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SOME EXTENDED MEANS AND HERMITE-HADAMARD INEQUALITY FOR r-PREINVEX FUNCTIONS ON INVEX SET

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Abstract. The necessary and sufficient conditions of weakly r-preinvex functions on invex set are obtained and some generalizations of Hermite-Hadamard inequality for weakly r-preinvex functions on invex set are established. From the obtained results, some better extended mean inequalities are also given.

1. Introduction

The classical Hermite-Hadamard inequality for convex functions states that if $f : [a, b] \to R$ is convex, then

$$\frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}.$$

The generalizations of Hermite-Hadamard inequality to the integral power mean of positive convex and r-convex functions on an interval $[a, b]$ are obtained in [5, 12, 13, 14, 19, 21]. The concept of r-convexity plays an important rose in statistics, see [16]. In [15], Sun extended the Hermite-Hadamard inequality that subsumes the relation between two-parameter mean and positive, twice-differentiable and convex function on [a,b].

In [6], Hanson introduced the invex functions as a generalization of convex function. Hanson's result inspired a great deal of subsequence work which has greatly found the role and applications of invexity in nonlinear optimization and related fields. In[4], Ben-Israel and Mond introduced the concept of the preinvex function and showed that preinvexity implies invexity. The properties of preinvex function in optimization, equilibrium problems and inequalities of variation were studied by Noor [9, 10] and Weir and Mond [18]. Antczak [1, 2] introduced the concept of r-invex and r-preinvex function and give a new method to solve nonlinear mathematical programing problems. In [22], Zhao et al., obtained a characterization for r-preinvex function. In [11], Noor gave some Hermite-Hadamard inequality for the preinvex and log-preinvex functions. Further, in [17], Wasim Ui-Haq and Javed Iqbal introduced Hermite-Hadamard inequality for r-preinvex functions.

The main purposes of this paper are to generalize the Hermite-Hadamard inequality that subsumes the relation between extended means and weakly r-preinvex functions on invex set. The main methods are, via characteristic of weakly r-preinvex functions on invex set, to establish the necessary and sufficient condition of weakly r-preinvex functions on invex set. From the obtained results, we will get

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some new extended two-parameter mean inequalities for weakly r-preinvex functions on invex set. Also, we noted that the obtained results are better than the results given in [11, 17].

2. Preliminary definitions and results for weakly r-preinvex functions

We shall also use a definition of an invex set with respect to η.

Definition 1. *Let $K \subset R^n$ be a nonempty set, $\eta : K \times K \to R^n$ and $u \in K$. Then the set K is said to be invex at u with respect to η, if*

$$u + \lambda\eta(v, u) \in K$$

for every $v \in K$ and $\lambda \in [0, 1]$. K is said to be an invex set with respect to η, if K is invex at each $u \in K$ with respect to the same function η.

We note that the Definition 1 essentially says that there is a path starting from u which is contained in K. It is not required that v should be an end point of the path. If we demand that v should be an end point of the path for every pair u, v then $\eta(v, u) = v - u$, and invexity reduces to convexity. Under this demand, we have that every convex set is also an invex set with respect to $\eta(v, u) = v - u$, but the converse is not true, see [7, 9].

In [3], Antczak introduced the following definition of an η-path on the basis of the considerations of the invex sets.

Definition 2. *Let $K \subset R^n$ be a nonempty invex set with respect to η, $u, v \in K$. A set P_{ux} is said to be a closed η-path joining the points u and $x = u + \eta(v, u)$ with $x \in K$ if*

$$P_{ux} := \{u + \lambda\eta(v, u) : \lambda \in [0, 1]\},$$

and P^0_{ux} is said to be a open η-path joining the points u and $x = u + \eta(v, u)$ with $x \in K$ if

$$P^0_{ux} := \{u + \lambda\eta(v, u) : \lambda \in (0, 1)\}$$

We note that if $\eta(v, u) = v-u$ then the set $P_{ux} = P_{uv} = \{\lambda v+(1-\lambda)u : \lambda \in [0, 1]\}$ is a definition of segment line with the end points u and v.

In [4], Ben-Israel and Mond introduced the class of preinvex function with respect to η on the optimization theory.

Definition 3. *Let $K \subset R^n$ be a nonempty invex set with respect to η. A function $f : K \to R$ is said to be preinvex with respect to η, if there is a vector-value function $\eta : K \times K \to R^n$ such that*

$$f(u + \lambda\eta(v, u)) \leq \lambda f(v) + (1 - \lambda)f(u)$$

for every $u, v \in K$ and $\lambda \in [0, 1]$.

We note that every convex function is preinvex function with respect to $\eta(v, u) = v - u$, but the convese may not always be true.

We begin by recalling some relative definitions about r-preinvex function. The detailed description of r-preinvex function was given by Antczak in [1]. The definition of r-preinvex functions is introduced as follows.

