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REPRESENTATION OF OPERATOR PERSPECTIVES FOR CONTINUOUSLY n-TIME DIFFERENTIABLE FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper we obtain some representations of operator perspectives for continuously n-time differentiable functions. Applications for weighted operator geometric mean, relative operator entropy and some exponential perspectives, are also provided.

1. Introduction

Let f be a continuous function defined on the interval I of real numbers, B a selfadjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right)\subset \mathring{I}$. Then by using the continuous functional calculus, we can define the *perspective* $\mathcal{P}_f(B,A)$ by setting

$$\mathcal{P}_f(B,A):=A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

If A and B are commutative, then

$$\mathcal{P}_f(B,A)=Af\left(BA^{-1}\right)$$

provided $\operatorname{Sp}\left(BA^{-1}\right)\subset \mathring{I}$.

It is well known that (see [9] and [8] or [10]), if f is an *operator convex function* defined in the positive half-line, then the mapping

$$(B,A)\mapsto \mathcal{P}_f(B,A)$$

defined in pairs of positive definite operators, is operator convex.

In the recent paper [2] we established the following reverse inequality for the perspective $\mathcal{P}_f(B,A)$.

Let $f:[m,M]\to\mathbb{R}$ be a *convex function* on the real interval $[m,M]$, A a positive invertible operator and B a selfadjoint operator such that

$$mA\le B\le MA, \tag{1.1}$$

then we have

$$\begin{aligned}0&\le \frac{1}{M-m}\left[f(m)(MA-B)+f(M)(B-mA)\right]-\mathcal{P}_f(B,A)\\&\le \frac{f'_-(M)-f'_+(m)}{M-m}\left(MA^{1/2}-BA^{-1/2}\right)\left(A^{-1/2}B-mA^{1/2}\right)\\&\le \frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]A.\end{aligned} \tag{1.2}$$

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Let $f : J \subset \mathbb{R} \to \mathbb{R}$ be a *twice differentiable function* on the interval $\mathring{J}$, the interior of J. Suppose that there exists the constants d, D such that

$$d \leq f''(t) \leq D \text{ for any } t \in \mathring{J}. \tag{1.3}$$

If A is a positive invertible operator and B a selfadjoint operator such that the condition (1.1) is valid with $[m, M] \subset \mathring{J}$, then we have the following result as well [3]

$$\begin{aligned} &\frac{1}{2} d \left(MA^{1/2} - BA^{-1/2}\right)\left(A^{-1/2}B - mA^{1/2}\right) \\ &\leq \frac{1}{M-m}\left[f(m)(MA - B) + f(M)(B - mA)\right] - \mathcal{P}_f(B, A) \\ &\leq \frac{1}{2} D \left(MA^{1/2} - BA^{-1/2}\right)\left(A^{-1/2}B - mA^{1/2}\right). \end{aligned} \tag{1.4}$$

If $d > 0$, then the first inequality in (1.4) is better than the same inequality in (1.2).

If $f_\nu : [0, \infty) \to [0, \infty)$, $f_\nu(t) = t^\nu$, $\nu \in [0, 1]$, then

$$P_{f_\nu}(B, A) := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2} =: A\sharp_\nu B,$$

is the *weighted operator geometric mean* of the positive invertible operators A and B with the weight ν.

We define the *weighted operator arithmetic mean* by

$$A\nabla_\nu B := (1 - \nu) A + \nu B, \ \nu \in [0, 1].$$

It is well known that the following *Young's type inequality* holds:

$$A\sharp_\nu B \leq A\nabla_\nu B$$

for any $\nu \in [0, 1]$.

If we take the function $f = \ln$, then

$$P_{\ln}(B, A) := A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} =: S(A|B),$$

is the *relative operator entropy*, for positive invertible operators A and B.

Kamei and Fujii [11], [12] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, which is a relative version of the operator entropy considered by Nakamura-Umegaki [25].

In the recent paper [6] we established the following representation result:

Theorem 1. *Let A be a positive invertible operator, B a selfadjoint operator such that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right) \subset \mathring{I}$ and $f : I \to \mathbb{C}$ be a continuously differentiable function on $\mathring{I}$. Then for any $a \in I$ and $\mu \in \mathbb{C}$ we have*

$$P_f(B, A) = f(a) A + \mu(B - aA) + \left(BA^{-1} - aI\right)\int_0^1 P_{f' - \mu}\left((aA)\nabla_t B, A\right) dt. \tag{1.5}$$

Using the representation (1.5), we have for positive invertible operators A, B that

$$\begin{aligned} A\sharp_\nu B &= a^\nu A + \mu(B - aA) \\ &+ \left(BA^{-1} - aI\right) \\ &\times \int_0^1 A^{1/2}\left[\nu\left(A^{-1/2}\left[(aA)\nabla_t B\right]A^{-1/2}\right)^{\nu-1} - \mu I\right]A^{1/2} dt, \end{aligned} \tag{1.6}$$

for any $a > 0$ and $\mu \in \mathbb{R}$.

