{-1} \text{ and } T_1(A|B)A^{-1}=BA^{-1}-1_H,$$

then we get

$$\begin{aligned}\tag{1.11} &\frac{1}{2n\max^{2n}\{1,M\}}\left(BA^{-1}-1_H\right)^{2n}A \\ &\leq \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}\left(BA^{-1}-1_H\right)^kA-S(A|B) \\ &\leq \frac{1}{2n\min^{2n}\{1,m\}}\left(BA^{-1}-1_H\right)^{2n}A\end{aligned}$$

and

$$\begin{aligned}\tag{1.12} \frac{1}{2n\max^{2n}\{1,M\}}\left(BA^{-1}-1_H\right)^{2n}A &\leq S(A|B)-\sum_{k=1}^{2n-1}\frac{1}{k}\left(1_H-AB^{-1}\right)^kA \\ &\leq \frac{1}{2n\min^{2n}\{1,m\}}\left(BA^{-1}-1_H\right)^{2n}A,\end{aligned}$$

where A, B are two positive invertible operators satisfying the condition (1.8).

Motivated by the above results we establish in this paper some new results providing Taylor's like expansion bounds for the relative operator entropy $S(A|B)$ in the case of positive invertible operators A, B that satisfy the boundedness condition $mA\leq B\leq MA$ with $0<m<M$ in terms of Tsallis' relative entropy $T_{\pm t}(A|B)$ with $t>0$.

2. Some Preliminary Results

The following theorem is well known in the literature as Taylor's theorem with the integral remainder.

Theorem 3. *Let $I\subset\mathbb{R}$ be a closed interval, $a\in I$ and let m be a positive integer. If $f:I\longrightarrow\mathbb{R}$ is such that $f^{(m)}$ is absolutely continuous on I, then for each $x\in I$*

$$f(x)=T_m(f;a,x)+R_m(f;a,x), \tag{2.1}$$

where $T_m(f;a,x)$ is Taylor's polynomial, i.e.,

$$T_m(f;a,x):=\sum_{k=0}^{m}\frac{(x-a)^k}{k!}f^{(k)}(a).$$

Note that $f^{(0)} := f$ *and* $0! := 1$, *and the remainder is given by*

$$R_m(f; a, x) := \frac{1}{m!} \int_a^x (x-t)^m f^{(m+1)}(t)\, dt.$$

We need the following result that has been obtained in [5]:

Lemma 1. *For any* $a,\ b > 0$ *we have that*

$$\frac{1}{2b \max\{a,b\}} (b-a)^2 \le \ln b - \ln a - \frac{b-a}{b} \le \frac{1}{2b \min\{a,b\}} (b-a)^2 \tag{2.2}$$

and

$$\frac{1}{2a \max\{a,b\}} (b-a)^2 \le \frac{b-a}{a} - \ln b + \ln a \le \frac{1}{2a \min\{a,b\}} (b-a)^2 . \tag{2.3}$$

If $n \ge 1$, *then for any* $a,\ b > 0$ *we have that*

$$\begin{aligned} &\frac{(b-a)^{2n+2}}{(2n+1)(2n+2)\, b \max^{2n+1}\{a,b\}} \\ &\le \ln b - \ln a - \frac{b-a}{b} + \frac{1}{b} \sum_{k=2}^{2n+1} \frac{(-1)^{k-1}}{k(k-1)} \frac{(b-a)^k}{a^{k-1}} \\ &\le \frac{(b-a)^{2n+2}}{(2n+1)(2n+2)\, b \min^{2n+1}\{a,b\}} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &\frac{(b-a)^{2n+2}}{(2n+1)(2n+2)\, a \max^{2n+1}\{a,b\}} \\ &\le \frac{b-a}{a} - \frac{1}{a} \sum_{k=2}^{2n+1} \frac{1}{k(k-1)} \frac{(b-a)^k}{b^{k-1}} - \ln b + \ln a \\ &\le \frac{(b-a)^{2n+2}}{(2n+1)(2n+2)\, a \min^{2n+1}\{a,b\}}. \end{aligned} \tag{2.5}$$

Proof. For the sake of completeness we give here a simple proof.

For any $a,\ b > 0$ we claim that

$$\ln b - \ln a - \frac{b-a}{b} = \frac{1}{b} \int_a^b \frac{b-t}{t} dt, \ \text{[3]} \tag{2.6}$$

and for any $m \ge 2$ and any $a,\ b > 0$

$$\ln b - \ln a - \frac{b-a}{b} + \frac{1}{b} \sum_{k=2}^{m} \frac{(-1)^{k-1}}{k(k-1)} \frac{(b-a)^k}{a^{k-1}} = \frac{(-1)^{m-1}}{mb} \int_a^b \frac{(b-t)^m}{t^m} dt. \tag{2.7}$$

Consider the function $f : (0,\infty) \longrightarrow \mathbb{R},\ f(x) = x \ln x$, then

$$f'(x) = \ln x + 1 \text{ and } f''(x) = \frac{1}{x}$$

and, in general, for $m \ge 2$ we have

$$f^{(m)}(x) = \frac{(-1)^m (m-2)!}{x^{m-1}},$$

where $0! := 1$.

If we use Taylor's representation (2.1) for $m = 1$, then we have

$$f(x) = f(a) + (x-a) f'(a) + \int_a^x (x-t) f''(t)\, dt$$

for any $x,\ a \in I$.

If we write this equality for $f(x) = x \ln x$ and $x = b$, we get

$$b \ln b = a \ln a + (b-a)(\ln a + 1) + \int_a^b \frac{b-t}{t} dt$$

namely

$$b \ln b = b \ln a + b - a + \int_a^b \frac{b-t}{t} dt$$

for any $a,\ b > 0$ that is equivalent to (2.6).

If we use Taylor's representation (2.1) for $m \geq 1$, then we have

$$f(x) = f(a) + (x-a) f'(a) + \sum_{k=2}^{m} \frac{(x-a)^k}{k!} f^{(k)}(a) + \frac{1}{m!} \int_a^x (x-t)^m f^{(m+1)}(t)\, dt$$

for any $x,\ a \in I$.

If we write this equality for $f(x) = x \ln x$ and $x = b$ we get

$$\begin{aligned} b \ln b = a \ln a + (b-a)(\ln a + 1) + \sum_{k=2}^{m} \frac{(-1)^k}{k(k-1)} \frac{(b-a)^k}{a^{k-1}} \\ + \frac{(-1)^{m-1}}{m} \int_a^b \frac{(b-t)^m}{t^m} dt, \end{aligned}$$

namely

$$\begin{aligned} b \ln b = b \ln a + b - a + \sum_{k=2}^{m} \frac{(-1)^k}{k(k-1)} \frac{(b-a)^k}{a^{k-1}} \\ + \frac{(-1)^{m-1}}{m} \int_a^b \frac{(b-t)^m}{t^m} dt \end{aligned}$$

for any $a,\ b > 0$ that is equivalent to (2.7).

Now, let $b > a > 0$, then

$$\frac{1}{a} \int_a^b (b-t)\, dt \geq \int_a^b \frac{b-t}{t} dt \geq \frac{1}{b} \int_a^b (b-t)\, dt$$

giving that

$$\frac{1}{2a}(b-a)^2 \geq \int_a^b \frac{b-t}{t} dt \geq \frac{1}{2b}(b-a)^2. \tag{2.8}$$

Let $a > b > 0$, then

$$\frac{1}{b} \int_b^a (t-b)\, dt \geq \int_a^b \frac{b-t}{t} dt = \int_b^a \frac{t-b}{t} dt \geq \frac{1}{a} \int_b^a (t-b)\, dt$$

giving that

$$\frac{1}{2b}(b-a)^2 \geq \int_a^b \frac{b-t}{t} dt \geq \frac{1}{2a}(b-a)^2. \tag{2.9}$$

Therefore, by (2.2) and (2.3) we get

$$\frac{1}{2\min\{a,b\}}(b-a)^2 \geq \int_a^b \frac{b-t}{t}dt \geq \frac{1}{2\max\{a,b\}}(b-a)^2,$$

for any $a,\, b>0$.