Definition 4. *Let $K \subset R^n$ be a nonempty invex set with respect to η. A function $f : K \to R^+$ is said to be r-preinvex with respect to η, if there is a vector-value function $\eta : K \times K \to R^n$ such that*

$$f(u + \lambda\eta(v,u)) \le \begin{cases} (\lambda f(v)^r + (1-\lambda)f(u)^r)^{\frac{1}{r}}, & \text{if } r \neq 0, \\ f(v)^\lambda f(u)^{1-\lambda}, & \text{if } r = 0. \end{cases}$$

for every $v, u \in K$ and $\lambda \in [0,1]$.

Note that 0-preinvex functions are logarithmic preinvex and 1-preinvex functions are preinvex functions. It is obvious that if f is r-preinvex , then f^r is preinvex function for positive r.

In [8], Mohan and Neogy showed that a differentiable invex function is also preinvex under the following Condition C.

Condition 1. *Let $K \subset R^n$ be a nonempty invex set with respect to $\eta : K \times K \to R^n$. We say that the function η satisfies the Condition C if for any $u, v \in K$ and $\lambda \in [0,1]$, the following two identities hold.*

$$< i > \eta(u, u + \lambda\eta(v,u)) = -\lambda\eta(v,u);$$

$$< ii > \eta(v, u + \lambda\eta(v,u)) = (1-\lambda)\eta(v,u).$$

Applying Condition C, we have the following Lemma.

Lemma 1. *Let $K \subset R^n$ be a nonempty invex set with respect to $\eta : K \times K \to R^n$, and the function η satisfies the Condition C. Then the following identity holds*

$$(\alpha - \beta)\eta(v,u) = \eta(u + \alpha\eta(v,u), u + \beta\eta(v,u))$$

for every $u, v \in K$ and $\alpha, \beta \in [0,1]$.

Proof. When $\alpha = \beta$ the identity holds, obviously. We will prove the case $\alpha > \beta$. Now, $0 < 1 - \beta \le 1$ and $0 < \frac{\alpha-\beta}{1-\beta} \le 1$, by $< ii >$ and $< i >$ in the Condition C, we obtain

$$\begin{aligned}(\alpha-\beta)\eta(v,u) &= \frac{(\alpha-\beta)}{1-\beta}\eta(v, u+\beta\eta(v,u)) \\ &= \eta\big(u + \beta\eta(v,u) + \frac{\alpha-\beta}{1-\beta}\eta(v, u+\beta\eta(v,u)), u + \beta\eta(v,u)\big).\end{aligned}$$

Using $< i >$ in the Condition C again, we get

$$\frac{1}{1-\beta}\eta(v, u+\beta\eta(v,u)) = \eta(v,u).$$

From the above two identities, the desired identity is obtained, immedetely. The proof of the case $\alpha < \beta$ is similar. This completes the proof of lemma. ∎

In [20], Yang et. al. gave the following Condition D to discuss the characterization of prequasi-invex function.

Condition 2. *Let $K \subset R^n$ be a nonempty invex set with respect to $\eta : K \times K \to R^n$, and let $f : K \to R$ be invex with respect to the same η. we say that the function f satisfies the Condition D if for any $u, v \in K$, the following inequality*

$$f(u + \eta(v,u)) \leq f(v)$$

holds.

Recall that the integral power mean M_p of a positive function on $[a,b]$ is a function given by

$$M_p(f; a, b) = \begin{cases} \left[\frac{1}{b-a}\int_a^b f^p(t)dt\right]^{\frac{1}{p}}, & \text{if } p \neq 0, \\ \exp\left[\frac{1}{b-a}\int_a^b \ln f(t)dt\right], & \text{if } p = 0, \end{cases}$$

and the power mean $M_r(x, y; \lambda)$ of order r of positive numbers x, y which is defined by

$$M_r(x, y; \lambda) = \begin{cases} (\lambda x^r + (1-\lambda)y^r)^{\frac{1}{r}}, & \text{if } r \neq 0, \\ x^\lambda y^{1-\lambda}, & \text{if } r = 0, \end{cases}$$

see [7]. In [7, 14], Stolarsky introduces the following mean values $E(r, s; x, y)$ to generalize the extended logarithmic mean $L_p(x, y)$ and alternative extended logarithmic mean $F_r(x, y)$. This is given by $E(r, s; x, x) = x$ if $x = y > 0$ and for distinct numbers x, y by

$$\begin{aligned} E(r,s;x,y) &= \left[\frac{s}{r}\frac{y^r - x^r}{y^s - x^s}\right]^{\frac{1}{(r-s)}}, \quad rs(r-s) \neq 0, \\ E(r,0;x,y) &= E(0,r;x,y) = \left[\frac{1}{r}\frac{y^r - x^r}{\ln y - \ln x}\right]^{\frac{1}{r}}, \quad r \neq 0, \\ E(r,r;x,y) &= e^{\frac{-1}{r}}\left[\frac{x^{x^r}}{y^{y^r}}\right]^{\frac{1}{(x^r - y^r)}}, \quad r \neq 0, \\ E(0,0;x,y) &= \sqrt{xy}. \end{aligned}$$

Clearly, for two positive numbers x, y, $E(p+1, 1; x, y) = L_p(x, y)$ and $E(r + 1, r; x, y) = F_r(x, y)$.

