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ON SOME INEQUALITIES FOR THE (p,k)-ANALOGUES OF THE DIGAMMA AND POLYGAMMA FUNCTIONS

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Abstract. In this paper, we establish several inequalities involving the (p,k)-analogues of the Digamma and Polygamma functions. Consequently, we recover some previous results as particular cases of the results of this paper.

1. Introduction

In [4], the authors introduced a new two-parameter deformation of the classical Gamma function, called the (p,k)-analogue of the Gamma function. It is defined for $p \in \mathbb{N}$, $k > 0$ and $x \in \mathbb{R}^+$ as

$$\begin{aligned}\Gamma_{p,k}(x) &= \int_0^p t^{x-1}\left(1-\frac{t^k}{pk}\right)^p dt\\ &= \frac{(p+1)!k^{p+1}(pk)^{\frac{x}{k}-1}}{x(x+k)(x+2k)\dots(x+pk)}\end{aligned}$$

satisfying the identities:

$$\begin{aligned}\Gamma_{p,k}(x+k) &= \frac{pkx}{x+pk+k}\Gamma_{p,k}(x), \qquad (1)\\ \Gamma_{p,k}(ak) &= \frac{p+1}{p}k^{a-1}\Gamma_p(a), \quad a\in\mathbb{R}^+\\ \Gamma_{p,k}(k) &= 1.\end{aligned}$$

The (p,k)-analogue of the Digamma function is defined as the logarithmic derivative of $\Gamma_{p,k}(x)$. That is

$$\begin{aligned}\psi_{p,k}(x) &= \frac{d}{dx}\ln\Gamma_{p,k}(x)\\ &= \frac{1}{k}\ln(pk) - \sum_{n=0}^{p}\frac{1}{(nk+x)} \qquad (2)\\ &= \frac{1}{k}\ln(pk) - \int_0^\infty \frac{1-e^{-k(p+1)t}}{1-e^{-kt}}e^{-xt}\,dt \qquad (3)\end{aligned}$$

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Also, the (p, k)-analogue of the Polygamma functions is defined as

$$\begin{aligned}\psi_{p,k}^{(m)}(x) &= \frac{d^m}{dx^m}\psi_{p,k}(x) \\ &= \sum_{n=0}^{p} \frac{(-1)^{m+1}m!}{(nk+x)^{m+1}} \qquad (4) \\ &= (-1)^{m+1}\int_0^\infty \left(\frac{1-e^{-k(p+1)t}}{1-e^{-kt}}\right) t^m e^{-xt}\,dt \qquad (5)\end{aligned}$$

for $m \in \mathbb{N}$, where $\psi_{p,k}^{(0)}(x) \equiv \psi_{p,k}(x)$. It follows easily from (4) that,

$$\psi_{p,k}^{(m)}(x) = \begin{cases} > 0 & \text{if } m \text{ is odd,} \\ < 0 & \text{if } m \text{ is even.} \end{cases} \tag{6}$$

From the identity (1), the following relations are established.

$$\begin{aligned}\psi_{p,k}(x+k) - \psi_{p,k}(x) &= \frac{1}{x} - \frac{1}{x+pk+k}, \\ \psi_{p,k}^{(m)}(x+k) - \psi_{p,k}^{(m)}(x) &= \frac{(-1)^m m!}{x^{m+1}} - \frac{(-1)^m m!}{(x+pk+k)^{m+1}}, \qquad m \in \mathbb{N}.\end{aligned}$$

Also, from (4), the following properties are deduced for $x > 0$.

(i) $\psi_{p,k}(x)$ is increasing.
(ii) $\psi_{p,k}^{(m)}(x)$ is decreasing if m is odd.
(iii) $\psi_{p,k}^{(m)}(x)$ is increasing if m is even.

2. Main Results

We now present the main findings of the paper in this section.

Theorem 2.1. *Let $x > 0$, $0 < y < 1$, $p \in \mathbb{N}$ and $k > 0$. Then the inequality*

$$\psi_{p,k}(x+y) > \psi_{p,k}(x) + \psi_{p,k}(y) \tag{7}$$

holds true.