By utilising the equality (2.6) we get the desired result (2.2).

Let $m=2n+1$ with $n\geq 1$. Then by (2.7) we have

$$\ln b-\ln a-\frac{b-a}{b}+\frac{1}{b}\sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k(k-1)}\frac{(b-a)^k}{a^{k-1}} = \frac{1}{(2n+1)b}\int_a^b \frac{(b-t)^{2n+1}}{t^{2n+1}}dt. \tag{2.10}$$

Let $b>a>0$, then

$$\frac{(b-a)^{2n+2}}{a^{2n+1}(2n+2)} \geq \int_a^b \frac{(b-t)^{2n+1}}{t^{2n+1}}dt \geq \frac{(b-a)^{2n+2}}{b^{2n+1}(2n+2)}. \tag{2.11}$$

If $a>b>0$, then

$$\int_a^b \frac{(b-t)^{2n+1}}{t^{2n+1}}dt=\int_b^a \frac{(t-b)^{2n+1}}{t^{2n+1}}dt$$

and

$$\frac{(b-a)^{2n+2}}{b^{2n+1}(2n+2)} \geq \int_b^a \frac{(t-b)^{2n+1}}{t^{2n+1}}dt \geq \frac{(b-a)^{2n+2}}{a^{2n+1}(2n+2)}. \tag{2.12}$$

Using (2.11) and (2.12) we get

$$\frac{(b-a)^{2n+2}}{\min^{2n+1}\{a,b\}(2n+2)} \geq \int_a^b \frac{(b-t)^{2n+1}}{t^{2n+1}}dt \geq \frac{(b-a)^{2n+2}}{\max^{2n+1}\{a,b\}(2n+2)} \tag{2.13}$$

for any $a,\, b>0$.

Finally, on utilising the representation (2.10) and the inequality (2.13) we get the desired result (2.4).

The inequality (2.5) follows from (2.4) by replacing a with b. □

Remark 1. *Since the lower bounds in (2.4) and (2.5) are positive, then we have*

$$\frac{b-a}{b}+\frac{1}{b}\sum_{k=2}^{2n+1}\frac{(-1)^k}{k(k-1)}\frac{(b-a)^k}{a^{k-1}} \leq \ln b-\ln a \leq \frac{b-a}{a}-\frac{1}{a}\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}\frac{(b-a)^k}{b^{k-1}} \tag{2.14}$$

for any $a,\, b>0$ and $n\geq 1$.

For $n\geq 0$ and $y>0$ we consider the bounds

$$B_1(y,n):=\frac{(y-1)^{2n+2}}{y\max^{2n+1}\{1,y\}}=(\min\{1,y\})^{2n+1}\left(\frac{y-1}{y}\right)^{2n+2} = \frac{1}{\min\{1,y\}}\left(1-\frac{\min\{1,y\}}{\max\{1,y\}}\right)^{2n+2},$$

$$\begin{aligned}B_2(y,n) &:= \frac{(y-1)^{2n+2}}{y\min^{2n+1}\{1,y\}} = (\max\{1,y\})^{2n+1}\left(\frac{y-1}{y}\right)^{2n+2} \\ &= \frac{1}{\max\{1,y\}}\left(\frac{\max\{1,y\}}{\min\{1,y\}}-1\right)^{2n+2},\end{aligned}$$

$$\begin{aligned}C_1(y,n) &:= \frac{(y-1)^{2n+2}}{\max^{2n+1}\{1,y\}} = y(\min\{1,y\})^{2n+1}\left(\frac{y-1}{y}\right)^{2n+2} \\ &= \max\{1,y\}\left(1-\frac{\min\{1,y\}}{\max\{1,y\}}\right)^{2n+2}\end{aligned}$$

and

$$\begin{aligned}C_2(y,n) &:= \frac{(y-1)^{2n+2}}{\min^{2n+1}\{1,y\}} = y(\max\{1,y\})^{2n+1}\left(\frac{y-1}{y}\right)^{2n+2} \\ &= \min\{1,y\}\left(\frac{\max\{1,y\}}{\min\{1,y\}}-1\right)^{2n+2}.\end{aligned}$$

From Lemma 1 we have, by taking $b=y\in(0,\infty)$ and $a=1$, that

$$\frac{1}{2}B_1(y,0)\le \ln y-\frac{y-1}{y}\le \frac{1}{2}B_2(y,0) \tag{2.15}$$

and

$$\frac{1}{2}C_1(y,0)\le y-1-\ln y\le C_2(y,0) \tag{2.16}$$

that have been obtained in [3] as well.

If $n\ge 1$, then for any $y>0$ we have from (2.4) and (2.5) that

$$\begin{aligned}\frac{1}{(2n+1)(2n+2)}B_1(y,n) &\le \ln y-\frac{y-1}{y}+\frac{1}{y}\sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k(k-1)}(y-1)^k \\ &\le \frac{1}{(2n+1)(2n+2)}B_2(y,n)\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}\frac{1}{(2n+1)(2n+2)}C_1(y,n) &\le y-1-\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}\frac{(y-1)^k}{y^{k-1}}-\ln y \\ &\le \frac{1}{(2n+1)(2n+2)}C_2(y,n).\end{aligned} \tag{2.18}$$

From (2.14) we also have

$$\frac{y-1}{y}+\frac{1}{y}\sum_{k=2}^{2n+1}\frac{(-1)^k}{k(k-1)}(y-1)^k\le \ln y\le y-1-\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}\frac{(y-1)^k}{y^{k-1}} \tag{2.19}$$

for any $y>0$ and $n\ge 1$.

If $y\in[v,V]\subset(0,\infty)$ and since

$$\min\{1,v\}\le\min\{1,y\}\le\min\{1,V\}$$

and

$$\max\{1,v\}\le\max\{1,y\}\le\max\{1,V\},$$

then we have the following further local bounds for $B_1(y,n)$, $B_2(y,n)$, $C_1(y,n)$ and $C_2(y,n)$

$$\frac{(y-1)^{2n+2}}{V\max^{2n+1}\{1,V\}} \le \frac{(y-1)^{2n+2}}{\max^{2n+1}\{1,V\}\,y} \le B_1(y,n), \tag{2.20}$$

$$B_2(y,n) \le \frac{(y-1)^{2n+2}}{\min^{2n+1}\{1,v\}\,y} \le \frac{(y-1)^{2n+2}}{v\min^{2n+1}\{1,v\}}, \tag{2.21}$$

$$\begin{aligned} & v(\min\{1,v\})^{2n+1}\left(\frac{y-1}{y}\right)^{2n+2} \\ & \le y(\min\{1,v\})^{2n+1}\left(\frac{y-1}{y}\right)^{2n+2} \le C_1(y,n) \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} C_2(y,n) &\le y(\max\{1,V\})^{2n+1}\left(\frac{y-1}{y}\right)^{2n+2} \\ &\le V(\max\{1,V\})^{2n+1}\left(\frac{y-1}{y}\right)^{2n+2} \end{aligned} \tag{2.23}$$

for $y\in[v,V]$ and $n\ge 0$.