In order to obtain our results, we will introduce the following new definitions.

Definition 5. *Let $K \subset R^n$ be a nonempty invex set with respect to η. A function $f : K \to R$ is said to be weakly preinvex with respect to η, if there is a vector-value function $\eta : K \times K \to R^n$ such that*

$$f(u + \lambda\eta(v,u)) \leq \lambda f(u + \eta(v,u)) + (1-\lambda)f(u),$$

for every $v, u \in K$ and $\lambda \in [0,1]$.

A natural idea of weakly r-preinvexity may be investigated via power means.

Definition 6. *Let $K \subset R^n$ be a nonempty invex set with respect to η. A function $f : K \to R^+$ is said to be weakly r-preinvex with respect to η, if there is a vector-value function $\eta : K \times K \to R^n$ such that*

$$f(u + \lambda\eta(v,u)) \leq M_r(f(u + \eta(v,u)), f(u); \lambda)$$

for every $v, u \in K$ *and* $\lambda \in [0, 1]$.

We note that if f is weakly r-preinvex function, then f^r is weakly preinvex function for positive r, and if f is weakly 0-preinvex function, then $log \circ f$ is weakly preinvex function. We also note that, in Definition 5 and Definition 6, if f further satisfies the Condition D, then f is preinvex function and r-preinvex function, respectively.

The extended mean of two-prameters for weakly r-preinvex function on invex set is defined as follows.

Definition 7. *Let* $K \subset R^n$ *be a nonempty invex set with respect to a vector-value function* $\eta : K \times K \to R^n$*, let* $f : K \to R^+$ *be an integrable on* η*-path* P_{ux} *for* $v, u \in K$*,* $\lambda \in [0, 1]$ *and* $x = u + \eta(v, u)$*. We define the two-prameters mean of the function* $f(u + \lambda\eta(v, u))$ *on* $[0, 1]$ *with respect to* λ *by*

$$M_{p,q}(f; u, u + \eta(v, u)) = \begin{cases} \left[\frac{\int_0^1 f^p(u+\lambda\eta(v,u))d\lambda}{\int_0^1 f^q(u+\lambda\eta(v,u))d\lambda}\right]^{\frac{1}{(p-q)}}, & \text{if } p \neq q, \\ \exp \frac{\int_0^1 f^q(u+\lambda\eta(v,u)) \ln f(u+\lambda\eta(v,u))d\lambda}{\int_0^1 f^q(u+\lambda\eta(v,u))d\lambda}, & \text{if } p = q. \end{cases}$$

In particular, when $q = 0$, denote $M_{p,0}(f; u, u + \eta(v, u)) = M_p(f; u, u + \eta(v, u))$ is the integral power mean.

In order to prove our main results, the following properties of weakly r-preinvex function are necessary.

Proposition 1. *Let* $K \subset R^n$ *be a nonempty invex set with respect to* $\eta : K \times K \to R^n$*,* η *satisfies Condition C and* $u \in K$*, and let* $f : P_{ux} \to R$ *for every* $v \in K$*,* $\lambda \in [0, 1]$ *and* $x = u+\eta(v, u) \in K$*. Suppose that* $r \geq 0$*, then* f *is a weakly* r*-preinvex function with respect to* η *if and only if* f^r *is convex function with respect to* λ*.*

Proof. Let $\phi(\lambda) = f^r(u + \lambda\eta(v, u))$ for $u, v \in K$, $\lambda \in [0, 1]$, $u + \lambda\eta(v, u) \in K$, and $r \geq 0$. The first, assume that f is a weakly r-preinvex function with respect to η and η satisfies Condition C. Obviously, f^r is weakly preinvex function with respect to the same η. Now, we will prove that $\phi(\lambda)$ is convex on $[0, 1]$. By lemma 1 and f^r is weakly preinvex, given $\alpha, \beta \in [0, 1]$ and for any $\lambda \in [0, 1]$, we obtain

$$\begin{aligned} & \phi(\beta + \lambda(\alpha - \beta)) \\ = \; & f^r\big(u + (\beta + \lambda(\alpha - \beta))\eta(v, u)\big) \\ = \; & f^r\big(u + \beta\eta(v, u) + \lambda(\alpha - \beta)\eta(v, u)\big) \\ = \; & f^r\big(u + \beta\eta(v, u) + \lambda(\eta(u + \alpha\eta(v, u), u + \beta\eta(v, u)))\big)(\text{by lemma 1}) \\ \leq \; & \lambda f^r\big(u + \beta\eta(v, u) + \eta(u + \alpha\eta(v, u), u + \beta\eta(v, u))\big) + (1 - \lambda)f^r\big(u + \beta\eta(v, u)\big) \\ = \; & \lambda f^r\big(u + \alpha\eta(v, u)\big) + (1 - \lambda)f^r\big(u + \beta\eta(v, u)\big)(\text{by lemma 1}) \end{aligned}$$

