Received 31/08/16

HERMITE-HADAMARD TYPE INEQUALITIES FOR CONFORMABLE FRACTIONAL INTEGRALS

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Abstract. In this paper, first we prove an identity for conformable fractional integrals. Second by using this identity we will present some integral inequalities connected with the left hand side of the Hermite-Hadamard type inequalities for conformable fractional integral. At the end applications to some special means and error estimates for the midpoint formula are discussed.

1. Introduction

The following class of functions is well known in the literature and is usually defined in the following way: a function $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \tag{1}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$. Also we say that f is concave, if the inequality in (1) is reversed. Many important inequalities have been obtained for this class of functions but here we will present only one of them in following:

If $f : I \to \mathbb{R}$ is a convex function on the interval I, then for any $a, b \in I$ with $a \neq b$, we have the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

Both inequalities hold in the reversed direction if f is concave. This remarkable result was given in ([13], 1893) and is well known in the literature as the Hermite-Hadamard inequality. Since, its discovery this inequality has become the center of interest for many prolific researchers and has received a considerable attention. Also a number of extensions, generalizations and variants of (2) have been appeared in the theory of Mathematical inequalities, for example see [1, 4–6, 8–12, 19–24, 26] and the references cited therein.

In [7] Dragomir and Agarwal proved the following results connected with the right hand part of H-H inequality.

2010 *Mathematics Subject Classification.* 26D15, 26A51, 26A33, 26A42.

Key words and phrases. Convex function, Hermite-Hadamard inequality, Conformable derivative, Conformable integrals, Special means, Midpoint formula.

Lemma 1 ([7])**.** *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then the following identity holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt. \tag{3}$$

Theorem 1. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$ and $|f'|$ is convex on $[a, b]$, then we have the following inequality:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{4}\left(\frac{|f'(a)|+|f'(b)|}{2}\right). \tag{4}$$

In [18], U. S. Kirmaci gave the following results.

Lemma 2 ([18])**.** *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \\ = \ &(b-a)\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (t-1)f'(ta+(1-t)b)dt\right]. \end{aligned} \tag{5}$$

Theorem 2 ([18])**.** *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$ and $|f'|$ is convex on $[a, b]$, then we have the following inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{6}$$

The following definitions and theorems with respect to conformable fractional derivative and integral were referred in [2, 14–17].

Definition 1. (**Conformable fractional derivative**). Given a function $f : [0, \infty) \to \mathbb{R}$. Then the "conformable fractional derivative" of f of order α is defined by

$$D_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon} \tag{7}$$

for all $t > 0, \alpha \in (0, 1)$. If f is α-differentiable in some $(0, a)$, $\alpha > 0$ $\lim_{t\to 0^+} f^\alpha(t)$ exist, then define

$$f^\alpha(0) = \lim_{t \to 0^+} f^\alpha(t). \tag{8}$$

We can write $f^\alpha(t)$ for $D_\alpha(f)(t)$ to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is α-differentiable.

Theorem 3. *Let $\alpha \in (0, 1]$ and f, g be α-differentiable at a point $t > 0$. Then*

i. $\frac{d_\alpha}{d_\alpha t}(t^n) = nt^{n-\alpha}$, *for all $n \in \mathbb{R}$.*
ii. $\frac{d_\alpha}{d_\alpha t}(c) = 0$, *for all constant functions $f(t) = c$.*
iii. $\frac{d_\alpha}{d_\alpha t}(af(t)+bg(t)) = a\frac{d_\alpha}{d_\alpha t}(f(t)) + b\frac{d_\alpha}{d_\alpha t}(g(t))$, *for all $a, b \in \mathbb{R}$.*
iv. $\frac{d_\alpha}{d_\alpha t}(f(t).g(t)) = f(t)\frac{d_\alpha}{d_\alpha t}(g(t)) + g(t)\frac{d_\alpha}{d_\alpha t}(f(t))$.

v. $\frac{d_\alpha}{d_\alpha t}\left(\frac{f(t)}{g(t)}\right) = \frac{f(t)\frac{d_\alpha}{d_\alpha t}(f(t)) - g(t)\frac{d_\alpha}{d_\alpha t}(f(t))}{(g(t))^2}$.

vi. $\frac{d_\alpha}{d_\alpha t}\left((f \circ g)(t)\right) = f'(g(t))\frac{d_\alpha}{d_\alpha t}(g(t))$, *for f differentiable at $g(t)$.*

If in addition f is differentiable, then

$$\frac{d_\alpha}{d_\alpha t}(f(t)) = t^{1-\alpha}\frac{d}{dt}(f(t)). \tag{9}$$

Also it is important to note the following:

(1) $\frac{d_\alpha}{d_\alpha t}(1) = 0$.
(2) $\frac{d_\alpha}{d_\alpha t}(e^{ax}) = ax^{1-\alpha}e^{ax}, a \in \mathbb{R}$.
(3) $\frac{d_\alpha}{d_\alpha t}(\sin(ax)) = ax^{1-\alpha}\cos(ax), a \in \mathbb{R}$.
(4) $\frac{d_\alpha}{d_\alpha t}(\cos(ax)) = -ax^{1-\alpha}\sin(ax), a \in \mathbb{R}$.
(5) $\frac{d_\alpha}{d_\alpha t}(\frac{1}{\alpha}t^\alpha) = 1$.
(6) $\frac{d_\alpha}{d_\alpha t}\left(\sin\left(\frac{t^\alpha}{\alpha}\right)\right) = \cos\left(\frac{t^\alpha}{\alpha}\right)$.
(7) $\frac{d_\alpha}{d_\alpha t}\left(\cos\left(\frac{t^\alpha}{\alpha}\right)\right) = -\sin\left(\frac{t^\alpha}{\alpha}\right)$.
(8) $\frac{d_\alpha}{d_\alpha t}\left(e^{\left(\frac{t^\alpha}{\alpha}\right)}\right) = e^{\left(\frac{t^\alpha}{\alpha}\right)}$.