We also have the local bounds

$$(\min\{1,v\})^{2n+1}\left(\frac{y-1}{y}\right)^{2n+2} \le B_1(y,n), \tag{2.24}$$

$$B_2(y,n) \le (\max\{1,V\})^{2n+1}\left(\frac{y-1}{y}\right)^{2n+2}, \tag{2.25}$$

$$\frac{1}{\max^{2n+1}\{1,V\}}(y-1)^{2n+2} \le C_1(y,n) \tag{2.26}$$

and

$$C_2(y,n) \le \frac{1}{\min^{2n+1}\{1,v\}}(y-1)^{2n+2} \tag{2.27}$$

for $y\in[v,V]$ and $n\ge 0$.

Observe also that, for $y\in[v,V]$ we have

$$0 \le 1-\frac{\min\{1,V\}}{\max\{1,v\}} \le 1-\frac{\min\{1,y\}}{\max\{1,y\}}$$

and

$$0 \le \frac{\max\{1,y\}}{\min\{1,y\}}-1 \le \frac{\max\{1,V\}}{\min\{1,v\}}-1.$$

We also have the global bounds:

$$\frac{1}{\min\{1,V\}}\left(1-\frac{\min\{1,V\}}{\max\{1,v\}}\right)^{2n+2} \le B_1(y,n), \tag{2.28}$$

$$B_2(y,n) \le \frac{1}{\max\{1,v\}}\left(\frac{\max\{1,V\}}{\min\{1,v\}}-1\right)^{2n+2}, \tag{2.29}$$

$$\max\{1,v\}\left(1-\frac{\min\{1,V\}}{\max\{1,v\}}\right)^{2n+2}\le C_1(y,n) \tag{2.30}$$

and

$$C_2(y,n)\le\min\{1,V\}\left(\frac{\max\{1,V\}}{\min\{1,v\}}-1\right)^{2n+2} \tag{2.31}$$

for $y\in[v,V]$ and $n\ge 0$.

By the use of these results we provide in the following various local and global inequalities for the relative operator entropy $S(A|B)$.

3. Local Operator Inequalities

The following operator inequality holds:

Theorem 4. *Let A, B be two positive invertible operators satisfying the boundedness condition (1.8) for some constants $m,M>0$ with $m<M$. Then for any $t>0$ and $n\ge 1$ we have*

$$\begin{aligned}
&\frac{t}{2M^t\max\{1,M^t\}}\left(T_t(A|B)A^{-1}\right)^2A \\
&\le\frac{t}{2\max\{1,M^t\}}\left(T_t(A|B)A^{-1}\right)^2A\left(A\sharp_tB\right)^{-1}A\le S(A|B)-T_{-t}(A|B) \\
&\le\frac{t}{2\min\{1,m^t\}}\left(T_t(A|B)A^{-1}\right)^2A\left(A\sharp_tB\right)^{-1}A \\
&\le\frac{t}{2m^t\min\{1,m^t\}}\left(T_t(A|B)A^{-1}\right)^2A,
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
&\frac{1}{2}m^t\min\left\{1,m^t\right\}t\left(T_{-t}(A|B)A^{-1}\right)^2A \\
&\le\frac{1}{2}\min\left\{1,m^t\right\}t\left(T_{-t}(A|B)A^{-1}\right)^2A\sharp_tB\le T_t(A|B)-S(A|B) \\
&\le\frac{1}{2}\max\left\{1,M^t\right\}t\left(T_{-t}(A|B)A^{-1}\right)^2A\sharp_tB \\
&\le\frac{1}{2}M^t\max\left\{1,M^t\right\}t\left(T_{-t}(A|B)A^{-1}\right)^2A,
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&\frac{t^{2n+1}}{(2n+1)(2n+2)M^t\max^{2n+1}\{1,M^t\}}\left(T_t(A|B)A^{-1}\right)^{2n+2}A \\
&\le\frac{t^{2n+1}}{(2n+1)(2n+2)\max^{2n+1}\{1,M^t\}}\left(T_t(A|B)A^{-1}\right)^{2n+2}A\left(A\sharp_tB\right)^{-1}A \\
&\le S(A|B)-T_{-t}(A|B) \\
&+\sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k(k-1)}t^{k-1}\left(T_t(A|B)A^{-1}\right)^kA\left(A\sharp_tB\right)^{-1}A \\
&\le\frac{t^{2n+1}}{(2n+1)(2n+2)\min^{2n+1}\{1,m^t\}}\left(T_t(A|B)A^{-1}\right)^{2n+2}A\left(A\sharp_tB\right)^{-1}A \\
&\le\frac{t^{2n+1}}{(2n+1)(2n+2)m^t\min^{2n+1}\{1,m^t\}}\left(T_t(A|B)A^{-1}\right)^{2n+2}A
\end{aligned} \tag{3.3}$$

and

$$
\begin{aligned}
(3.4)\quad & \frac{1}{(2n+1)(2n+2)} m^t \left(\min\left\{1, m^t\right\}\right)^{2n+1} t^{2n+1} \left(T_{-t}(A|B) A^{-1}\right)^{2n+2} A \\
& \leq \frac{1}{(2n+1)(2n+2)} \left(\min\left\{1, m^t\right\}\right)^{2n+1} t^{2n+1} \left(T_{-t}(A|B) A^{-1}\right)^{2n+2} A\sharp_t B \\
& \leq T_t(A|B) - \sum_{k=2}^{2n+1} \frac{1}{k(k-1)} t^{k-1} \left(T_{-t}(A|B) A^{-1}\right)^k A\sharp_t B - S(A|B) \\
& \leq \frac{1}{(2n+1)(2n+2)} \left(\max\left\{1, M^t\right\}\right)^{2n+1} t^{2n+1} \left(T_{-t}(A|B) A^{-1}\right)^{2n+2} A\sharp_t B \\
& \leq \frac{1}{(2n+1)(2n+2)} M^t \left(\max\left\{1, M^t\right\}\right)^{2n+1} t^{2n+1} \left(T_{-t}(A|B) A^{-1}\right)^{2n+2} A.
\end{aligned}
$$

Proof. Let $x \in [m, M] \subset (0, \infty)$ and for $t > 0$ put $y = x^t \in [m^t, M^t]$. Then by (2.15), (2.16) and (2.20)-(2.23) for $v = m^t$ and $V = M^t$ we have

$$
\begin{aligned}
(3.5)\quad \frac{(x^t-1)^2}{2M^t \max\{1, M^t\}} & \leq \frac{(x^t-1)^2 x^{-t}}{2\max\{1, M^t\}} \leq \frac{1}{2} B_1\left(x^t, 0\right) \\
& \leq \ln x^t - \frac{x^t - 1}{x^t} \leq \frac{1}{2} B_2\left(x^t, 0\right) \\
& \leq \frac{(x^t-1)^2 x^{-t}}{2\min\{1, m^t\}} \leq \frac{(x^t-1)^2}{2m^t \min\{1, m^t\}}
\end{aligned}
$$

and

$$
\begin{aligned}
(3.6)\quad \frac{1}{2} m^t \min\left\{1, m^t\right\} \left(\frac{x^t-1}{x^t}\right)^2 & \leq \frac{1}{2} \min\left\{1, m^t\right\} \left(\frac{x^t-1}{x^t}\right)^2 x^t \\
& \leq \frac{1}{2} C_1(y, 0) \leq x^t - 1 - \ln x^t \\
& \leq C_2(y, 0) \leq \frac{1}{2} \max\left\{1, M^t\right\} \left(\frac{x^t-1}{x^t}\right)^2 x^t \\
& \leq \frac{1}{2} M^t \max\left\{1, M^t\right\} \left(\frac{x^t-1}{x^t}\right)^2
\end{aligned}
$$

for $x \in [m, M] \subset (0, \infty)$ and for $t > 0$.