for $r > 0$, and, similarly,

$$\begin{aligned} & \phi(\beta + \lambda(\alpha - \beta)) \\ \leq \; & f^\lambda\big(u + \beta\eta(v, u) + \eta(u + \alpha\eta(v, u), u + \beta\eta(v, u))\big)f^{1-\lambda}\big(u + \beta\eta(v, u)\big) \\ = \; & f^\lambda\big(u + \alpha\eta(v, u)\big)f^{1-\lambda}\big(u + \beta\eta(v, u)\big) \end{aligned}$$

for $r = 0$. Therefore, we have

$$\phi(\beta + \lambda(\alpha - \beta)) \leq \begin{cases} \lambda\phi(\alpha) + (1-\lambda)\phi(\beta), & \text{if } r > 0, \\ \phi^{\lambda}(\alpha)\phi^{1-\lambda}(\beta), & \text{if } r = 0. \end{cases}$$

It is proved that $f^r(u + \lambda\eta(v,u))$ is convex function with respect to λ.

The second, assume that $f^r(u + \lambda\eta(v,u))$ is convex function with respect to λ. We will prove that $f(u + \lambda\eta(v,u))$ is a weakly r-preinvex function with respect to η. By $\phi(\lambda) = f^r(u + \lambda\eta(v,u))$ is convex function with respect to λ, we obtain

$$\phi(\lambda \cdot 1 + (1-\lambda) \cdot 0) \leq \begin{cases} \lambda\phi(1) + (1-\lambda)\phi(0), & \text{if } r > 0, \\ \phi^{\lambda}(1)\phi^{1-\lambda}(0), & \text{if } r = 0, \end{cases}$$

and then

$$f^r(u + \lambda\eta(v,u)) \leq \begin{cases} \lambda f^r(u + \eta(v,u)) + (1-\lambda) f^r(u), & \text{if } r > 0, \\ f^{\lambda}(u + \eta(v,u)) f^{1-\lambda}(u), & \text{if } r = 0. \end{cases}$$

We have f is weakly r-preinvex with respect to η. This completes the proof of proposition. ∎

Proposition 2. *Suppose that assumption in Proposition 1 is satisfied, and further, let f be continuous on P_{ux} and twice-differentiable on P_{ux}^0. Then f is a weakly r-preinvex function with respect to η if and only if*

$$rf^{r-2}(u)\{(r-1)[\eta(v,u)^T\nabla f(u)]^2 + f(u)\eta(v,u)^T\nabla^2 f(u)\eta(v,u)\} \geq 0$$

for $r > 0$, and

$$\{\eta(v,u)^T\nabla^2 f(u)\eta(v,u)f(u) - [\eta(v,u)^T\nabla f(u)]^2\}/f^2(u) \geq 0$$

for $r = 0$.

Proof. Let $\phi(\lambda) = f^r(u + \lambda\eta(v,u))$ for $u, v \in K$, $\lambda \in [0,1]$, $u + \lambda\eta(v,u) \in K$, and $r \geq 0$. Applying f is continuous and twice-differentiable weakly r-preinvex function with respect to η, we obtain

$$\phi'(\lambda) = \begin{cases} rf^{r-1}(u + \lambda\eta(v,u))\eta(v,u)^T\nabla f(u + \lambda\eta(v,u)), & \text{if } r > 0, \\ \eta(v,u)^T\nabla f(u + \lambda\eta(v,u))/f(u + \lambda\eta(v,u)), & \text{if } r = 0, \end{cases}$$

and

$$\phi''(\lambda) = \begin{cases} rf^{r-2}(u + \lambda\eta(v,u))\{(r-1)[\eta(v,u)^T\nabla f(u + \lambda\eta(v,u))]^2 \\ +f(u + \lambda\eta(v,u))\eta(v,u)^T\nabla^2 f(u + \lambda\eta(v,u))\eta(v,u)\}, & \text{if } r > 0, \\ \{\eta(v,u)^T\nabla^2 f(u + \lambda\eta(v,u))\eta(v,u)f(u + \lambda\eta(v,u)) \\ -[\eta(v,u)^T\nabla f(u + \lambda\eta(v,u))]^2\}/f^2(u + \lambda\eta(v,u)), & \text{if } r = 0, \end{cases}$$

and let $\lambda \to 0^+$, we obtain

$$\phi''(\lambda) = \begin{cases} rf^{r-2}(u)\{(r-1)[\eta(v,u)^T\nabla f(u)]^2 \\ +f(u)\eta(v,u)^T\nabla^2 f(u)\eta(v,u)\}, & \text{if } r > 0, \\ \{\eta(v,u)^T\nabla^2 f(u)\eta(v,u)f(u) \\ -[\eta(v,u)^T\nabla f(u)]^2\}/f^2(u), & \text{if } r = 0. \end{cases}$$

Applying Proposition 1 for $r \geq 0$, we get $\phi(\lambda) = f^r(u + \lambda\eta(v,u))$ is convex function with respect to λ and then $\phi''(\lambda) \geq 0$. This completes the desire.