Theorem 4. ***(Mean value theorem for conformable fractional differentiable functions)****. Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be a continuous on $[a,b]$ and an α-fractional differentiable mapping on (a,b) with $0 \le a < b$. Then, there exists $c \in (a,b)$, such that*

$$D_\alpha(f)(c) = \frac{f(b) - f(a)}{\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}}.$$

Definition 2. (**Conformable fractional integral**). Let $\alpha \in (0,1]$ and $0 \le a < b$. A function $f : [a,b] \to \mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral

$$\int_a^b f(x)d_\alpha x := \int_a^b f(x)x^{\alpha-1}dx \tag{10}$$

exists and is finite. All α-fractional integrable functions on $[a,b]$ is indicated by $L^1_\alpha([a,b])$.

Remark 1.

$$I^a_\alpha(f)(t) = I^a_1(t^{\alpha-1}f) = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx, \tag{11}$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

Theorem 5. *Let $f : (a,b) \to \mathbb{R}$ be differentiable and $0 < \alpha \le 1$. Then, for all $t > a$ we have*

$$I^a_\alpha D^a_\alpha(f)(t) = f(t) - f(a). \tag{12}$$

Theorem 6. ***(Integration by parts)*** *Let $f, g : [a,b] \to \mathbb{R}$ be two functions such that fg is differentiable. Then*

$$\int_a^b f(x)D^a_\alpha(g)(x)d_\alpha x = fg|_a^b - \int_a^b g(x)D^a_\alpha(f)(x)d_\alpha x. \tag{13}$$

Theorem 7. *Assume that $f:[a,\infty)\to R$ such that $f^{(n)}(t)$ is continuous and $\alpha\in(n,n+1]$. Then, for all $t>a$ we have*

$$D_\alpha^a f(t) I_\alpha^a = f(t).$$

Theorem 8. *Let $\alpha\in(0,1]$ and $f:[a,b]\to\mathbb{R}$ be a continuous on $[a,b]$ with $0\le a<b$. Then,*

$$|I_\alpha^a(f)(x)| \le I_\alpha^a|f|(x).$$

More recently in [3] D. R. Anderson investigated the following conformable integral version of Hermite-Hadamard inequality:

Theorem 9 ([3])**.** *If $\alpha\in(0,1]$ and $f:[a,b]\to\mathbb{R}$ is an α-fractional differentiable function such that $D_\alpha f$ is increasing, then we the following inequality*

$$\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(t)d_\alpha t \le \frac{f(a)+f(b)}{2}. \tag{14}$$

Moreover if the function f is decreasing on $[a,b]$, then we have

$$f\left(\frac{a+b}{2}\right) \le \frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(t)d_\alpha t. \tag{15}$$

If $\alpha=1$, then this reduces to the classical Hermite-Hadamard inequality.

In this paper, we prove an identity for conformable fractional integrals. Applying this identity we establish some new Hermite-Hadamard type inequalities for conformable fractional integral connected with the Hermite-Hadamard type inequalities for conformable fractional integral. The obtained results are further used to get new bounds for the special means of real numbers and to derive some error estimates for the midpoint formula.

2. Main Results

We begin this section with the following lemma associated with the inequality (15), which is essential for the derivation of our main results.

Lemma 3. *Let $a,b\in\mathbb{R}$ with $0\le a<b$, and let $f:[a,b]\to\mathbb{R}$ be an α-fractional differentiable function on (a,b) for $\alpha\in(0,1]$. If $D_\alpha(f)\in L_\alpha^1([a,b])$, then the following identity holds:*

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) - \frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(s)d_\alpha s \\
= \ & \frac{(b-a)}{b^\alpha-a^\alpha}\Bigg[\int_0^{\frac{1}{2}}\left(((1-t)a+tb)^{2\alpha-1}-a^\alpha((1-t)a+tb)^{\alpha-1}\right) \\
\times \ & D_\alpha(f)((1-t)a+tb)t^{1-\alpha}d_\alpha t + \int_{\frac{1}{2}}^1\left(((1-t)a+tb)^{2\alpha-1}-b^\alpha((1-t)a+tb)^{\alpha-1}\right) \\
\times \ & D_\alpha(f)((1-t)a+tb)t^{1-\alpha}d_\alpha t\Bigg].
\end{aligned} \tag{16}$$

Proof. Integrating by parts, we have

$$
\begin{aligned}
I = & \int_0^{\frac{1}{2}} \left(((1-t)a+tb)^{2\alpha-1} - a^{\alpha}((1-t)a+tb)^{\alpha-1}\right) D_{\alpha}(f)((1-t)a+tb)dt \\
+ & \int_{\frac{1}{2}}^{1} \left(((1-t)a+tb)^{2\alpha-1} - b^{\alpha}((1-t)a+tb)^{\alpha-1}\right) D_{\alpha}(f)((1-t)a+tb)dt \\
= & \int_0^{\frac{1}{2}} \left(((1-t)a+tb)^{\alpha} - a^{\alpha}\right) f'((1-t)a+tb)dt \\
+ & \int_{\frac{1}{2}}^{1} \left(((1-t)a+tb)^{\alpha} - b^{\alpha}\right) f'((1-t)a+tb)dt \\
= & \left(((1-t)a+tb)^{\alpha} - a^{\alpha}\right) \frac{f((1-t)a+tb)}{b-a}\Bigg|_0^{\frac{1}{2}} \\
- & \int_0^{\frac{1}{2}} \alpha((1-t)a+tb)^{\alpha-1}(b-a)\frac{f((1-t)a+tb)}{b-a}dt \\
+ & \left(((1-t)a+tb)^{\alpha} - b^{\alpha}\right) \frac{f((1-t)a+tb)}{b-a}\Bigg|_{\frac{1}{2}}^{1} \\
- & \int_{\frac{1}{2}}^{1} \alpha((1-t)a+tb)^{\alpha-1}(b-a)\frac{f((1-t)a+tb)}{b-a}dt \\
= & \frac{1}{b-a}\left[\left(\left(\frac{a+b}{2}\right)^{\alpha} - a^{\alpha}\right) f\left(\frac{a+b}{2}\right) - \alpha \int_a^{\frac{a+b}{2}} f(s)d_{\alpha}s\right] \\
+ & \frac{1}{b-a}\left[\left(b^{\alpha} - \left(\frac{a+b}{2}\right)^{\alpha}\right) f\left(\frac{a+b}{2}\right) - \alpha \int_{\frac{a+b}{2}}^{b} f(s)d_{\alpha}s\right] \\
= & \frac{b^{\alpha}-a^{\alpha}}{b-a} f\left(\frac{a+b}{2}\right) - \frac{\alpha}{b-a}\int_a^b f(s)d_{\alpha}s,
\end{aligned}
$$