From (2.17), (2.18) and (2.20)-(2.23) we also have

$$
\begin{aligned}
&\frac{1}{(2n+1)(2n+2)M^t\max^{2n+1}\{1,M^t\}}\left(x^t-1\right)^{2n+2} \\
&\le \frac{1}{(2n+1)(2n+2)\max^{2n+1}\{1,M^t\}}\left(x^t-1\right)^{2n+2}x^{-t} \\
&\le \frac{1}{(2n+1)(2n+2)}B_1\left(x^t,n\right) \\
&\le \ln x^t-\frac{x^t-1}{x^t}+\sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k(k-1)}\left(x^t-1\right)^k x^{-t} \\
&\le \frac{1}{(2n+1)(2n+2)}B_2\left(x^t,n\right) \\
&\le \frac{1}{(2n+1)(2n+2)\min^{2n+1}\{1,m^t\}}\left(x^t-1\right)^{2n+2}x^{-t} \\
&\le \frac{1}{(2n+1)(2n+2)m^t\min^{2n+1}\{1,m^t\}}\left(x^t-1\right)^{2n+2}
\end{aligned}
\tag{3.7}
$$

and

$$
\begin{aligned}
&\frac{1}{(2n+1)(2n+2)}m^t\left(\min\left\{1,m^t\right\}\right)^{2n+1}\left(\frac{x^t-1}{x^t}\right)^{2n+2} \\
&\le \frac{1}{(2n+1)(2n+2)}\left(\min\left\{1,m^t\right\}\right)^{2n+1}\left(\frac{x^t-1}{x^t}\right)^{2n+2}x^t \\
&\le \frac{1}{(2n+1)(2n+2)}C_1\left(x^t,n\right) \\
&\le x^t-1-\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}\left(\frac{x^t-1}{x^t}\right)^k x^t-\ln x^t \\
&\le \frac{1}{(2n+1)(2n+2)}C_2\left(x^t,n\right) \\
&\le \frac{1}{(2n+1)(2n+2)}\left(\max\left\{1,M^t\right\}\right)^{2n+1}\left(\frac{x^t-1}{x^t}\right)^{2n+2}x^t \\
&\le \frac{1}{(2n+1)(2n+2)}M^t\left(\max\left\{1,M^t\right\}\right)^{2n+1}\left(\frac{x^t-1}{x^t}\right)^{2n+2}
\end{aligned}
\tag{3.8}
$$

for $x\in[m,M]\subset(0,\infty)$, $n\ge 1$ and for $t>0$.

If we use the functions $T_{\pm t}$, then we have by (3.7) and (3.8) that

$$
\begin{aligned}
&\frac{t^{2n+1}}{(2n+1)(2n+2)M^t\max^{2n+1}\{1,M^t\}}(T_t(x))^{2n+2} \\
&\le \frac{t^{2n+1}}{(2n+1)(2n+2)\max^{2n+1}\{1,M^t\}}(T_t(x))^{2n+2}x^{-t} \\
&\le \ln x - T_{-t}(x) + \sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k(k-1)}t^{k-1}(T_t(x))^k x^{-t} \\
&\le \frac{t^{2n+1}}{(2n+1)(2n+2)\min^{2n+1}\{1,m^t\}}(T_t(x))^{2n+2}x^{-t} \\
&\le \frac{t^{2n+1}}{(2n+1)(2n+2)m^t\min^{2n+1}\{1,m^t\}}(T_t(x))^{2n+2}
\end{aligned} \tag{3.9}
$$

and

$$
\begin{aligned}
&\frac{1}{(2n+1)(2n+2)}m^t\left(\min\left\{1,m^t\right\}\right)^{2n+1}t^{2n+1}(T_{-t}(x))^{2n+2} \\
&\le \frac{1}{(2n+1)(2n+2)}\left(\min\left\{1,m^t\right\}\right)^{2n+1}t^{2n+1}(T_{-t}(x))^{2n+2}x^t \\
&\le T_t(x) - \sum_{k=2}^{2n+1}\frac{1}{k(k-1)}t^{k-1}(T_{-t}(x))^k x^t - \ln x \\
&\le \frac{1}{(2n+1)(2n+2)}\left(\max\left\{1,M^t\right\}\right)^{2n+1}t^{2n+1}(T_{-t}(x))^{2n+2}x^t \\
&\le \frac{1}{(2n+1)(2n+2)}M^t\left(\max\left\{1,M^t\right\}\right)^{2n+1}t^{2n+1}(T_{-t}(x))^{2n+2}
\end{aligned} \tag{3.10}
$$

for $x \in [m,M] \subset (0,\infty)$, $n \ge 1$ and for $t > 0$.

Using the continuous functional calculus for the positive invertible operator X with spectrum in $[m,M]$ we have by (3.9) that

$$
\begin{aligned}
&\frac{t^{2n+1}}{(2n+1)(2n+2)M^t\max^{2n+1}\{1,M^t\}}(T_t(X))^{2n+2} \\
&\le \frac{t^{2n+1}}{(2n+1)(2n+2)\max^{2n+1}\{1,M^t\}}(T_t(X))^{2n+2}(X)^{-t} \\
&\le \ln x - T_{-t}(X) + \sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k(k-1)}t^{k-1}(T_t(X))^k (X)^{-t} \\
&\le \frac{t^{2n+1}}{(2n+1)(2n+2)\min^{2n+1}\{1,m^t\}}(T_t(X))^{2n+2}(X)^{-t} \\
&\le \frac{t^{2n+1}}{(2n+1)(2n+2)m^t\min^{2n+1}\{1,m^t\}}(T_t(X))^{2n+2}
\end{aligned} \tag{3.11}
$$

for any $t > 0$ and $n \ge 1$.

If in the inequality (1.8) we multiply both sides by $A^{-1/2}$ we get $m1_H \leq A^{-1/2}BA^{-1/2} \leq M1_H$ and by (3.11) we obtain

$$\begin{aligned}
&\frac{t^{2n+1}}{(2n+1)(2n+2)M^t \max^{2n+1}\{1,M^t\}}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+2} \\
&\leq \frac{t^{2n+1}}{(2n+1)(2n+2)\max^{2n+1}\{1,M^t\}}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+2}\left(A^{-1/2}BA^{-1/2}\right)^{-t} \\
&\leq \ln\left(A^{-1/2}BA^{-1/2}\right) - T_{-t}\left(A^{-1/2}BA^{-1/2}\right) \\
&+ \sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k(k-1)}t^{k-1}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^k\left(A^{-1/2}BA^{-1/2}\right)^{-t} \\
&\leq \frac{t^{2n+1}}{(2n+1)(2n+2)\min^{2n+1}\{1,m^t\}}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+2}\left(A^{-1/2}BA^{-1/2}\right)^{-t} \\
&\leq \frac{t^{2n+1}}{(2n+1)(2n+2)m^t \min^{2n+1}\{1,m^t\}}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+2}
\end{aligned}$$

for any $t > 0$ and $n \geq 1$.