Conversely, assume that for every $u, v \in K$,

$$rf^{r-2}(u)\{(r-1)[\eta(v,u)^T\nabla f(u)]^2 + f(u)\eta(v,u)^T\nabla^2 f(u)\eta(v,u)\} \geq 0$$

for $r > 0$, and

$$\{\eta(v,u)^T\nabla^2 f(u)\eta(v,u)f(u) - [\eta(v,u)^T\nabla f(u)]^2\}/f^2(u) \geq 0$$

for $r = 0$. We will prove that f is a weakly r-preinvex function with respect to η. Applying the assumption, we obtain, for every $u, v \in K$, λ in $[0,1]$, and $u + \lambda\eta(v,u) \in K$,

$$\begin{aligned} & rf^{r-2}(u+\lambda\eta(v,u))\{(r-1)[\eta(v,u+\lambda\eta(v,u))^T\nabla f(u+\lambda\eta(v,u))]^2 \\ + \quad & f(u+\lambda\eta(v,u))\eta(v,u+\lambda\eta(v,u))^T\nabla^2 f(u+\lambda\eta(v,u))\eta(v,u+\lambda\eta(v,u))\} \geq 0 \end{aligned}$$

for $r > 0$, and

$$\begin{aligned} & \{\eta(v,u+\lambda\eta(v,u))^T\nabla^2 f(u+\lambda\eta(v,u))\eta(v,u+\lambda\eta(v,u))f(u+\lambda\eta(v,u+\lambda\eta(v,u))) \\ - \quad & [\eta(v,u+\lambda\eta(v,u))^T\nabla f(u+\lambda\eta(v,u))]^2\}/f^2(u+\lambda\eta(v,u)) \geq 0 \end{aligned}$$

for $r = 0$. By $< ii >$ in Condition C, we get

$$\begin{aligned} & rf^{r-2}(u+\lambda\eta(v,u))\{(r-1)[(1-\lambda)\eta(v,u)^T\nabla f(u+\lambda\eta(v,u))]^2 \\ + \quad & f(u+\lambda\eta(v,u))(1-\lambda)\eta(v,u)^T\nabla^2 f(u+\lambda\eta(v,u))(1-\lambda)\eta(v,u)\} \geq 0 \end{aligned}$$

for $r > 0$, and

$$\begin{aligned} & \{(1-\lambda)^2\eta(v,u)^T\nabla^2 f(u+\lambda\eta(v,u))\eta(v,u)f(u+\lambda\eta(v,u+\lambda\eta(v,u))) \\ - \quad & [(1-\lambda)\eta(v,u)^T\nabla f(u+\lambda\eta(v,u))]^2\}/f^2(u+\lambda\eta(v,u)) \geq 0 \end{aligned}$$

for $r = 0$. Thus, we have

$$\begin{aligned} \phi''(\lambda) \quad = \quad & rf^{r-2}(u+\lambda\eta(v,u))\{(r-1)[\eta(v,u)^T\nabla f(u+\lambda\eta(v,u))]^2 \\ & +f(u+\lambda\eta(v,u))\eta(v,u)^T\nabla^2 f(u+\lambda\eta(v,u))\eta(v,u)\} \geq 0 \end{aligned}$$

for $r > 0$, and

$$\begin{aligned} \phi''(\lambda) \quad = \quad & \{\eta(v,u)^T\nabla^2 f(u+\lambda\eta(v,u))\eta(v,u)f(u+\lambda\eta(v,u+\lambda\eta(v,u))) \\ & -[\eta(v,u)^T\nabla f(u+\lambda\eta(v,u))]^2\}/f^2(u+\lambda\eta(v,u)) \geq 0 \end{aligned}$$

for $r = 0$. This implies that $\phi(\lambda) = f^r(u+\lambda\eta(v,u))$ is convex function with respect to λ. Applying Proposition 1, we have that f is a weakly r-preinvex function with respect to η. The proof of propositon is complete. ∎

3. Hermite-Hadamard inequality for weakly r-preinvex function

For simplicity, in this section, we assume that $K \subset R^n$ be a nonempty invex set with respect to a vector value function $\eta : K \times K \to R^n$. The following theorem is our main result.