where, we have used the change of variable $s = (1-t)a + tb$ and then multiplying both sides by $\frac{b-a}{b^{\alpha}-a^{\alpha}}$ to get the desired result in (16).

Remark 2. By setting $\alpha = 1$, the identity in (16) reduces to (5).

□

Theorem 10. *Let $a, b \in \mathbb{R}$ with $0 \leq a < b$, and let $f : [a,b] \to \mathbb{R}$ be an α-differentiable function on (a,b) for $\alpha \in (0,1]$. If $D_{\alpha}(f) \in L_{\alpha}^{1}([a,b])$ and $|f'|$ is convex on $[a,b]$, then we have the following inequality:*

$$
\begin{aligned}
& \left| f\left(\frac{a+b}{2}\right) - \frac{\alpha}{b^{\alpha}-a^{\alpha}} \int_a^b f(s)d_{\alpha}s \right| \qquad (17) \\
\leq \; & \frac{|f'(a)|}{192}\left[13b^{\alpha} - 35a^{\alpha}\right] + \frac{|f'(b)|}{192}\left[19b^{\alpha} - 29a^{\alpha}\right] + \left(ab^{\alpha-1} + a^{\alpha-1}b\right)\left[\frac{11|f'(a)| + 5|f'(b)|}{192}\right].
\end{aligned}
$$

Proof. First of all we consider Lemma 3 and then using the convexity of $x^{\alpha-1}$ and $-x^{\alpha}$ $(x>0)$ for $\alpha \in (0,1]$. Also since the function $|f'|$ is convex, therefore we have

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{\alpha}{b^{\alpha}-a^{\alpha}} \int_a^b f(s) d_{\alpha} s \right| \\
\leq\ & \frac{b-a}{b^{\alpha}-a^{\alpha}} \left[\int_0^{\frac{1}{2}} \left(((1-t)a+tb)^{\alpha} - a^{\alpha} \right) \left| f'((1-t)a+tb) \right| dt \right. \\
+\ & \left. \int_{\frac{1}{2}}^1 \left(b^{\alpha} - ((1-t)a+tb)^{\alpha} \right) \left| f'((1-t)a+tb) \right| dt \right] \\
=\ & \frac{b-a}{b^{\alpha}-a^{\alpha}} \left[\int_0^{\frac{1}{2}} \left(((1-t)a+tb)^{\alpha-1+1} - a^{\alpha} \right) \left| f'((1-t)a+tb) \right| dt \right. \\
+\ & \left. \int_{\frac{1}{2}}^1 \left(b^{\alpha} - ((1-t)a+tb)^{\alpha} \right) \left| f'((1-t)a+tb) \right| dt \right] \\
\leq\ & \frac{b-a}{b^{\alpha}-a^{\alpha}} \left[\int_0^{\frac{1}{2}} \left(((1-t)a+tb)^{\alpha-1}((1-t)a+tb) - a^{\alpha} \right) \left| f'((1-t)a+tb) \right| dt \right. \\
+\ & \left. \int_{\frac{1}{2}}^1 \left(b^{\alpha} - ((1-t)a^{\alpha}+tb^{\alpha}) \right) \left| f'((1-t)a+tb) \right| dt \right] \\
\leq\ & \frac{b-a}{b^{\alpha}-a^{\alpha}} \left[\int_0^{\frac{1}{2}} \left(((1-t)a^{\alpha-1}+tb^{\alpha-1})((1-t)a+tb) - a^{\alpha} \right) \left| f'((1-t)a+tb) \right| dt \right. \\
+\ & \left. \int_{\frac{1}{2}}^1 \left(b^{\alpha} - ((1-t)a^{\alpha}+tb^{\alpha}) \right) \left| f'((1-t)a+tb) \right| dt \right] \\
\leq\ & \frac{b-a}{b^{\alpha}-a^{\alpha}} \left[\int_0^{\frac{1}{2}} \left(((1-t)a^{\alpha-1}+tb^{\alpha-1})((1-t)a+tb) - a^{\alpha} \right) \left[(1-t)|f'(a)| + t|f'(b))| \right] dt \right. \\
+\ & \left. \int_{\frac{1}{2}}^1 \left(b^{\alpha} - ((1-t)a^{\alpha}+tb^{\alpha}) \right) \left[(1-t)|f'(a)| + t|f'(b))| \right] dt \right].
\end{aligned}$$