If we multiply this inequality in both sides by $A^{1/2}$, then we get

$$\begin{aligned}
&\frac{t^{2n+1}}{(2n+1)(2n+2)M^t \max^{2n+1}\{1,M^t\}} \\
&\times A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+2}A^{1/2} \\
&\leq \frac{t^{2n+1}}{(2n+1)(2n+2)\max^{2n+1}\{1,M^t\}} \\
&\times A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+2}\left(A^{-1/2}BA^{-1/2}\right)^{-t}A^{1/2} \\
&\leq A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2} - A^{1/2}T_{-t}\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} \\
&+ \sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k(k-1)}t^{k-1}A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^k\left(A^{-1/2}BA^{-1/2}\right)^{-t}A^{1/2} \\
&\leq \frac{t^{2n+1}}{(2n+1)(2n+2)\min^{2n+1}\{1,m^t\}} \\
&\times A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+2}\left(A^{-1/2}BA^{-1/2}\right)^{-t}A^{1/2} \\
&\leq \frac{t^{2n+1}}{(2n+1)(2n+2)m^t \min^{2n+1}\{1,m^t\}} \\
&\times A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+2}A^{1/2}
\end{aligned} \tag{3.12}$$

for any $t > 0$ and $n \geq 1$.

For $k \geq 1$ we have

$$\begin{aligned}
&A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^k A^{1/2}\\
&= A^{1/2}\left(A^{-1/2}A^{1/2}T_t\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}A^{-1/2}\right)^k A^{1/2}\\
&= A^{1/2}\left(A^{-1/2}T_t\left(A|B\right)A^{-1/2}\right)^k A^{1/2}\\
&= A^{1/2}A^{-1/2}T_t\left(A|B\right)A^{-1/2}...A^{-1/2}T_t\left(A|B\right)A^{-1/2}A^{1/2}\\
&= T_t\left(A|B\right)A^{-1}...T_t\left(A|B\right)A^{-1/2}A^{1/2}\\
&= T_t\left(A|B\right)A^{-1}...T_t\left(A|B\right)A^{-1}A = \left(T_t\left(A|B\right)A^{-1}\right)^k A.
\end{aligned}$$

Then we have

$$A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+2} A^{1/2} = \left(T_t\left(A|B\right)A^{-1}\right)^{2n+2} A,$$

$$\begin{aligned}
&A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+2}\left(A^{-1/2}BA^{-1/2}\right)^{-t}A^{1/2}\\
&= A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+2}A^{1/2}A^{-1/2}\left(A^{-1/2}BA^{-1/2}\right)^{-t}A^{-1/2}A\\
&= A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+2}A^{1/2}\left(A\sharp_t B\right)^{-1}A\\
&= \left(T_t\left(A|B\right)A^{-1}\right)^{2n+2}A\left(A\sharp_t B\right)^{-1}A
\end{aligned}$$

and

$$\begin{aligned}
&A^{1/2}\left(T_t\left(A^{-1/2}BA^{-1/2}\right)\right)^{k}\left(A^{-1/2}BA^{-1/2}\right)^{-t}A^{1/2}\\
&= \left(T_t\left(A|B\right)A^{-1}\right)^{k}A\left(A\sharp_t B\right)^{-1}A
\end{aligned}$$

and by (3.12) we get the desired result (3.3).

Using the functional calculus and the inequality (3.10) we have

$$\begin{aligned}
&\frac{1}{(2n+1)(2n+2)}m^t\left(\min\left\{1,m^t\right\}\right)^{2n+1}t^{2n+1}\left(T_{-t}\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+2}\\
&\leq \frac{1}{(2n+1)(2n+2)}\left(\min\left\{1,m^t\right\}\right)^{2n+1}t^{2n+1}\\
&\times\left(T_{-t}\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+2}\left(A^{-1/2}BA^{-1/2}\right)^{t}\\
&\leq T_t\left(A^{-1/2}BA^{-1/2}\right) - \ln A^{-1/2}BA^{-1/2}\\
&-\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}t^{k-1}\left(T_{-t}\left(A^{-1/2}BA^{-1/2}\right)\right)^{k}\left(A^{-1/2}BA^{-1/2}\right)^{t}\\
&\leq \frac{1}{(2n+1)(2n+2)}\left(\max\left\{1,M^t\right\}\right)^{2n+1}t^{2n+1}\\
&\times\left(T_{-t}\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+2}\left(A^{-1/2}BA^{-1/2}\right)^{t}\\
&\leq \frac{1}{(2n+1)(2n+2)}M^t\left(\max\left\{1,M^t\right\}\right)^{2n+1}t^{2n+1}\left(T_{-t}\left(A^{-1/2}BA^{-1/2}\right)\right)^{2n+2}
\end{aligned}$$

for any $t > 0$ and $n \geq 1$.

If we multiply this inequality in both sides by $A^{1/2}$, then we get

$$\begin{aligned}
(3.13) \quad & \frac{1}{(2n+1)(2n+2)} m^t \left(\min\left\{1, m^t\right\}\right)^{2n+1} t^{2n+1} \\
& \times A^{1/2} \left(T_{-t}\left(A^{-1/2} B A^{-1/2}\right)\right)^{2n+2} A^{1/2} \\
& \leq \frac{1}{(2n+1)(2n+2)} \left(\min\left\{1, m^t\right\}\right)^{2n+1} t^{2n+1} \\
& \times A^{1/2} \left(T_{-t}\left(A^{-1/2} B A^{-1/2}\right)\right)^{2n+2} \left(A^{-1/2} B A^{-1/2}\right)^t A^{1/2} \\
& \leq A^{1/2} T_t \left(A^{-1/2} B A^{-1/2}\right) A^{1/2} - A^{1/2} \left(\ln\left(A^{-1/2} B A^{-1/2}\right)\right) A^{1/2} \\
& - \sum_{k=2}^{2n+1} \frac{1}{k(k-1)} t^{k-1} A^{1/2} \left(T_{-t}\left(A^{-1/2} B A^{-1/2}\right)\right)^k \left(A^{-1/2} B A^{-1/2}\right)^t A^{1/2} \\
& \leq \frac{1}{(2n+1)(2n+2)} \left(\max\left\{1, M^t\right\}\right)^{2n+1} t^{2n+1} \\
& \times A^{1/2} \left(T_{-t}\left(A^{-1/2} B A^{-1/2}\right)\right)^{2n+2} \left(A^{-1/2} B A^{-1/2}\right)^t A^{1/2} \\
& \leq \frac{1}{(2n+1)(2n+2)} M^t \left(\max\left\{1, M^t\right\}\right)^{2n+1} t^{2n+1} \\
& \times A^{1/2} \left(T_{-t}\left(A^{-1/2} B A^{-1/2}\right)\right)^{2n+2} A^{1/2}
\end{aligned}$$

for any $t > 0$ and $n \geq 1$.

We have

$$A^{1/2} \left(T_{-t}\left(A^{-1/2} B A^{-1/2}\right)\right)^{2n+2} A^{1/2} = \left(T_{-t}(A|B) A^{-1}\right)^{2n+2} A,$$

$$\begin{aligned}
& A^{1/2} \left(T_{-t}\left(A^{-1/2} B A^{-1/2}\right)\right)^{2n+2} \left(A^{-1/2} B A^{-1/2}\right)^t A^{1/2} \\
& = A^{1/2} \left(T_{-t}\left(A^{-1/2} B A^{-1/2}\right)\right)^{2n+2} A^{1/2} A^{-1} A^{1/2} \left(A^{-1/2} B A^{-1/2}\right)^t A^{1/2} \\
& = \left(T_{-t}(A|B) A^{-1}\right)^{2n+2} A A^{-1} A \sharp_t B = \left(T_{-t}(A|B) A^{-1}\right)^{2n+2} A \sharp_t B
\end{aligned}$$

and for $k \geq 1$

$$A^{1/2} \left(T_{-t}\left(A^{-1/2} B A^{-1/2}\right)\right)^k \left(A^{-1/2} B A^{-1/2}\right)^t A^{1/2} = \left(T_{-t}(A|B) A^{-1}\right)^k A \sharp_t B,$$

where $t > 0$ and $n \geq 1$. By using (3.13) we deduce the desired result (3.4).