Proof. Let $F(x) = \psi_{p,k}(x+y) - \psi_{p,k}(x) - \psi_{p,k}(y)$ for a fixed y. Then,

$$\begin{aligned}F'(x) = \psi'_{p,k}(x+y) - \psi'_{p,k}(x) &= \sum_{n=0}^{p}\left[\frac{1}{(nk+x+y)^2} - \frac{1}{(nk+x)^2}\right] \\ &< 0.\end{aligned}$$

That implies F is decreasing. Further,

$$\begin{aligned}\lim_{x\to\infty} F(x) &= \lim_{x\to\infty}\left[\psi_{p,k}(x+y)-\psi_{p,k}(x)-\psi_{p,k}(y)\right]\\ &= \lim_{x\to\infty}\left[-\frac{1}{k}\ln(pk)-\sum_{n=0}^{p}\frac{1}{nk+x+y}+\sum_{n=0}^{p}\frac{1}{nk+x}+\sum_{n=0}^{p}\frac{1}{nk+y}\right]\\ &= -\frac{1}{k}\ln(pk)+\sum_{n=0}^{p}\frac{1}{nk+y} > 0.\end{aligned}$$

Therefore $F(x)\geq 0$ yielding the result (7).

Theorem 2.2. *Let $x>0$, $y>0$, $p\in\mathbb{N}$ and $k>0$. Then for a positive odd integer m, the inequality*

$$\psi_{p,k}^{(m)}(x+y) < \psi_{p,k}^{(m)}(x)+\psi_{p,k}^{(m)}(y) \tag{8}$$

holds true.

Proof. Let $G(x)=\psi_{p,k}^{(m)}(x+y)-\psi_{p,k}^{(m)}(x)-\psi_{p,k}^{(m)}(y)$ for a fixed y. Then,

$$\begin{aligned}G'(x) &= \psi_{p,k}^{(m+1)}(x+y)-\psi_{p,k}^{(m+1)}(x)\\ &= \sum_{n=0}^{p}\frac{(-1)^{m+2}(m+1)!}{(nk+x+y)^{m+2}}-\sum_{n=0}^{p}\frac{(-1)^{m+2}(m+1)!}{(nk+x)^{m+2}}\\ &= (-1)^{m+2}(m+1)!\sum_{n=0}^{p}\left[\frac{1}{(nk+x+y)^{m+2}}-\frac{1}{(nk+x)^{m+2}}\right]\\ &= -(m+1)!\sum_{n=0}^{p}\left[\frac{1}{(nk+x+y)^{m+2}}-\frac{1}{(nk+x)^{m+2}}\right] > 0\end{aligned}$$

since m is odd. That implies G is increasing. Further,

$$\begin{aligned}\lim_{x\to\infty} G(x) &= \lim_{x\to\infty}\sum_{n=0}^{p}\left[\frac{(-1)^{m+1}m!}{(nk+x+y)^{m+1}}-\frac{(-1)^{m+1}m!}{(nk+x)^{m+1}}-\frac{(-1)^{m+1}m!}{(nk+y)^{m+1}}\right]\\ &= -\sum_{n=0}^{p}\frac{(-1)^{m+1}m!}{(nk+y)^{m+1}}\\ &= -\sum_{n=0}^{p}\frac{m!}{(nk+y)^{m+1}} < 0\end{aligned}$$

Therefore $G(x)\leq 0$ yielding the result (8).

Theorem 2.3. *Let $x>0$, $y>0$, $p\in\mathbb{N}$ and $k>0$. Then for a positive even integer m, the inequality*

$$\psi_{p,k}^{(m)}(x+y) > \psi_{p,k}^{(m)}(x)+\psi_{p,k}^{(m)}(y) \tag{9}$$

holds true.