Evaluating all the above integrals, we have the following

$$\begin{aligned}
& \frac{b-a}{b^{\alpha}-a^{\alpha}} \left[\int_0^{\frac{1}{2}} \left(((1-t)a^{\alpha-1}+tb^{\alpha-1})((1-t)a+tb) - a^{\alpha} \right) \left[(1-t)|f'(a)| + t|f'(b))| \right] dt \right. \\
+\ & \left. \int_{\frac{1}{2}}^1 \left(b^{\alpha} - ((1-t)a^{\alpha}+tb^{\alpha}) \right) \left[(1-t)|f'(a)| + t|f'(b))| \right] dt \right] \\
=\ & \frac{b-a}{b^{\alpha}-a^{\alpha}} \left[\frac{15}{64} a^{\alpha}|f'(a)| + \frac{11}{192} a^{\alpha}|f'(b)| + \frac{11}{192} ab^{\alpha-1}|f'(a)| + \frac{5}{192} ab^{\alpha-1}|f'(b)| \right.
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{11}{192}a^{\alpha-1}b|f'(a)| + \frac{5}{192}a^{\alpha-1}b|f'(b)| + \frac{5}{192}b^{\alpha}|f'(a)| + \frac{1}{64}b^{\alpha}|f'(b)| - \frac{1}{24}a^{\alpha}|f'(a)| \\
&- \frac{1}{12}a^{\alpha}|f'(b)| - \frac{1}{12}b^{\alpha}|f'(a)| - \frac{7}{24}b^{\alpha}|f'(b)| - \frac{3}{8}a^{\alpha}|f'(a)| - \frac{1}{8}a^{\alpha}|f'(b)| + \frac{1}{8}b^{\alpha}|f'(a)| \\
&+ \left.\frac{3}{8}b^{\alpha}|f'(b)|\right] \\
&= \frac{|f'(a)|}{192}\left[13b^{\alpha} - 35a^{\alpha}\right] + \frac{|f'(b)|}{192}\left[19b^{\alpha} - 29a^{\alpha}\right] + (ab^{\alpha-1} + a^{\alpha-1}b)\left[\frac{11|f'(a)| + 5|f'(b)|}{192}\right].
\end{aligned}
$$

□

Remark 3. By putting $\alpha = 1$ in (17), we get the inequality in (6).

Theorem 11. *Let $a, b \in \mathbb{R}$ with $0 \leq a < b$, and let $f : [a, b] \to \mathbb{R}$ be an α-differentiable function on (a, b) for $\alpha \in (0, 1]$. If $D_{\alpha}(f) \in L^1_{\alpha}([a, b])$ and $|f'|^q$ is convex on $[a, b]$ for $q > 1$, then the following inequality holds:*

$$
\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{\alpha}{b^{\alpha} - a^{\alpha}} \int_a^b f(s) d_{\alpha}s \right| \\
\leq\ & \frac{(b-a)}{b^{\alpha} - a^{\alpha}} \left[(A_1(\alpha))^{1-\frac{1}{q}} \left\{ A_2(\alpha)|f'(a)|^q + A_3(\alpha)|f'(b)|^q \right\}^{\frac{1}{q}} \right. \\
+\ & \left. (B_1(\alpha))^{1-\frac{1}{q}} \left\{ B_2(\alpha)|f'(a)|^q + B_3(\alpha)|f'(b)|^q \right\}^{\frac{1}{q}} \right], \qquad (18)
\end{aligned}
$$

where

$$
\begin{aligned}
A_1(\alpha) &= \left[\frac{(a+b)^{\alpha+1} - (2a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)}\right] - \frac{a^{\alpha}}{2}, \quad B_1(\alpha) = \frac{b^{\alpha}}{2} - \left[\frac{(2b)^{\alpha+1} - (a+b)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)}\right], \\
A_2(\alpha) &= \frac{(a+b)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)}\left[\frac{(b-a)(\alpha+2) - (a+b)}{2(b-a)(\alpha+2)}\right], \\
&- \frac{a^{\alpha+1}}{(b-a)(\alpha+2)}\left[\frac{(a+b)^{\alpha+1} - (2a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)}\right] - \frac{3a^{\alpha}}{8},
\end{aligned}
$$

$$
\begin{aligned}
B_2(\alpha) &= \frac{(a+b)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)}\left[\frac{2(b-a)(\alpha+2) - (a+b)}{2(b-a)(\alpha+2)}\right] - \frac{b^{\alpha+1}}{(b-a)(\alpha+1)(\alpha+2)} - \frac{3b^{\alpha}}{8}, \\
A_3(\alpha) &= \frac{(a+b)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)}\left[\frac{(b-a)(\alpha+2) - (a+b)}{2(b-a)(\alpha+2)}\right] - \frac{a^{\alpha+2}}{(b-a)^2(\alpha+1)(\alpha+2)} - \frac{a^{\alpha}}{2}, \\
B_3(\alpha) &= \frac{(a+b)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)}\left[\frac{2(b-a)(\alpha+2) + (a+b)}{2(b-a)(\alpha+2)}\right] \\
&- \frac{b^{\alpha+1}}{(b-a)(\alpha+1)}\left[\frac{(a+b)^{\alpha+1} - b}{(\alpha+2)(b-a)}\right] + \frac{b^{\alpha}}{2}.
\end{aligned}
$$

Proof. Using Lemma 3 it follows that

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(s)d_\alpha s\right|\\
=&\left|\frac{(b-a)}{b^\alpha-a^\alpha}\left[\int_0^{\frac{1}{2}}\left(((1-t)a+tb)^{2\alpha-1}-a^\alpha((1-t)a+tb)^{\alpha-1}\right)D_\alpha(f)((1-t)a+tb)dt\right.\right.\\
+&\left.\left.\int_{\frac{1}{2}}^1\left(((1-t)a+tb)^{2\alpha-1}-b^\alpha((1-t)a+tb)^{\alpha-1}\right)D_\alpha(f)((1-t)a+tb)dt\right]\right|\\
\leq&\frac{(b-a)}{b^\alpha-a^\alpha}\left[\int_0^{\frac{1}{2}}\left(((1-t)a+tb)^\alpha-a^\alpha\right)\left|f'((1-t)a+tb)\right|dt\right.\\
+&\left.\int_{\frac{1}{2}}^1\left(b^\alpha-((1-t)a+tb)^\alpha\right)\left|f'((1-t)a+tb)\right|dt\right].
\end{aligned}$$