The inequalities (3.1) and (3.2) follow in a similar way by using (3.5) and (3.6) and we omit the details. $\square$

Remark 2. *If we take in the inequalities (3.1)-(3.4) $t = 1$ and since*

$$T_{-1}(A|B) A^{-1} = 1_H - AB^{-1} \text{ and } T_1(A|B) A^{-1} = BA^{-1} - 1_H,$$

then we get

$$\begin{aligned}
&\frac{1}{2M\max\{1,M\}}\left(BA^{-1}-1_H\right)^2A \\
&\le \frac{1}{2\max\{1,M\}}\left(BA^{-1}-1_H\right)^2AB^{-1}A \le S(A|B)-\left(1_H-AB^{-1}\right)A \\
&\le \frac{1}{2\min\{1,m\}}\left(BA^{-1}-1_H\right)^2AB^{-1}A \\
&\le \frac{1}{2m\min\{1,m\}}\left(BA^{-1}-1_H\right)^2A,
\end{aligned}\tag{3.14}$$

$$\begin{aligned}
&\frac{1}{2}m\min\{1,m\}\left(1_H-AB^{-1}\right)^2A \\
&\le \frac{1}{2}\min\{1,m\}\left(1_H-AB^{-1}\right)^2B \le B-A-S(A|B) \\
&\le \frac{1}{2}\max\{1,M\}\left(1_H-AB^{-1}\right)^2B \\
&\le \frac{1}{2}M\max\{1,M\}\left(1_H-AB^{-1}\right)^2A,
\end{aligned}\tag{3.15}$$

$$\begin{aligned}
&\frac{1}{(2n+1)(2n+2)M\max^{2n+1}\{1,M\}}\left(BA^{-1}-1_H\right)^{2n+2}A \\
&\le \frac{1}{(2n+1)(2n+2)\max^{2n+1}\{1,M\}}\left(BA^{-1}-1_H\right)^{2n+2}AB^{-1}A \\
&\le S(A|B)-\left(1_H-AB^{-1}\right)A \\
&+\sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k(k-1)}\left(BA^{-1}-1_H\right)^kAB^{-1}A \\
&\le \frac{1}{(2n+1)(2n+2)\min^{2n+1}\{1,m\}}\left(BA^{-1}-1_H\right)^{2n+2}AB^{-1}A \\
&\le \frac{1}{(2n+1)(2n+2)m\min^{2n+1}\{1,m\}}\left(BA^{-1}-1_H\right)^{2n+2}A
\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}
&\frac{1}{(2n+1)(2n+2)}m\left(\min\{1,m\}\right)^{2n+1}\left(1_H-AB^{-1}\right)^{2n+2}A \\
&\le \frac{1}{(2n+1)(2n+2)}\left(\min\{1,m\}\right)^{2n+1}\left(1_H-AB^{-1}\right)^{2n+2}B \\
&\le B-A-\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}\left(1_H-AB^{-1}\right)^kB-S(A|B) \\
&\le \frac{1}{(2n+1)(2n+2)}\left(\max\{1,M\}\right)^{2n+1}\left(1_H-AB^{-1}\right)^{2n+2}B \\
&\le \frac{1}{(2n+1)(2n+2)}M\left(\max\{1,M\}\right)^{2n+1}\left(1_H-AB^{-1}\right)^{2n+2}A.
\end{aligned}\tag{3.17}$$

We have:

Theorem 5. *Let A, B be two positive invertible operators satisfying the boundedness condition (1.8) for some constants m, $M>0$ with $m<M$. Then for any $t>0$ and $n\geq 1$ we have*

$$\begin{aligned}&\frac{1}{2}\min\left\{1,m^{t}\right\}t\left(T_{-t}\left(A|B\right)A^{-1}\right)^{2}A\\&\leq S\left(A|B\right)-T_{-t}\left(A|B\right)\\&\leq\frac{1}{2}\max\left\{1,M^{t}\right\}t\left(T_{-t}\left(A|B\right)A^{-1}\right)^{2}A,\end{aligned}\tag{3.18}$$

$$\begin{aligned}\frac{t}{2\max\left\{1,M^{t}\right\}}\left(T_{t}\left(A|B\right)A^{-1}\right)^{2}A&\leq T_{t}\left(A|B\right)-S\left(A|B\right)\\&\leq\frac{t}{2\min\left\{1,m^{t}\right\}}\left(T_{t}\left(A|B\right)A^{-1}\right)^{2}A,\end{aligned}\tag{3.19}$$

$$\begin{aligned}&\frac{t^{2n+1}}{\left(2n+1\right)\left(2n+2\right)}\left(\min\left\{1,m^{t}\right\}\right)^{2n+1}\left(T_{-t}\left(A|B\right)A^{-1}\right)^{2n+2}A\\&\leq S\left(A|B\right)-T_{-t}\left(A|B\right)\\&+\sum_{k=2}^{2n+1}\frac{\left(-1\right)^{k-1}}{k\left(k-1\right)}t^{k-1}\left(T_{t}\left(A|B\right)A^{-1}\right)^{k}A\left(A\sharp_{t}B\right)^{-1}A\\&\leq\frac{t^{2n+1}}{\left(2n+1\right)\left(2n+2\right)}\left(\max\left\{1,M^{t}\right\}\right)^{2n+1}\left(T_{-t}\left(A|B\right)A^{-1}\right)^{2n+2}A\end{aligned}\tag{3.20}$$

and

$$\begin{aligned}&\frac{1}{\left(2n+1\right)\left(2n+2\right)}\frac{1}{\max^{2n+1}\left\{1,M^{t}\right\}}\left(T_{t}\left(A|B\right)A^{-1}\right)^{2n+2}A\\&\leq T_{t}\left(A|B\right)-\sum_{k=2}^{2n+1}\frac{1}{k\left(k-1\right)}t^{k-1}\left(T_{-t}\left(A|B\right)A^{-1}\right)^{k}A\sharp_{t}B-S\left(A|B\right)\\&\leq\frac{1}{\left(2n+1\right)\left(2n+2\right)}\frac{1}{\min^{2n+1}\left\{1,m^{t}\right\}}\left(T_{t}\left(A|B\right)A^{-1}\right)^{2n+2}A.\end{aligned}\tag{3.21}$$