Proof. Let $H(x) = \psi_{p,k}^{(m)}(x+y) - \psi_{p,k}^{(m)}(x) - \psi_{p,k}^{(m)}(y)$ for a fixed y. Then,

$$\begin{aligned} H'(x) &= \psi_{p,k}^{(m+1)}(x+y) - \psi_{p,k}^{(m+1)}(x) \\ &= \sum_{n=0}^{p} \frac{(-1)^{m+2}(m+1)!}{(nk+x+y)^{m+2}} - \sum_{n=0}^{p} \frac{(-1)^{m+2}(m+1)!}{(nk+x)^{m+2}} \\ &= (-1)^{m+2}(m+1)! \sum_{n=0}^{p} \left[\frac{1}{(nk+x+y)^{m+2}} - \frac{1}{(nk+x)^{m+2}} \right] \\ &= (m+1)! \sum_{n=0}^{p} \left[\frac{1}{(nk+x+y)^{m+2}} - \frac{1}{(nk+x)^{m+2}} \right] < 0 \end{aligned}$$

since m is even. Thus, H is decreasing. Further,

$$\begin{aligned} \lim_{x\to\infty} H(x) &= \lim_{x\to\infty} \sum_{n=0}^{p} \left[\frac{(-1)^{m+1}m!}{(nk+x+y)^{m+1}} - \frac{(-1)^{m+1}m!}{(nk+x)^{m+1}} - \frac{(-1)^{m+1}m!}{(nk+y)^{m+1}} \right] \\ &= -\sum_{n=0}^{p} \frac{(-1)^{m+1}m!}{(nk+y)^{m+1}} \\ &= \sum_{n=0}^{p} \frac{m!}{(nk+y)^{m+1}} > 0 \end{aligned}$$

Therefore $H(x) \geq 0$ yielding the result (9).

Theorem 2.4. *Let $x, y > 0$, $p \in \mathbb{N}$ and $k > 0$. Then for $m \in \mathbb{N}$, the inequality*

$$\psi_{p,k}^{(m)}(x)\psi_{p,k}^{(m)}(y) > \left[\psi_{p,k}^{(m)}(x+y)\right]^2 \tag{10}$$

holds true.

Proof. Suppose that m is odd. Then,

$$\begin{aligned} \psi_{p,k}^{(m)}(x) - \psi_{p,k}^{(m)}(x+y) &= (-1)^{m+1}m! \sum_{n=0}^{p} \left[\frac{1}{(nk+x)^{m+1}} - \frac{1}{(nk+x+y)^{m+1}} \right] \\ &> 0 \end{aligned}$$

Hence,

$$\psi_{p,k}^{(m)}(x) > \psi_{p,k}^{(m)}(x+y) > 0. \tag{11}$$

By a similar procedure, we obtain

$$\psi_{p,k}^{(m)}(y) > \psi_{p,k}^{(m)}(x+y) > 0. \tag{12}$$

Then by multiplying (11) and (12) we obtain

$$\psi_{p,k}^{(m)}(x)\psi_{p,k}^{(m)}(y) > \left[\psi_{p,k}^{(m)}(x+y)\right]^2.$$

Next, suppose that m is even. Then

$$\psi_{p,k}^{(m)}(x) < \psi_{p,k}^{(m)}(x+y) < 0, \tag{13}$$

$$\psi_{p,k}^{(m)}(y) < \psi_{p,k}^{(m)}(x+y) < 0. \tag{14}$$

Similarly, by multiplying (13) and (14) we obtain

$$\psi_{p,k}^{(m)}(x)\psi_{p,k}^{(m)}(y) > \left[\psi_{p,k}^{(m)}(x+y)\right]^2$$

concluding the proof.

Remark 2.5. Let $p \to \infty$ in (7), (8), (9) and (10), then we recover the results of [5] as a particular case.

Remark 2.6. Let $k \to 1$ in (7), (8), (9) and (10), then we recover the results of [6] as a particular case.

Remark 2.7. Let $p \to \infty$ as $k \to 1$ in (7), (8), (9) and (10), then we recover the results of Theorems 2.2, 2.3 and 2.4 of [8].