Theorem 1. *Let f be a weakly r-preinvex function on invex K with $r \geq 0$. Assume that f be a positive and continuous function on P_{ax} and twice-differentiable on P^0_{ax} for every $a, b \in K$, $\lambda \in [0,1]$ and $a < x = a + \eta(b,a)$, and let η satisfy Condition C. Further, let $g_1, g_2 : (0,\infty) \to R, g_2$ be a positive integrable on $[m, M]$ and the ratio*

g_1/g_2 integrable on $[m, M]$, where m and M are the minimum and maximun of f on P_{ax}, respectively. If g_1/g_2 is increasing on $[m, M]$, then

$$\frac{\int_0^1 g_1(f(a+\lambda\eta(b,a)))d\lambda}{\int_0^1 g_2(f(a+\lambda\eta(b,a)))d\lambda} \leq \frac{\int_{f(a)}^{f(a+\eta(b,a))} x^{r-1}g_1(x)dx}{\int_{f(a)}^{f(a+\eta(b,a))} x^{r-1}g_2(x)dx} \tag{3.1}$$

for $f(a) \neq f(a+\eta(b,a))$, the right-hand side of (3.1) is defined by $g_1(f(a))/g_2(f(a))$ for $f(a) = f(a+\eta(b,a))$, while if g_1/g_2 is decreasing, the inequality (3.1) is reversed.

Proof. Let $\phi(\lambda) = f^r(a+\lambda\eta(b,a))$ for $r \neq 0$ and $\phi(\lambda) = \ln f(a+\lambda\eta(b,a))$ for $r = 0$. For convenience, let $\psi(\lambda) = f(a+\lambda\eta(b,a))$. Applying f is weakly r-preinvex function with respect to η, by Proposition 2, we have

$$\phi''(\lambda) \quad = \quad rf^{(r-2)}(a)\{(r-1)[\eta(b,a)^T\nabla f(a)]^2 + f(a)\eta(b,a)^T\nabla^2 f(a)\eta(b,a)\}.$$

is positive. We give only the proof in the case of $r > 0$ and g_1/g_2 being increasing, since the proof in the other is analogous.

When $f(a) \neq f(a+\eta(b,a))$. The inequality (3.1) is equivalent to

$$\frac{\int_0^1 g_1(\psi(\lambda)d\lambda}{\int_0^1 g_2(\psi(\lambda)d\lambda} \leq \frac{\int_0^1 \psi^{r-1}(\lambda)g_1(\psi(\lambda))\psi'(\lambda)d\lambda}{\int_0^1 \psi^{r-1}(\lambda)g_2(\psi(\lambda))\psi'(\lambda)d\lambda}. \tag{3.2}$$

We take notation

$$\begin{aligned} I = &\int_0^1 g_1(\psi(\lambda))d\lambda \int_0^1 \psi^{r-1}(\mu)g_2(\psi(\mu))\psi'(\mu)d\mu \\ &- \int_0^1 g_2(\psi(\lambda))d\lambda \int_0^1 \psi^{r-1}(\mu)g_1(\psi(\mu))\psi'(\mu)d\mu \\ = &\int_0^1 \int_0^1 g_2(\psi(\lambda))g_2(\psi(\mu))\psi^{r-1}(\mu)\psi'(\mu)\Big[\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\Big]d\lambda d\mu. \end{aligned} \tag{3.3}$$

Replacing λ and μ by each other in (3.3), we get

$$I = \int_0^1 \int_0^1 g_2(\psi(\lambda))g_2(\psi(\mu))\psi^{r-1}(\lambda)\psi'(\lambda)\Big[\frac{g_1(\psi(\mu))}{g_2(\psi(\mu))} - \frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))}\Big]d\lambda d\mu. \tag{3.4}$$

Now, adding (3.3) and (3.4), we obtain

(3.5)
$$I = \frac{1}{2r}\int_0^1 \int_0^1 g_2(\psi(\lambda))g_2(\psi(\mu))\big[(\psi^r(\mu))' - (\psi^r(\lambda))'\big]\Big[\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\Big]d\lambda d\mu.$$

If the derivative $\phi'(\lambda) = (\psi^r(\lambda))' \geq 0$ for all $\lambda \in (0,1)$, from $\phi''(\lambda) = (\psi^r(\lambda))'' \geq 0$, we have

$$\frac{1}{r}\big[(\psi^r(\mu))' - (\psi^r(\lambda))'\big]\Big[\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\Big] \leq 0.$$