Proof. Using the inequalities (2.15)-(2.18) and (2.24)-(2.27) we can state that

$$\begin{aligned}\frac{1}{2}\min\left\{1,v\right\}\left(\frac{y-1}{y}\right)^{2}&\leq\frac{1}{2}B_{1}\left(y,0\right)\leq\ln y-\frac{y-1}{y}\leq\frac{1}{2}B_{2}\left(y,0\right)\\&\leq\frac{1}{2}\max\left\{1,V\right\}\left(\frac{y-1}{y}\right)^{2}\end{aligned}\tag{3.22}$$

$$\begin{aligned}\frac{1}{2\max\left\{1,V\right\}}\left(y-1\right)^{2}&\leq\frac{1}{2}C_{1}\left(y,0\right)\leq y-1-\ln y\leq C_{2}\left(y,0\right)\\&\leq\frac{1}{2\min\left\{1,v\right\}}\left(y-1\right)^{2}\end{aligned}\tag{3.23}$$

$$\begin{aligned}
(3.24)\qquad & \frac{1}{(2n+1)(2n+2)}\left(\min\{1,v\}\right)^{2n+1}\left(\frac{y-1}{y}\right)^{2n+2} \\
&\leq \frac{1}{(2n+1)(2n+2)}B_1(y,n) \\
&\leq \ln y-\frac{y-1}{y}+\frac{1}{y}\sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k(k-1)}(y-1)^k \\
&\leq \frac{1}{(2n+1)(2n+2)}B_2(y,n) \\
&\leq \frac{1}{(2n+1)(2n+2)}\left(\max\{1,V\}\right)^{2n+1}\left(\frac{y-1}{y}\right)^{2n+2}
\end{aligned}$$

and

$$\begin{aligned}
(3.25)\qquad & \frac{1}{(2n+1)(2n+2)}\frac{1}{\max^{2n+1}\{1,V\}}(y-1)^{2n+2} \\
&\leq \frac{1}{(2n+1)(2n+2)}C_1(y,n) \\
&\leq y-1-\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}\frac{(y-1)^k}{y^{k-1}}-\ln y \\
&\leq \frac{1}{(2n+1)(2n+2)}C_2(y,n) \\
&\leq \frac{1}{(2n+1)(2n+2)}\frac{1}{\min^{2n+1}\{1,v\}}(y-1)^{2n+2},
\end{aligned}$$

where $y\in[v,V]$ and $n\geq 1$.

Now, by using a similar argument to the one presented in the proof of Theorem 4 we deduce the desired results. □

Remark 3. *If we take in the inequalities (3.18)-(3.21) $t=1$ then we get*

$$\begin{aligned}
(3.26)\qquad & \frac{1}{2}\min\{1,m\}\left(1_H-AB^{-1}\right)^2A\leq S(A|B)-\left(1_H-AB^{-1}\right)A \\
&\qquad\leq \frac{1}{2}\max\{1,M\}\left(1_H-AB^{-1}\right)^2A,
\end{aligned}$$

$$\begin{aligned}
(3.27)\qquad & \frac{1}{2\max\{1,M\}}\left(BA^{-1}-1_H\right)^2A\leq B-A-S(A|B) \\
&\qquad\leq \frac{1}{2\min\{1,m\}}\left(BA^{-1}-1_H\right)^2A,
\end{aligned}$$

$$\begin{aligned}
(3.28)\qquad & \frac{1}{(2n+1)(2n+2)}\left(\min\{1,m\}\right)^{2n+1}\left(1_H-AB^{-1}\right)^{2n+2}A \\
&\leq S(A|B)-\left(1_H-AB^{-1}\right)A \\
&+\sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k(k-1)}\left(BA^{-1}-1_H\right)^kAB^{-1}A \\
&\leq \frac{1}{(2n+1)(2n+2)}\left(\max\{1,M\}\right)^{2n+1}\left(1_H-AB^{-1}\right)^{2n+2}A
\end{aligned}$$

and

$$\begin{aligned}(3.29)\qquad &\frac{1}{(2n+1)(2n+2)\max^{2n+1}\{1,M\}}\left(BA^{-1}-1_H\right)^{2n+2}A\\ &\leq B-A-\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}\left(1_H-AB^{-1}\right)^{k}B-S(A|B)\\ &\leq \frac{1}{(2n+1)(2n+2)\min^{2n+1}\{1,m\}}\left(BA^{-1}-1_H\right)^{2n+2}A\end{aligned}$$

for $n\geq 1$.

4. Global Operator Inequalities

We have:

Theorem 6. *Let* A, B *be two positive invertible operators satisfying the boundedness condition (1.8) for some constants* m, $M>0$ *with* $m<M$. *Then for any* $t>0$ *and* $n\geq 1$ *we have*

$$\begin{aligned}(4.1)\qquad &\frac{t}{2\min\{1,M^t\}}T_t^2\left(\frac{\min\{1,M\}}{\max\{1,m\}}\right)A\\ &\leq S(A|B)-T_{-t}(A|B)\\ &\leq \frac{t}{2\max\{1,m^t\}}T_t^2\left(\frac{\max\{1,M\}}{\min\{1,m\}}\right)A,\end{aligned}$$

$$\begin{aligned}(4.2)\qquad &\frac{1}{2}\max\left\{1,m^t\right\}tT_t^2\left(\frac{\min\{1,M\}}{\max\{1,m\}}\right)A\\ &\leq T_t(A|B)-S(A|B)\\ &\leq \frac{1}{2}\min\left\{1,M^t\right\}tT_t^2\left(\frac{\min\{1,M\}}{\max\{1,m\}}\right)A,\end{aligned}$$

$$\begin{aligned}(4.3)\qquad &\frac{t^{2n+1}}{(2n+1)(2n+2)\min\{1,M^t\}}T_t^{2n+2}\left(\frac{\min\{1,M\}}{\max\{1,m\}}\right)A\\ &\leq S(A|B)-T_{-t}(A|B)\\ &+\sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k(k-1)}t^{k-1}\left(T_t(A|B)A^{-1}\right)^{k}A\left(A\sharp_t B\right)^{-1}A\\ &\leq \frac{t^{2n+1}}{(2n+1)(2n+2)\max\{1,m^t\}}T_t^{2n}\left(\frac{\max\{1,M\}}{\min\{1,m\}}\right)A\end{aligned}$$

and

$$\begin{aligned}(4.4)\qquad &\frac{t^{2n+1}\max\{1,m^t\}}{(2n+1)(2n+2)}T_t^{2n+2}\left(\frac{\min\{1,M\}}{\max\{1,m\}}\right)A\\ &\leq T_t(A|B)-\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}t^{k-1}\left(T_{-t}(A|B)A^{-1}\right)^{k}A\sharp_t B-S(A|B)\\ &\leq \frac{t^{2n+1}\min\{1,M^t\}}{(2n+1)(2n+2)}T_t^{2n}\left(\frac{\max\{1,M\}}{\min\{1,m\}}\right)A,\end{aligned}$$

where T_t *is defined by (1.3).*

Proof. Let $x \in [m, M] \subset (0, \infty)$ and for $t > 0$ put $y = x^t \in [m^t, M^t]$. Then by (2.15)-(2.18) and (2.28)-(2.31) for $v = m^t$ and $V = M^t$ we have

$$\begin{aligned} &\frac{1}{2} \frac{1}{\min\{1, M^t\} t} \left(1 - \left(\frac{\min\{1, M\}}{\max\{1, m\}}\right)^t\right)^2 \\ &\leq \ln x - \frac{x^t - 1}{tx^t} \\ &\leq \frac{1}{2} \frac{1}{\max\{1, m^t\} t} \left(\left(\frac{\max\{1, M\}}{\min\{1, m\}}\right)^t - 1\right)^2, \end{aligned}$$

$$\begin{aligned} &\frac{1}{2t} \max\left\{1, m^t\right\} \left(1 - \left(\frac{\min\{1, M\}}{\max\{1, m\}}\right)^t\right)^2 \\ &\leq \frac{x^t - 1}{t} - \ln x \\ &\leq \frac{1}{2t} \min\left\{1, M^t\right\} \left(\left(\frac{\max\{1, M\}}{\min\{1, m\}}\right)^t - 1\right)^2, \end{aligned}$$