Theorem 2.8. *Let $p \in \mathbb{N}$, $k > 0$, $\alpha \in \mathbb{N}$ and $x_i > 0$ for each $i = 1, 2, \ldots, \alpha$. If m is a positive odd integer, then the inequality*

$$\prod_{i=1}^{\alpha} \psi_{p,k}^{(m)}(x_i) \geq \left[\psi_{p,k}^{(m)}\left(\sum_{i=1}^{\alpha} x_i\right)\right]^{\alpha} \tag{15}$$

holds true.

Proof. We proceed as follows.

$$\begin{aligned}
\psi_{p,k}^{(m)}(x_1) - \psi_{p,k}^{(m)}\left(\sum_{i=1}^{\alpha} x_i\right) &= \sum_{n=0}^{p}\left[\frac{(-1)^{m+1}m!}{(nk+x_1)^{m+1}} - \frac{(-1)^{m+1}m!}{(nk+\sum_{i=1}^{\alpha} x_i)^{m+1}}\right] \\
&= m!\sum_{n=0}^{p}\left[\frac{1}{(nk+x_1)^{m+1}} - \frac{1}{(nk+\sum_{i=1}^{\alpha} x_i)^{m+1}}\right] \geq 0.
\end{aligned}$$

Hence,

$$\psi_{p,k}^{(m)}(x_1) \geq \psi_{p,k}^{(m)}\left(\sum_{i=1}^{\alpha} x_i\right) > 0.$$

Continuing with this technique, we obtain the following.

$$\begin{aligned}
\psi_{p,k}^{(m)}(x_2) &\geq \psi_{p,k}^{(m)}\left(\sum_{i=1}^{\alpha} x_i\right) > 0, \\
\psi_{p,k}^{(m)}(x_3) &\geq \psi_{p,k}^{(m)}\left(\sum_{i=1}^{\alpha} x_i\right) > 0, \\
&\vdots \qquad\qquad \vdots \\
\psi_{p,k}^{(m)}(x_\alpha) &\geq \psi_{p,k}^{(m)}\left(\sum_{i=1}^{\alpha} x_i\right) > 0.
\end{aligned}$$

Then multiplying these inequalities yields,

$$\prod_{i=1}^{\alpha} \psi_{p,k}^{(m)}(x_i) \geq \left[\psi_{p,k}^{(m)}\left(\sum_{i=1}^{\alpha} x_i\right)\right]^{\alpha}$$

as required.

Remark 2.9. Let $\alpha = 2$, $x_1 = x$ and $x_2 = y$ in (15), then we recover the result (10).

Theorem 2.10. *Let $x, y, a > 1$ such that $\frac{1}{x} + \frac{1}{y} \leq 1$ and $\frac{1}{a} + \frac{1}{b} = 1$. Then for $p \in \mathbb{N}$ and $k > 0$, the inequality*

$$\psi_{p,k}^{(m)}(xy) \leq \left(\psi_{p,k}^{(m)}(x)\right)^{\frac{1}{a}} \left(\psi_{p,k}^{(m)}(y)\right)^{\frac{1}{b}} \tag{16}$$

is valid for a positive odd integer m.

Proof. From the hypothesis, it follows that $xy \geq x + y$. Then since $\psi_{p,k}^{(m)}(x)$ is decreasing for odd m, and by using the Hölder's inequality for finite sums, we obtain

$$\begin{aligned}
\psi_{p,k}^{(m)}(xy) \leq \psi_{p,k}^{(m)}(x+y) &= \sum_{n=0}^{p} \frac{m!}{(nk+x+y)^{m+1}} \\
&= \sum_{n=0}^{p} \frac{(m!)^{\frac{1}{a}}(m!)^{\frac{1}{b}}}{(nk+x+y)^{\frac{m+1}{a}}(nk+x+y)^{\frac{m+1}{b}}} \\
&\leq \sum_{n=0}^{p} \frac{(m!)^{\frac{1}{a}}}{(nk+x)^{\frac{m+1}{a}}} \cdot \frac{(m!)^{\frac{1}{b}}}{(nk+y)^{\frac{m+1}{b}}} \\
&\leq \left(\sum_{n=0}^{p} \frac{m!}{(nk+x)^{m+1}}\right)^{\frac{1}{a}} \left(\sum_{n=0}^{p} \frac{m!}{(nk+y)^{m+1}}\right)^{\frac{1}{b}} \\
&= \left(\psi_{p,k}^{(m)}(x)\right)^{\frac{1}{a}} \left(\psi_{p,k}^{(m)}(y)\right)^{\frac{1}{b}}
\end{aligned}$$

which completes the proof.