From (3.5), we get $I \leq 0$. This implies that the inequality (3.2) holds and then (3.1) holds. If the derivative $\phi'(\lambda) = (\psi^r(\lambda))' \leq 0$ for all $\lambda \in (0,1)$, by the argument similar to that used in the case of $\phi'(\lambda) = (\psi^r(\lambda))' \geq 0$, we have $I \geq 0$, This implies that the inequality (3.1) holds. If the sign of derivative $\phi'(\lambda) = (\psi^r(\lambda))'$ can be changed, and $\phi(0) < \phi(1)$ and then $\psi^r(0) \leq \psi^r(1)$, there exist a point $\alpha \in (0,1)$ such that $\phi'(\alpha) = (\psi^r(\alpha))' = 0$, and $(\psi^r(\lambda))' \leq 0$ for all $\lambda \in [0,\alpha]$ and $(\psi^r(\lambda))' \geq 0$

for all $\lambda \in [\alpha, 1]$. Therefore, there exist $\beta \in (\alpha, 1)$ such that $\psi(0) = \psi(\beta)$. Thus, we get

$$\int_0^{\beta} \psi^{r-1}(\lambda) g_1(\psi(\lambda))\psi'(\lambda) d\lambda = \int_{\psi(0)}^{\psi(\alpha)} x^{r-1} g_1(x) dx + \int_{\psi(\alpha)}^{\psi(\beta)} x^{r-1} g_1(x) dx = 0,$$

and similarly,

$$\int_0^{\beta} \psi^{r-1}(\lambda) g_2(\psi(\lambda))\psi'(\lambda) d\lambda = 0,$$

and then we have that inequality (3.1) is equivalent to

$$\frac{\int_0^1 g_1(\psi(\lambda))d\lambda}{\int_0^1 g_1(\psi(\lambda))d\lambda} \le \frac{\int_\beta^1 \psi^{r-1}(\lambda) g_1(\psi(\lambda))\psi'(\lambda) d\lambda}{\int_\beta^1 \psi^{r-1}(\lambda) g_2(\psi(\lambda))\psi'(\lambda) d\lambda}. \tag{3.6}$$

We take notation

$$\begin{aligned} I_2 = & \int_0^1 g_1(\psi(\lambda))d\lambda \int_\beta^1 \psi^{r-1}(\mu) g_2(\psi(\mu))\psi'(\mu) d\mu \\ & - \int_0^1 g_2(\psi(\lambda))d\lambda \int_\beta^1 \psi^{r-1}(\mu) g_1(\psi(\mu))\psi'(\mu) d\mu \\ = & \frac{1}{r}\int_0^1 \int_\beta^1 g_2(\psi(\lambda)) g_2(\psi(\mu))\psi^{r-1}(\mu)\psi'(\mu)\Big[\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\Big] d\lambda d\mu, \end{aligned} \tag{3.7}$$

and considering

$$I_{21} = \frac{1}{r}\int_0^\beta \int_\beta^1 g_2(\psi(\lambda)) g_2(\psi(\mu))\psi^{r-1}(\mu)\psi'(\mu)\Big[\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\Big] d\lambda d\mu,$$

and

$$I_{22} = \frac{1}{r}\int_\beta^1 \int_\beta^1 g_2(\psi(\lambda)) g_2(\psi(\mu))\psi^{r-1}(\mu)\psi'(\mu)\Big[\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\Big] d\lambda d\mu.$$

When $(\lambda, \mu) \in [0, \beta] \times [\beta, 1]$, we have $\lambda \le \mu$ and $(\psi^r(\mu))' = r\psi^{r-1}(\mu)\psi'(\mu) \ge 0$ for all $\mu \in (\beta, 1)$, and then $\psi'(\mu) \ge 0$ for all $\mu \in (\beta, 1)$, thus

$$\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} \le \frac{g_1(\psi(\beta))}{g_2(\psi(\beta))} \le \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}.$$

Thus, we have that $I_{21} \le 0$. By the result proved in case of $\phi'(\lambda) = (\psi^r(\lambda))' \ge 0$, we can get $I_{22} \le 0$. Therefore, $I_2 = I_{21} + I_{22} \le 0$, and then (3.6) holds. It follows that (3.1) holds. If the sign of the derivative $\phi'(\lambda) = (\psi^r(\lambda))'$ can be changed and $\psi(0) \ge \psi(1)$. Using the proof similar to case of $\phi(0) < \phi(1)$, we can derive that inequaluty (3.1) holds.

When $f(a) = f(a + \eta(b, a))$, we have $\psi(0) = \psi(1)$, and then $\phi(0) = \phi(1)$. Since $\phi'' = (\psi^r(\lambda))'' \ge 0$, we derive that $\phi' = (\psi^r(\lambda))'$ is continous and increasing for $\lambda \in (0, 1)$. There exist a point $\alpha \in (0, 1)$ such that $(\psi^r(\alpha))' = 0$ and $(\psi^r(\lambda))' \le 0$ for all $\lambda \in (0, \alpha)$, and $(\psi^r(\lambda))' \ge 0$ for all $\lambda \in (\alpha, 1)$. Hence

$$\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} \le \frac{g_1(\psi(1))}{g_2(\psi(1))},$$

for all $\lambda \in (0, 1)$. It follows that

$$\int_0^1 g_1(\psi(\lambda))d\lambda \le \frac{g_1(\psi(1))}{g_2(\psi(1))}\int_0^1 g_2(\psi(\lambda))d\lambda.$$