$$\begin{aligned} &\frac{1}{(2n+1)(2n+2)} \frac{1}{\min\{1, M^t\} t} \left(1 - \left(\frac{\min\{1, M\}}{\max\{1, m\}}\right)^t\right)^{2n+2} \\ &\leq \ln x - \frac{x^t - 1}{tx^t} + \sum_{k=2}^{2n+1} \frac{(-1)^{k-1}}{k(k-1)} t^{k-1} \left(\frac{x^t - 1}{t}\right)^k \left(x^t\right)^{-1} \\ &\leq \frac{1}{(2n+1)(2n+2)} \frac{1}{\max\{1, m^t\} t} \left(\left(\frac{\max\{1, M\}}{\min\{1, m\}}\right)^t - 1\right)^{2n+2} \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{(2n+1)(2n+2)t} \max\left\{1, m^t\right\} \left(1 - \left(\frac{\min\{1, M\}}{\max\{1, m\}}\right)^t\right)^{2n+2} \\ &\leq \frac{x^t - 1}{t} - \sum_{k=2}^{2n+1} \frac{1}{k(k-1)} t^{k-1} \left(\frac{x^t - 1}{tx^t}\right)^k x^t - \ln x \\ &\leq \frac{1}{(2n+1)(2n+2)t} \min\left\{1, M^t\right\} \left(\left(\frac{\max\{1, M\}}{\min\{1, m\}}\right)^t - 1\right)^{2n+2}, \end{aligned}$$

for any $x \in [m, M] \subset (0, \infty)$, $n \geq 1$ and for $t > 0$.

These inequalities may be written in terms of $T_{\pm t}$ as

$$\begin{aligned} &\frac{t}{2 \min\{1, M^t\}} T_t^2 \left(\frac{\min\{1, M\}}{\max\{1, m\}}\right) \\ &\leq \ln x - T_{-t}(x) \\ &\leq \frac{t}{2 \max\{1, m^t\}} T_t^2 \left(\frac{\max\{1, M\}}{\min\{1, m\}}\right), \end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\max\left\{1,m^{t}\right\}tT_{t}^{2}\left(\frac{\min\left\{1,M\right\}}{\max\left\{1,m\right\}}\right)\\&\leq T_{t}\left(x\right)-\ln x\\&\leq\frac{1}{2}\min\left\{1,M^{t}\right\}tT_{t}^{2}\left(\frac{\min\left\{1,M\right\}}{\max\left\{1,m\right\}}\right),\end{aligned}$$

$$\begin{aligned}&\frac{t^{2n+1}}{\left(2n+1\right)\left(2n+2\right)\min\left\{1,M^{t}\right\}}T_{t}^{2n+2}\left(\frac{\min\left\{1,M\right\}}{\max\left\{1,m\right\}}\right)\\&\leq\ln x-T_{-t}\left(x\right)+\sum_{k=2}^{2n+1}\frac{\left(-1\right)^{k-1}}{k\left(k-1\right)}t^{k-1}\left(T_{t}\left(x\right)\right)^{k}\left(x^{t}\right)^{-1}\\&\leq\frac{t^{2n+1}}{\left(2n+1\right)\left(2n+2\right)\max\left\{1,m^{t}\right\}}T_{t}^{2n}\left(\frac{\max\left\{1,M\right\}}{\min\left\{1,m\right\}}\right)\end{aligned}$$

and

$$\begin{aligned}&\frac{t^{2n+1}\max\left\{1,m^{t}\right\}}{\left(2n+1\right)\left(2n+2\right)}T_{t}^{2n+2}\left(\frac{\min\left\{1,M\right\}}{\max\left\{1,m\right\}}\right)\\&\leq T_{t}\left(x\right)-\sum_{k=2}^{2n+1}\frac{1}{k\left(k-1\right)}t^{k-1}\left(T_{-t}\left(x\right)\right)^{k}x^{t}-\ln x\\&\leq\frac{t^{2n+1}\min\left\{1,M^{t}\right\}}{\left(2n+1\right)\left(2n+2\right)}T_{t}^{2n}\left(\frac{\max\left\{1,M\right\}}{\min\left\{1,m\right\}}\right),\end{aligned}$$

for any $x\in[m,M]\subset(0,\infty)$, $n\geq 1$ and for $t>0$.

Now, by using a similar argument to the one presented in the proof of Theorem 4 we deduce the desired results. $\square$

Remark 4. *If we take in the inequalities (4.1)-(4.4) $t=1$, then we get*

$$\begin{aligned}&\frac{1}{2\min\left\{1,M\right\}}\left(1-\frac{\min\left\{1,M\right\}}{\max\left\{1,m\right\}}\right)^{2}A\\&\leq S\left(A|B\right)-\left(1_{H}-AB^{-1}\right)A\\&\leq\frac{1}{2\max\left\{1,m\right\}}\left(\frac{\max\left\{1,M\right\}}{\min\left\{1,m\right\}}-1\right)^{2}A,\end{aligned}\tag{4.5}$$

$$\begin{aligned}&\frac{1}{2}\max\left\{1,m\right\}\left(1-\frac{\min\left\{1,M\right\}}{\max\left\{1,m\right\}}\right)^{2}A\\&\leq B-A-S\left(A|B\right)\\&\leq\frac{1}{2}\min\left\{1,M\right\}\left(\frac{\max\left\{1,M\right\}}{\min\left\{1,m\right\}}-1\right)^{2}A,\end{aligned}\tag{4.6}$$

$$\frac{1}{(2n+1)(2n+2)\min\{1,M\}}\left(1-\frac{\min\{1,M\}}{\max\{1,m\}}\right)^{2n+2}A \tag{4.7}$$
$$\leq S(A|B)-\left(1_H-AB^{-1}\right)A$$
$$+\sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k(k-1)}\left(BA^{-1}-1_H\right)^k AB^{-1}A$$
$$\leq \frac{1}{(2n+1)(2n+2)\max\{1,m\}}\left(\frac{\max\{1,M\}}{\min\{1,m\}}-1\right)^{2n+2}A$$

and

$$\frac{1}{(2n+1)(2n+2)}\max\{1,m\}\left(1-\frac{\min\{1,M\}}{\max\{1,m\}}\right)^{2n+2}A \tag{4.8}$$
$$\leq B-A-\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}\left(1_H-AB^{-1}\right)^k B-S(A|B)$$
$$\frac{1}{(2n+1)(2n+2)}\min\{1,M\}\left(\frac{\max\{1,M\}}{\min\{1,m\}}-1\right)^{2n+2}$$

for $n\geq 1$.

Finally, we can prove some inequalities that do not require the assumption (1.8) in order to get upper and lower bounds for the relative operator entropy:

Theorem 7. *For any* A, B *two positive invertible operators and for any* $t>0$ *and* $n\geq 1$ *we have*

$$T_{-t}(A|B)+\sum_{k=2}^{2n+1}\frac{(-1)^k}{k(k-1)}t^{k-1}\left(T_t(A|B)A^{-1}\right)^k A\left(A\sharp_t B\right)^{-1}A \tag{4.9}$$
$$\leq S(A|B)$$
$$\leq T_t(A|B)-\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}t^{k-1}\left(T_{-t}(A|B)A^{-1}\right)^k A\sharp_t B.$$

Proof. For $t>0$ put $y=x^t$ and use inequality (2.19) to get

$$\frac{x^t-1}{tx^t}+\sum_{k=2}^{2n+1}\frac{(-1)^k}{k(k-1)}t^{k-1}\left(\frac{x^t-1}{t}\right)^k x^{-t} \tag{4.10}$$
$$\leq \ln x$$
$$\leq \frac{x^t-1}{t}-\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}t^{k-1}\left(\frac{x^t-1}{tx^t}\right)^k x^t$$

for any $x>0$ and $n\geq 1$.

Now, by using the functional calculus for invertible positive operators, we obtain the desired result (4.9). □

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