Theorem 2.11. *Let $p \in \mathbb{N}$, $k > 0$, $s > 1$, $\frac{1}{s} + \frac{1}{t} = 1$ and $m, n \in \mathbb{N}$ such that $\frac{m}{s} + \frac{n}{t} \in \mathbb{N}$. Then, the inequality*

$$\left|\psi_{p,k}^{\left(\frac{m}{s}+\frac{n}{t}\right)}(x+y)\right| \leq \left|\psi_{p,k}^{(m)}(x)\right|^{\frac{1}{s}} \left|\psi_{p,k}^{(n)}(y)\right|^{\frac{1}{t}} \tag{17}$$

holds for $x > 0$ and $y > 0$.

Proof. From the series representation (4), we obtain

$$\begin{aligned}
\left|\psi_{p,k}^{\left(\frac{m}{s}+\frac{n}{t}\right)}(x+y)\right| &= \left(\frac{m}{s}+\frac{n}{t}\right)!\sum_{i=0}^{p}\frac{1}{(ik+x+y)^{\frac{m}{s}+\frac{n}{t}+1}}\\
&= \sum_{i=0}^{p}\frac{\left(\frac{m}{s}+\frac{n}{t}\right)!}{(ik+x+y)^{\frac{m+1}{s}}(ik+x+y)^{\frac{n+1}{t}}}\\
&\leq \sum_{i=0}^{p}\frac{\left(\frac{m}{s}+\frac{n}{t}\right)!}{(ik+x)^{\frac{m+1}{s}}(ik+y)^{\frac{n+1}{t}}}\\
&\leq \sum_{i=0}^{p}\frac{(m!)^{\frac{1}{s}}}{(ik+x)^{\frac{m+1}{s}}}\cdot\frac{(n!)^{\frac{1}{t}}}{(ik+y)^{\frac{n+1}{t}}}\\
&\leq \left(\sum_{i=0}^{p}\frac{m!}{(ik+x)^{m+1}}\right)^{\frac{1}{s}}\left(\sum_{i=0}^{p}\frac{n!}{(ik+y)^{n+1}}\right)^{\frac{1}{t}}\\
&= \left|\psi_{p,k}^{(m)}(x)\right|^{\frac{1}{s}}\left|\psi_{p,k}^{(n)}(y)\right|^{\frac{1}{t}}.
\end{aligned}$$

Note: In the proof, we have used the Hölder's inequality for finite sums and the fact that $\left(\frac{m}{s}+\frac{n}{t}\right)! \leq (m!)^{\frac{1}{s}}(n!)^{\frac{1}{t}}$, which follows from the logarithmic convexity of the factorial function.

Corollary 2.12. *Let $p \in \mathbb{N}$, $k > 0$ and $m \in \mathbb{N}$. Then the inequality*

$$\left|\psi_{p,k}^{(m)}(x)\right|\left|\psi_{p,k}^{(m+2)}(x)\right| \geq \left|\psi_{p,k}^{(m+1)}(2x)\right|^{2}$$

holds for $x > 0$.

Proof. Let $x = y$, $s = t = 2$ and $n = m + 2$ in Theorem 2.11.

Corollary 2.13. *Let $p \in \mathbb{N}$, $k > 0$, $m \in \mathbb{N}$ and $\frac{1}{s}+\frac{1}{t} = 1$. Then the inequality*

$$\left|\psi_{p,k}^{(m)}(x+y)\right| \leq \left|\psi_{p,k}^{(m)}(x)\right|^{\frac{1}{s}}\left|\psi_{p,k}^{(m)}(y)\right|^{\frac{1}{t}} \tag{18}$$

holds for $x > 0$ and $y > 0$.

Proof. Let $m = n$ in Theorem 2.11.