Now by the power-mean inequality

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left(((1-t)a+tb)^\alpha-a^\alpha\right)\left|f'((1-t)a+tb)\right|dt\\
\leq&\left(\int_0^{\frac{1}{2}}\left(((1-t)a+tb)^\alpha-a^\alpha\right)dt\right)^{1-\frac{1}{q}}\\
\times&\left(\int_0^{\frac{1}{2}}\left(((1-t)a+tb)^\alpha-a^\alpha\right)\left|f'((1-t)a+tb)\right|^q dt\right)^{\frac{1}{q}}
\end{aligned}$$

and similarly, we have

$$\begin{aligned}
&\int_{\frac{1}{2}}^1\left(b^\alpha-((1-t)a+tb)^\alpha\right)\left|f'((1-t)a+tb)\right|dt\\
\leq&\left(\int_{\frac{1}{2}}^1\left(b^\alpha-((1-t)a+tb)^\alpha\right)dt\right)^{1-\frac{1}{q}}\\
\times&\left(\int_{\frac{1}{2}}^1\left(b^\alpha-((1-t)a+tb)^\alpha\right)\left|f'((1-t)a+tb)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Now by the convexity $|f'|^q$ from above, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left(((1-t)a+tb)^\alpha-a^\alpha\right)\left|f'((1-t)a+tb)\right|^q dt\\
\leq&\int_0^{\frac{1}{2}}\left(((1-t)a+tb)^\alpha-a^\alpha\right)\left[(1-t)|f'(a)|^q+t|f'(b)|^q\right]dt
\end{aligned}$$

$$
\begin{aligned}
&= |f'(a)|^q \int_0^{\frac{1}{2}} \big(((1-t)a+tb)^\alpha - a^\alpha\big)(1-t)dt + |f'(b)|^q \int_0^{\frac{1}{2}} \big(((1-t)a+tb)^\alpha - a^\alpha\big)tdt \\
&= |f'(a)|^q \Bigg(\frac{(a+b)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)} \left[\frac{(b-a)(\alpha+2)-(a+b)}{2(b-a)(\alpha+2)}\right] \\
&- \frac{a^{\alpha+1}}{(b-a)(\alpha+2)} \left[\frac{(a+b)^{\alpha+1}-(2a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)}\right] - \frac{3a^\alpha}{8} \Bigg) \\
&+ |f'(b)|^q \Bigg(\frac{(a+b)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)} \left[\frac{(b-a)(\alpha+2)-(a+b)}{2(b-a)(\alpha+2)}\right] \\
&- \frac{a^{\alpha+2}}{(b-a)^2(\alpha+1)(\alpha+2)} - \frac{a^\alpha}{2} \Bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\frac{1}{2}}^1 \big(b^\alpha - ((1-t)a+tb)^\alpha\big) \left|f'((1-t)a+tb)\right|^q dt \\
&\leq \int_{\frac{1}{2}}^1 \big(b^\alpha - ((1-t)a+tb)^\alpha\big) \left[(1-t)|f'(a)|^q + t|f'(b)|^q\right] dt \\
&= |f'(a)|^q \int_{\frac{1}{2}}^1 \big(b^\alpha - ((1-t)a+tb)^\alpha\big)(1-t)dt + |f'(b)|^q \int_{\frac{1}{2}}^1 \big(b^\alpha - ((1-t)a+tb)^\alpha\big)tdt \\
&= |f'(a)|^q \Bigg(\frac{(a+b)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)} \left[\frac{2(b-a)(\alpha+2)-(a+b)}{2(b-a)(\alpha+2)}\right] - \frac{b^{\alpha+1}}{(b-a)(\alpha+1)(\alpha+2)} \\
&- \frac{3b^\alpha}{8} \Bigg) + |f'(b)|^q \Bigg(\frac{(a+b)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)} \left[\frac{2(b-a)(\alpha+2)+(a+b)}{2(b-a)(\alpha+2)}\right] \\
&- \frac{b^{\alpha+1}}{(b-a)(\alpha+1)} \left[\frac{(a+b)^{\alpha+1}-b}{(\alpha+2)(b-a)}\right] + \frac{b^\alpha}{2} \Bigg),
\end{aligned}
$$

where, we have also used the facts that

$$
\int_0^{\frac{1}{2}} \big(((1-t)a+tb)^\alpha - a^\alpha\big)dt = \left[\frac{(a+b)^{\alpha+1}-(2a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)}\right] - \frac{a^\alpha}{2}
$$

$$
\int_{\frac{1}{2}}^1 \big(b^\alpha - ((1-t)a+tb)^\alpha\big)dt = \frac{b^\alpha}{2} - \left[\frac{(2b)^{\alpha+1}-(a+b)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)}\right].
$$

Hence, we have the result in (18). $\square$

Remark 4. By setting $\alpha = 1$ in (18), we get the following inequality:

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(s)ds \right| \\
\leq\ & \left(\frac{b-a}{8}\right)^{1-\frac{1}{q}} \Bigg[\left\{ A_2(1)|f'(a)|^q + A_3(1)|f'(b)|^q \right\}^{\frac{1}{q}} \\
+\ & \left\{ B_2(1)|f'(a)|^q + B_3(1)|f'(b)|^q \right\}^{\frac{1}{q}} \Bigg], \qquad (19)
\end{aligned}$$

where

$$\begin{aligned}
A_2(1) &= \frac{(a+b)^2(b-2a) - a^2(b^2 - 3a^2 + 2ab) - 3a(b-a)^2}{24(b-a)^2}, \\
B_2(1) &= \frac{(a+b)^2(5b-7a) - 8b^2(b-a) - 18b(b-a)}{48(b-a)^2}, \\
A_3(1) &= \frac{(a+b)^2(b-2a) - 4a^3 - 12a(b-a)^2}{24(b-a)^2}, \\
B_3(1) &= \frac{(a+b)^2(7b-5a) - 8b^2(a+b)^2 + 8b^3 - 24b(b-a)^2}{48(b-a)^2}.
\end{aligned}$$