Therefore, the inequality (3.1) is valid. This completes the proof of Theorem. ∎

Remark 1. *If take $g_1(x) = x^p, g_2(x) = x^q$ for suitable real number p, q in* (3.1)*, we get the following extended mean inequality for the twice-differentiable and weakly r-preinvex function f on invex set with respect to η satisfied condition C.*

$$M_{p,q}(f; a, a+\eta(b,a)) \le E(p+r, q+r; f(a), f(a+\eta(b,a))). \tag{3.8}$$

Moreover, if take $q = 0$ in (3.8)*, we obtain*

$$M_p(f; a, a+\eta(b,a)) \le E(p+r, r; f(a), f(a+\eta(b,a))), \tag{3.9}$$

If take $r = 1$ in (3.9)*, we have*

$$M_p(f; a, a+\eta(b,a)) \le L_p(f(a), f(a+\eta(b,a))),$$

and take $p = 1$ in (3.9)*, we have*

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \le F_r(f(a), f(a+\eta(b,a))). \tag{3.10}$$

Further moreover, if f satisfies the Condition D, from (3.10)*, we obtain*

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \le F_r(f(a), f(a+\eta(b,a))) \le F_r(f(a), f(b)). \tag{3.11}$$

We note that the inequality (3.11) *is a refinement of the inequality given by Wasim Ui-Haq and Javed Iqbal in* [17]*. For $r = 1$ or $r = 0$ in* (3.11)*, we also note that the inequality* (3.11) *is a refinement of the inequality given by Noor in* [11]*.*

Theorem 2. *Let f be a weakly r-preinvex function on invex K with $r \ge 0$. Assume that f be a positive and continuous function on P_{ax} for given $a, b \in K$, $\lambda \in [0,1]$ and $a < x = a + \eta(b,a)$. Further, let $g : (0,\infty) \to R$ be a positive integrable on $[m, M]$, where m, M as in Theorem* (1)*. If g is increasing on $[m, M]$, then*

$$\int_0^1 g(f(a+\lambda\eta(b,a)))d\lambda \le \frac{r\int_{f(a)}^{f(a+\eta(b,a))} x^{r-1}g(x)dx}{f^r(a+\eta(b,a)) - f^r(a)} \tag{3.12}$$

for $f(a) \ne f(a+\eta(b,a))$, the right-hand side of (3.12) *is defined by $g(f(a))$ for $f(a) = f(a+\eta(b,a))$, while if g is decreasing, the inequality* (3.12) *is reversed.*

Proof. Here we consider only the case when $r > 0$ and g is increasing, the proof is analogous in other case. When $f(a) \ne f(a+\eta(b,a))$, by definition of weakly r-preinvex function, we obtain

$$\begin{aligned}
&\int_0^1 g(f(a+\lambda\eta(b,a)))d\lambda \\
\le &\int_0^1 g\big((\lambda f^r(a+\eta(b,a)) + (1-\lambda)f^r(a))^{1/r}\big)d\lambda \\
= &\frac{r}{f^r(a+\eta(b,a)) - f^r(a)} \int_{f(a)}^{f(a+\eta(b,a))} g(x)x^{r-1}dx.
\end{aligned}$$

Similarly, when $f(a) = f(a+\eta(b,a))$, we have

$$\begin{aligned}&\int_0^1 g(f(a+\lambda\eta(b,a)))d\lambda\\ \le&\int_0^1 g\big((\lambda f^r(a+\eta(b,a))+(1-\lambda)f^r(a))^{1/r}\big)d\lambda\\ =&g(f(a)),\end{aligned}$$

immediately. The proof of Theorem is complete. ∎

Remark 2. *We note that it is not necessary that the function f in Theorem 2 is twice-differentiable. Similar to the remark 1, if take $g(x) = x^p$ in* (3.12)*, we obtain the following extended mean inequality for the weakly r-preinvex function f on invex set with respect to η.*

$$M_p(f;a,a+\eta(b,a)) \le E(p+r,r;f(a),f(a+\eta(b,a))), \tag{3.13}$$

moreover, take $r = 1$ in (3.13)*, we have*

$$M_p(f;a,a+\eta(b,a)) \le L_p(f(a),f(a+\eta(b,a))),$$

and take $p = 1$ in (3.13)*, we have*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \le F_r(f(a),f(a+\eta(b,a))). \tag{3.14}$$

Further moreover, if f satisfies Condition D, from (3.14)*, we obtain*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \le F_r(f(a),f(a+\eta(b,a))) \le F_r(f(a),f(b)). \tag{3.15}$$

We note that the inequality (3.15) *is a refinement of the inequality given by Wasim Ui-Haq and Javed Iqbal in* [17] *and then is also a refinement of the inequality given by Noor in* [11].

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