Remark 2.14. It is interesting to note that, by letting $s = t = 2$ in (18), we obtain a result which coincides with (10).

Theorem 2.15. *Let $m \in \mathbb{N}$, $\beta \geq 1$ and $x > 0$. Then for $p \in \mathbb{N}$ and $k > 0$, the following inequalities*

$$\left(\exp\psi_{p,k}^{(m)}(x)\right)^{\beta} > \exp\psi_{p,k}^{(m+1)}(x).\exp\psi_{p,k}^{(m-1)}(x), \quad \textit{if } m \textit{ is odd} \tag{19}$$

$$\left(\exp\psi_{p,k}^{(m)}(x)\right)^{\beta} < \exp\psi_{p,k}^{(m+1)}(x).\exp\psi_{p,k}^{(m-1)}(x), \quad \textit{if } m \textit{ is even.} \tag{20}$$

are satisfied.

Proof. By relation (4), we obtain

$$\begin{aligned}
&\psi_{p,k}^{(m)}(x) - \psi_{p,k}^{(m+1)}(x) - \psi_{p,k}^{(m-1)}(x) \\
&= \sum_{n=0}^{p} \frac{(-1)^{m+1} m!}{(nk+x)^{m+1}} - \sum_{n=0}^{p} \frac{(-1)^{m+2}(m+1)!}{(nk+x)^{m+2}} - \sum_{n=0}^{p} \frac{(-1)^{m}(m-1)!}{(nk+x)^{m}} \\
&= (-1)^m \left[\sum_{n=0}^{p} \frac{-m!}{(nk+x)^{m+1}} - \sum_{n=0}^{p} \frac{(m+1)!}{(nk+x)^{m+2}} - \sum_{n=0}^{p} \frac{(m-1)!}{(nk+x)^{m}} \right] \\
&= (-1)^{m+1} \left[\sum_{n=0}^{p} \frac{m!}{(nk+x)^{m+1}} + \sum_{n=0}^{p} \frac{(m+1)!}{(nk+x)^{m+2}} + \sum_{n=0}^{p} \frac{(m-1)!}{(nk+x)^{m}} \right] \\
&> (<) 0
\end{aligned}$$

respectively for odd(even) m. This implies,

$$\psi_{p,k}^{(m)}(x) > \psi_{p,k}^{(m+1)}(x) + \psi_{p,k}^{(m-1)}(x)$$

and

$$\psi_{p,k}^{(m)}(x) < \psi_{p,k}^{(m+1)}(x) + \psi_{p,k}^{(m-1)}(x)$$

respectively for odd and even m. Then for $\beta \geq 1$, we have

$$\beta\psi_{p,k}^{(m)}(x) \geq \psi_{p,k}^{(m)}(x) > \psi_{p,k}^{(m+1)}(x) + \psi_{p,k}^{(m-1)}(x), \tag{21}$$

and

$$\beta\psi_{p,k}^{(m)}(x) \leq \psi_{p,k}^{(m)}(x) < \psi_{p,k}^{(m+1)}(x) + \psi_{p,k}^{(m-1)}(x). \tag{22}$$

By exponentiating the inequalities (21) and (22), we obtain the desired results.

Remark 2.16. Let $\beta = 2$ in Theorem 2.15, then we obtain the result of Theorem 2.7 of [4].

Remark 2.17. Let $\beta = 2$, $p \to \infty$ and $k \to 1$ in Theorem 2.15, then we obtain the result of Theorem 3.2 of [1].

Lemma 2.18. *Let a, b, c, d, α, β be positive real numbers such that $a + bx \leq c + dx$, $\beta d \leq \alpha b$, $\psi_{p,k}(a+bx) > 0$ and $\psi_{p,k}(c+dx) > 0$. Then*

$$\alpha b \psi_{p,k}(c+dx)\psi'_{p,k}(a+bx) - \beta d \psi_{p,k}(a+bx)\psi'_{p,k}(c+dx) \geq 0.$$