Theorem 12. *Let $a, b \in \mathbb{R}$ with $0 \leq a < b$, and let $f : [a,b] \to \mathbb{R}$ be an α-differentiable function on (a,b) for $\alpha \in (0,1]$. If $D_\alpha(f) \in L^1_\alpha([a,b])$ and $|f'|^q$ is concave on $[a,b]$ for $q > 1$, then the following inequality holds:*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(s)d_\alpha s \right| \\
\leq\ & \frac{(b-a)}{b^\alpha - a^\alpha}\left[A_1(\alpha) f'\left(\frac{C_1(\alpha)}{A_1(\alpha)}\right) + B_1(\alpha) f'\left(\frac{C_2(\alpha)}{B_1(\alpha)}\right) \right], \qquad (20)
\end{aligned}$$

where

$$\begin{aligned}
A_1(\alpha) &= \left[\frac{(a+b)^{\alpha+1} - (2a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)}\right] - \frac{a^\alpha}{2}, \quad B_1(\alpha) = \frac{b^\alpha}{2} - \left[\frac{(2b)^{\alpha+1} - (a+b)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)}\right], \\
C_1(\alpha) &= \frac{(a+b)^2}{4(b-a)}\left[\frac{(a+b)^\alpha - 2^{\alpha-1}a^\alpha(\alpha+2)}{2^\alpha(\alpha+2)}\right] - \frac{\alpha a^{\alpha+2}}{2(\alpha+2)(b-a)}, \\
C_2(\alpha) &= \frac{(a+b)^2}{4(b-a)}\left[\frac{(a+b)^\alpha - 2^{\alpha-1}b^\alpha(\alpha+2)}{2^\alpha(\alpha+2)}\right] - \frac{\alpha b^{\alpha+2}}{2(\alpha+2)(b-a)}.
\end{aligned}$$

Proof. By power mean inequality, we have

$$\begin{aligned}
\left(t|f'(a)| + (1-t)|f'(b)|\right)^q &\leq t|f'(a)|^q + (1-t)|f'(b)|^q \\
&\leq |f'(ta + (1-t)b)|^q, \text{ (by concavity of } |f'|^q \text{)}
\end{aligned}$$

and therefore

$$|f'(ta + (1-t)b)| \geq t|f'(a)| + (1-t)|f'(b)|,$$

which shows that $|f'|$ is also concave. Now taking into consideration Lemma 3 , it follows that

$$
\begin{aligned}
& \left| f\left(\frac{a+b}{2}\right) - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(s)d_\alpha s \right| \\
= & \left| \frac{(b-a)}{b^\alpha - a^\alpha}\left[\int_0^{\frac{1}{2}} \left(((1-t)a+tb)^{2\alpha-1} - a^\alpha((1-t)a+tb)^{\alpha-1}\right) D_\alpha(f)((1-t)a+tb)dt \right.\right. \\
+ & \left.\left. \int_{\frac{1}{2}}^1 \left(((1-t)a+tb)^{2\alpha-1} - b^\alpha((1-t)a+tb)^{\alpha-1}\right) D_\alpha(f)((1-t)a+tb)dt \right]\right| \\
\leq & \frac{(b-a)}{b^\alpha - a^\alpha}\left[\int_0^{\frac{1}{2}} \left(((1-t)a+tb)^{\alpha} - a^\alpha\right) \left|f'((1-t)a+tb)\right| dt \right. \\
+ & \left. \int_{\frac{1}{2}}^1 \left(b^\alpha - ((1-t)a+tb)^{\alpha}\right) \left|f'((1-t)a+tb)\right| dt \right].
\end{aligned}
$$

and applying Jensen's integral inequality, we have

$$
\begin{aligned}
& \int_0^{\frac{1}{2}} \left(((1-t)a+tb)^{\alpha} - a^\alpha\right) \left|f'((1-t)a+tb)\right| dt \\
\leq & \left(\int_0^{\frac{1}{2}} \left(((1-t)a+tb)^{\alpha} - a^\alpha\right) \right) f'\left(\frac{\int_0^{\frac{1}{2}} \left(((1-t)a+tb)^{\alpha} - a^\alpha\right)((1-t)a+tb)dt}{\int_0^{\frac{1}{2}} \left(((1-t)a+tb)^{\alpha} - a^\alpha\right)} \right) \\
= & A_1(\alpha) f'\left(\frac{C_1(\alpha)}{A_1(\alpha)}\right).
\end{aligned}
$$

Equivalently, we have

$$
\begin{aligned}
& \int_{\frac{1}{2}}^1 \left(b^\alpha - ((1-t)a+tb)^{\alpha}\right) \left|f'((1-t)a+tb)\right| dt \\
\leq & \left(b^\alpha - \int_0^{\frac{1}{2}} \left(((1-t)a+tb)^{\alpha}\right) \right) f'\left(\frac{\int_{\frac{1}{2}}^1 \left(b^\alpha - ((1-t)a+tb)^{\alpha}\right)((1-t)a+tb)dt}{\int_{\frac{1}{2}}^1 \left(b^\alpha - ((1-t)a+tb)^{\alpha}\right)dt} \right) \\
= & B_1(\alpha) f'\left(\frac{C_2(\alpha)}{B_1(\alpha)}\right).
\end{aligned}
$$

where, we have also used the following facts that

$$
\int_0^{\frac{1}{2}} \left(((1-t)a+tb)^{\alpha} - a^\alpha\right)dt = A_1(\alpha) = \left[\frac{(a+b)^{\alpha+1} - (2a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)}\right] - \frac{a^\alpha}{2},
$$

$$
\int_{\frac{1}{2}}^1 \left(b^\alpha - ((1-t)a+tb)^{\alpha}\right)dt = B_1(\alpha) = \frac{b^\alpha}{2} - \left[\frac{(2b)^{\alpha+1} - (a+b)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)}\right],
$$