Proof. Note that, $\psi_{p,k}(x)$ is increasing and $\psi'_{p,k}(x)$ is decreasing for $x > 0$. Then, since $0 < a + bx \leq c + dx$, we have,
$0 < \psi_{p,k}(a+bx) \leq \psi_{p,k}(c+dx)$ and $\psi'_{p,k}(a+bx) \geq \psi'_{p,k}(c+dx) > 0$.
That implies;
$\psi_{p,k}(c+dx)\psi'_{p,k}(a+bx) \geq \psi_{p,k}(c+dx)\psi'_{p,k}(c+dx) \geq \psi_{p,k}(a+bx)\psi'_{p,k}(c+dx)$.
Moreover, $\alpha b \geq \beta d > 0$ implies;
$\alpha b\psi_{p,k}(c+dx)\psi'_{p,k}(a+bx) \geq \alpha b\psi_{p,k}(a+bx)\psi'_{p,k}(c+dx) \geq \beta d\psi_{p,k}(a+bx)\psi'_{p,k}(c+dx)$.
Therefore,
$\alpha b\psi_{p,k}(c+dx)\psi'_{p,k}(a+bx) - \beta d\psi_{p,k}(a+bx)\psi'_{p,k}(c+dx) \geq 0$.

Theorem 2.19. *Define a function T for $p \in \mathbb{N}$ and $k > 0$ by*
$$T(x) = \frac{\psi_{p,k}(a+bx)^{\alpha}}{\psi_{p,k}(c+dx)^{\beta}}, \quad x \in [0, \infty)$$
where a, b, c, d, α, β are positive real numbers such that $a+bx \leq c+dx$, $\beta d \leq \alpha b$, $\psi_{p,k}(a+bx) > 0$ and $\psi_{p,k}(c+dx) > 0$. Then T is increasing on $x \in [0, \infty)$ and the inequality
$$\frac{\psi_{p,k}(a)^{\alpha}}{\psi_{p,k}(c)^{\beta}} \leq \frac{\psi_{p,k}(a+bx)^{\alpha}}{\psi_{p,k}(c+dx)^{\beta}} \leq \frac{\psi_{p,k}(a+b)^{\alpha}}{\psi_{p,k}(c+d)^{\beta}} \tag{23}$$
holds for $x \in [0, 1]$.

Proof. Let $\lambda(x) = \ln T(x)$. That is,
$$\lambda(x) = \ln \frac{\psi_{p,k}(a+bx)^{\alpha}}{\psi_{p,k}(c+dx)^{\beta}} = \alpha \ln \psi_{p,k}(a+bx) - \beta \ln \psi_{p,k}(c+dx).$$

Then,
$$\begin{aligned}\lambda'(x) &= \alpha b \frac{\psi'_{p,k}(a+bx)}{\psi_{p,k}(a+bx)} - \beta d \frac{\psi'_{p,k}(c+dx)}{\psi_{p,k}(c+dx)} \\ &= \frac{\alpha b \psi'_{p,k}(a+bx)\psi_{p,k}(c+dx) - \beta d \psi'_{p,k}(c+dx)\psi_{p,k}(a+bx)}{\psi_{p,k}(a+bx)\psi_{p,k}(c+dx)} \geq 0\end{aligned}$$
resulting from Lemma 2.18. That implies T is increasing on $x \in [0, \infty)$ and for every $x \in [0, 1]$ we have,
$$T(0) \leq T(x) \leq T(1)$$
yielding the result (23).

Remark 2.20. Let $p \to \infty$ in Theorem 2.19, then we obtain the k-analogue of (23) as presented in Theorem 3.7 of [7].

Remark 2.21. Let $k \to 1$ in Theorem 2.19, then we obtain the p-analogue of (23).

Remark 2.22. Let $p \to \infty$ as $k \to 1$ in Theorem 2.19, then we obtain Theorem 2.3 of [2].

Remark 2.23. Results similar to (23) can also be found in [3] for the (q, k) and (p, q) analogues of the Digamma function.

Lemma 2.24 ([4]). *Let m be a positive odd integer. Then the inequality*
$$\psi_{p,k}^{(m)}(x)\psi_{p,k}^{(m+2)}(x) - \left[\psi_{p,k}^{(m+1)}(x)\right]^2 \geq 0.$$
holds for $p \in \mathbb{N}$, $k > 0$ and $x > 0$.

Lemma 2.25. *For a positive odd integer m, let $H(x) = \frac{\psi_{p,k}^{(m+1)}(x)}{\psi_{p,k}^{(m)}(x)}$, where $p \in \mathbb{N}$ and $k > 0$. Then H is increasing for $x > 0$.*

Proof. Direct differentiation yields

$$H'(x) = \frac{\psi_{p,k}^{(m)}(x)\psi_{p,k}^{(m+2)}(x) - [\psi_{p,k}^{(m+1)}(x)]^2}{[\psi_{p,k}^{(m)}(x)]^2}.$$

Then by Lemma 2.24, we conclude that $H'(x) \geq 0$ ending the proof.

Lemma 2.26. *Let $u \geq w > 0$, $p \in \mathbb{N}$, $k > 0$ and m a positive odd integer. Then for $0 < x \leq y$, we have*

$$u\frac{\psi_{p,k}^{(m+1)}(x)}{\psi_{p,k}^{(m)}(x)} - w\frac{\psi_{p,k}^{(m+1)}(y)}{\psi_{p,k}^{(m)}(y)} \leq 0.$$

Proof. Let $H(x)$ be defined as in Lemma 2.25. Then for $0 < x \leq y$ we have, $H(x) \leq H(y) < 0$ since $H(x)$ is increasing. This together with the fact that $u \geq w > 0$ gives $uH(x) - wH(y) \leq 0$ yielding the desired result.

Theorem 2.27. *Define a function U for $p \in \mathbb{N}$, $k > 0$ and m a positive odd integer by*

$$U(x) = \frac{\psi_{p,k}^{(m)}(a+bx)^{\alpha}}{\psi_{p,k}^{(m)}(c+dx)^{\beta}}, \quad x \in [0, \infty)$$

where a, b, c, d, α, β are positive real numbers such that $a + bx \leq c + dx$ and $\beta d \leq \alpha b$. Then U is decreasing on $x \in [0, \infty)$ and the inequality

$$\frac{\psi_{p,k}^{(m)}(a)^{\alpha}}{\psi_{p,k}^{(m)}(c)^{\beta}} \geq \frac{\psi_{p,k}^{(m)}(a+bx)^{\alpha}}{\psi_{p,k}^{(m)}(c+dx)^{\beta}} \geq \frac{\psi_{p,k}^{(m)}(a+b)^{\alpha}}{\psi_{p,k}^{(m)}(c+d)^{\beta}} \tag{24}$$

is valid for $x \in [0, 1]$.

Proof. Let $\delta(x) = \ln U(x)$. That is,

$$\delta(x) = \alpha \ln \psi_{p,k}^{(m)}(a+bx) - \beta \ln \psi_{p,k}^{(m)}(c+dx).$$

Then,

$$\delta'(x) = \alpha b\frac{\psi_{p,k}^{(m+1)}(a+bx)}{\psi_{p,k}^{(m)}(a+bx)} - \beta d\frac{\psi_{p,k}^{(m+1)}(c+dx)}{\psi_{p,k}^{(m)}(c+dx)}$$

Since $0 < a + bx \leq c + dx$ and $0 < \beta d \leq \alpha b$, then by Lemma 2.26, we conclude that $\delta'(x) \leq 0$. Thus, $\delta(x)$ is decreasing on $x \in [0, \infty)$. Therefore, U is also decreasing on $x \in [0, \infty)$ and for $x \in [0, 1]$, we have $U(0) \geq U(x) \geq U(1)$ yielding the result (24).

Remark 2.28. Let $b = d = \alpha = \beta = 1$ in Theorem 2.27. Then by allowing $p \to \infty$ as $k \to 1$, we recover the result of Theorem 2.9 of [2].

3. Conclusion

In this work, we have established several inequalities involving the (p, k)-analogues of the Digamma and Polygamma functions. As a consequence, some previous results are recovered as particular cases of the results of this paper.

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