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \left(((1-t)a+tb)^\alpha - a^\alpha\right)((1-t)a+tb)dt \\
= \quad & C_1(\alpha) = \frac{(a+b)^2}{4(b-a)}\left[\frac{(a+b)^\alpha - 2^{\alpha-1}a^\alpha(\alpha+2)}{2^\alpha(\alpha+2)}\right] - \frac{\alpha a^{\alpha+2}}{2(\alpha+2)(b-a)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 \left(b^\alpha - ((1-t)a+tb)^\alpha\right)((1-t)a+tb)dt \\
= \quad & C_2(\alpha) = \frac{(a+b)^2}{4(b-a)}\left[\frac{(a+b)^\alpha - 2^{\alpha-1}b^\alpha(\alpha+2)}{2^\alpha(\alpha+2)}\right] - \frac{\alpha b^{\alpha+2}}{2(\alpha+2)(b-a)}.
\end{aligned}$$

□

Remark 5. If we set $\alpha = 1$ in (20), then we have the following

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(s)ds \right| \\
\leq \quad & \frac{(b-a)}{8}\left[f'\left(\frac{(a+b)^4 - 3a(a+b)^2 - 4a^3}{3(b-a)^2}\right)\right. \\
+ \quad & \left. f'\left(\frac{(a+b)^4 - 3b(a+b)^2 - 4b^3}{3(b-a)^2}\right)\right]. \qquad (21)
\end{aligned}$$

3. Applications to special means and midpoint formula

We begin this section by considering some particular means for two positive real numbers a, b ($a \neq b$) and for this purpose we recall the following well-known definitions in literature:
(1) The arithmetic mean:

$$A = A(a,b) = \frac{a+b}{2},\ a, b \in \mathbb{R}^+.$$

(2) The logarithmic mean:

$$L(a,b) = \frac{b-a}{\ln b - \ln a},\ a \neq b,\ a, b \in \mathbb{R}^+.$$

(3) The generalized logarithmic r-th mean:

$$L_{(\alpha,r)}(a,b) = \left[\frac{\alpha(b^{r+\alpha} - a^{r+\alpha})}{(b^\alpha - a^\alpha)(r+\alpha)}\right]^{\frac{1}{r}}, \quad a \neq b,\ r \neq 0, -\alpha,\ \alpha \in (0,1],\ r \in \mathbb{R}.$$

Now, by making use of the results obtained in section 2, we give some applications to special means of real numbers.

Proposition 1. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, $r > 1$ and $\alpha \in (0,1]$, then the following holds:*

$$\begin{aligned} & |A^r(a,b) - L^r_{(\alpha,r)}(a,b)| \\ \leq \ & \frac{(r-1)(b-a)}{b^\alpha - a^\alpha}\left[\frac{|a|^{(r-1)}}{192}\left[13b^\alpha - 35a^\alpha\right] + \frac{|b|^{(r-1)}}{192}\left[19b^\alpha - 29a^\alpha\right]\right. \\ + \ & \left.(ab^{\alpha-1} + a^{\alpha-1}b)\left\{\frac{11|a|^{(r-1)} + 5|b|^{(r-1)}}{192}\right\}\right]. \end{aligned}$$

Proof. The result follows from Theorem 10 for the convex function $f(x) = x^r, x > 0$. $\square$

Proposition 2. *Let $a, b \in \mathbb{R}$ with $0 < a < b$ and $r > 1$. Then for $q > 1$ and $\alpha \in (0,1]$, we have the following inequality:*

$$\begin{aligned} |A^r(a,b) - L^r_{(\alpha,r)}(a,b))| \ \leq \ & \frac{(r-1)(b-a)}{b^\alpha - a^\alpha}\left[(A_1(\alpha))^{1-\frac{1}{q}}\left\{A_2(\alpha)|a|^{(r-1)q} + A_3(\alpha)|b|^{(r-1)q}\right\}^{\frac{1}{q}}\right. \\ + \ & \left.(B_1(\alpha))^{1-\frac{1}{q}}\left\{B_2(\alpha)|a|^{(r-1)q} + B_3(\alpha)|b|^{(r-1)q}\right\}^{\frac{1}{q}}\right]. \end{aligned}$$

Proof. One can obtain the result from Theorem 11 by using the convex function $f(x) = x^r, x > 0$. $\square$

Proposition 3. *Let $a, b \in \mathbb{R}$ with $0 < a < b$ and $\alpha \in (0,1]$, then we have*

$$\begin{aligned} & |A^{-1}(a,b) - L^r_{(\alpha,-3)}(a,b)| \\ \leq \ & \frac{(b-a)}{b^\alpha - a^\alpha}\left[\frac{|a|^{-2}}{192}\left[13b^\alpha - 35a^\alpha\right] + \frac{|b|^{-2}}{192}\left[19b^\alpha - 29a^\alpha\right]\right. \\ + \ & \left.(ab^{\alpha-1} + a^{\alpha-1}b)\left[\frac{11|a|^{-2} + 5|b|^{-2}}{192}\right]\right]. \end{aligned}$$

Proof. The statement of results follows from Theorem 10 for the convex function $f(x) = \frac{1}{x}$ $x > 0$. $\square$

Proposition 4. *Let $a, b \in \mathbb{R}$ with $0 < a < b$. Then for $q > 1$ and $\alpha \in (0,1]$, we have*

$$\begin{aligned} |A^{-1}(a,b) - L^r_{(\alpha,-3)}(a,b)| \ \leq \ & \frac{(b-a)}{b^\alpha - a^\alpha}\left[(A_1(\alpha))^{1-\frac{1}{q}}\left\{A_2(\alpha)|a|^{-2q} + A_3(\alpha)|b|^{-2q}\right\}^{\frac{1}{q}}\right. \\ + \ & \left.(B_1(\alpha))^{1-\frac{1}{q}}\left\{B_2(\alpha)|a|^{-2q} + B_3(\alpha)|b|^{-2q}\right\}^{\frac{1}{q}}\right]. \end{aligned}$$

Proof. We can get the inequality from Theorem 11 by using the convex function $f(x) = \frac{1}{x}$ $x > 0$. $\square$

Let P be the partition of the points $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ of the interval $[a, b]$ and consider the quadrature formula

$$\int_a^b f(x)d_\alpha x = T_\alpha(f, P) + E_\alpha(f, P), \tag{22}$$

where

$$T_\alpha(f, P) = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) \frac{(x_{i+1}^\alpha - x_i^\alpha)}{\alpha}. \tag{23}$$

is the midpoint version and $E_\alpha(f, P)$ denotes the associated approximation error. Here, we are going to derive some new estimates for the midpoint formula.

Proposition 5. *Let $a, b \in \mathbb{R}$ with $0 \leq a < b$, and let $f : [a, b] \to \mathbb{R}$ be an α-differentiable function on (a, b) for $\alpha \in (0, 1]$. If $D_\alpha(f) \in L_\alpha^1([a, b])$ and $|f'|$ is convex on $[a, b]$, then we have*

$$\begin{aligned}
& |E_\alpha(f, P)| \\
\leq & \sum_{i=0}^{n-1} \frac{(x_{i+1}^\alpha - x_i^\alpha)}{\alpha} \left[\frac{|f'(x_i)|}{192} \left[13x_{i+1}^\alpha - 35x_i^\alpha\right] + \frac{|f'(x_{i+1})|}{192} \left[19x_{i+1}^\alpha - 29x_i^\alpha\right] \right. \\
+ & \left. (x_i x_{i+1}^{\alpha-1} + x_i^{\alpha-1} x_{i+1}) \left[\frac{11|f'(x_i)| + 5|f'(x_{i+1})|}{192}\right] \right].
\end{aligned}$$

Proof. Applying Theorem 10 on the subintervals $[x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$ of the partition P, we have

$$\begin{aligned}
& \left| f\left(\frac{x_i + x_{i+1}}{2}\right) \frac{(x_{i+1}^\alpha - x_i^\alpha)}{\alpha} - \int_{x_i}^{x_{i+1}} f(x)d_\alpha x \right| \\
\leq & \frac{(x_{i+1}^\alpha - x_i^\alpha)}{\alpha} \left[\frac{|f'(x_i)|}{192} \left[13x_{i+1}^\alpha - 35x_i^\alpha\right] + \frac{|f'(x_{i+1})|}{192} \left[19x_{i+1}^\alpha - 29x_i^\alpha\right] \right. \\
+ & \left. (x_i x_{i+1}^{\alpha-1} + x_i^{\alpha-1} x_{i+1}) \left[\frac{11|f'(x_i)| + 5|f'(x_{i+1})|}{192}\right] \right].
\end{aligned}$$

hence from above

$$\begin{aligned}
&\left|\int_a^b f(x)d_\alpha x - T_\alpha(f,P)\right| \\
= & \left|\sum_{i=0}^{n-1}\left\{\int_{x_i}^{x_{i+1}} f(x)d_\alpha x - f\left(\frac{x_i+x_{i+1}}{2}\right)\frac{\left(x_{i+1}^\alpha - x_i^\alpha\right)}{\alpha}\right\}\right| \\
\leq & \sum_{i=0}^{n-1}\left|\left\{\int_{x_i}^{x_{i+1}} f(x)d_\alpha x - f\left(\frac{x_i+x_{i+1}}{2}\right)\frac{\left(x_{i+1}^\alpha - x_i^\alpha\right)}{\alpha}\right\}\right| \\
\leq & \sum_{i=0}^{n-1}\frac{(x_{i+1}^\alpha - x_i^\alpha)}{\alpha}\left[\frac{|f'(x_i)|}{192}\left[13x_{i+1}^\alpha - 35x_i^\alpha\right] + \frac{|f'(x_{i+1})|}{192}\left[19x_{i+1}^\alpha - 29x_i^\alpha\right]\right. \\
+ & \left.(x_i x_{i+1}^{\alpha-1} + x_i^{\alpha-1}x_{i+1})\left[\frac{11|f'(x_i)| + 5|f'(x_{i+1})|}{192}\right]\right].
\end{aligned}$$

□

Proposition 6. *Let $a, b \in \mathbb{R}$ with $0 \leq a < b$, and let $f : [a,b] \to \mathbb{R}$ be an α-differentiable function on (a,b) for $\alpha \in (0,1]$. If $D_\alpha(f) \in L_\alpha^1([a,b])$ and $|f'|^q$ is convex on $[a,b]$ with $q > 1$, then we have*

$$\begin{aligned}
|E_\alpha(f,P)| \leq & \sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)}{\alpha}\left[(A_1(\alpha))^{1-\frac{1}{q}}\left\{A_2(\alpha)|f'(x_i)|^q + A_3(\alpha)|f'(x_{i+1})|^q\right\}^{\frac{1}{q}}\right. \\
+ & \left.(B_1(\alpha))^{1-\frac{1}{q}}\left\{B_2(\alpha)|f'(x_i)|^q + B_3(\alpha)|f'(x_{i+1})|^q\right\}^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. The proof is analogous to that of Proposition 5 only by using Theorem 11. □

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