If we take in this equality $a = 1$ and $\mu = \nu$, then we get the equality

$$A\sharp_\nu B = A\nabla_\nu B + \nu \left(BA^{-1} - I\right) \int_0^1 A^{1/2} \left[\left(A^{-1/2} \left(A\nabla_t B\right) A^{-1/2}\right)^{\nu-1} - I\right] A^{1/2} dt, \tag{1.7}$$

that is equivalent to

$$0 \leq A\nabla_\nu B - A\sharp_\nu B = \nu \left(I - BA^{-1}\right) \int_0^1 A^{1/2} \left[\left(A^{-1/2} \left(A\nabla_t B\right) A^{-1/2}\right)^{\nu-1} - I\right] A^{1/2} dt. \tag{1.8}$$

Using the identity (1.5) for $f = \ln$, we have for the positive invertible operators A, B that

$$\begin{aligned} S\left(A|B\right) &= \left(\ln a\right) A + \mu \left(B - aA\right) \\ &+ \left(BA^{-1} - aI\right) \int_0^1 A^{1/2} \left[A^{1/2} \left[\left(aA\right) \nabla_t B\right]^{-1} A^{1/2} - \mu I\right] A^{1/2} dt \\ &= \left(\ln a\right) A + \mu \left(B - aA\right) \\ &+ \left(B - aA\right) \left(\int_0^1 \left[\left[\left(aA\right) \nabla_t B\right]^{-1} - \mu A^{-1}\right] dt\right) A \end{aligned} \tag{1.9}$$

for any $a > 0$ and $\mu \in \mathbb{R}$.

If we take in (1.9) $a = 1$ and $\mu = 1$, then we have the simpler equality

$$S\left(A|B\right) = B - A + \left(B - A\right) \left(\int_0^1 \left[\left(A\nabla_t B\right)^{-1} - A^{-1}\right] dt\right) A. \tag{1.10}$$

Motivated by the above results, in this paper we obtain some representations for operator perspectives of continuously n-time differentiable functions. Applications for weighted operator geometric mean, relative operator entropy and some exponential perspectives, are also provided.

2. Some Representation Results

For the function $f_{c,k} : \mathbb{R} \to [0, \infty)$, $f_{c,k}\left(u\right) = \left(u - c\right)^k$ where $c \in \mathbb{R}$ and natural number $k \geq 1$ we consider the perspective

$$\mathcal{P}_{f_{c,k}}\left(B, A\right) := A^{1/2} f_{c,k}\left(A^{-1/2} B A^{-1/2}\right) A^{1/2} \tag{2.1}$$

where A is a positive invertible operator and B a selfadjoint operator on the Hilbert space H.

We observe that for any $k \geq 1$ we have

$$\begin{aligned} \mathcal{P}_{f_{c,k}}\left(B, A\right) &= A^{1/2} \left(A^{-1/2} B A^{-1/2} - c1_H\right)^k A^{1/2} \\ &= A^{1/2} \left(A^{-1/2} \left(B - cA\right) A^{-1/2}\right)^k A^{1/2} \\ &= A^{1/2} A^{-1/2} \left(B - cA\right) A^{-1/2} ... A^{-1/2} \left(B - cA\right) A^{-1/2} A^{1/2} \\ &= \left(B - cA\right) A^{-1} ... \left(B - cA\right) A^{-1} A = \left[\left(B - cA\right) A^{-1}\right]^k A \\ &= \left(BA^{-1} - c1_H\right)^k A. \end{aligned}$$

For $k=0$, we put $\mathcal{P}_{f_{c,0}}(B,A)=A$.

The following theorem is well known in the literature as Taylor's theorem with the integral remainder.

Theorem 2. *Let $I\subset\mathbb{R}$ be a closed interval, $a\in I$ and let n be a positive integer. If $f:I\longrightarrow\mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous on I, then for each $u\in I$*

$$f(u)=T_n(f;a,u)+R_n(f;a,u), \tag{2.2}$$

where $T_n(f;a,u)$ is Taylor's polynomial, i.e.,

$$T_n(f;a,u):=\sum_{k=0}^{n}\frac{(u-a)^k}{k!}f^{(k)}(a).$$

Note that $f^{(0)}:=f$ and $0!:=1$, and the remainder is given by

$$R_n(f;a,u):=\frac{1}{n!}\int_a^u(u-t)^n f^{(n+1)}(t)\,dt.$$

We have the following generalization of Theorem 1:

Theorem 3. *Let A be a positive invertible operator, B a selfadjoint operator such that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right)\subset\mathring{I}$ and $f:I\to\mathbb{C}$ be a n-time differentiable function on $\mathring{I}$ with the derivative $f^{(n+1)}$ continuous on $\mathring{I}$. Then for any $a\in I$ and $\mu\in\mathbb{C}$ we have*

$$\begin{aligned}P_f(B,A)=&\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(a)\mathcal{P}_{f_{a,k}}(B,A)+\frac{1}{(n+1)!}\mu\mathcal{P}_{f_{a,n+1}}(B,A)\\&+\frac{1}{n!}\mathcal{P}_{f_{a,n+1}}(B,A)A^{-1}\int_0^1(1-s)^n\mathcal{P}_{f^{(n+1)}-\mu}((aA)\nabla_sB,A)\,ds.\end{aligned}\tag{2.3}$$

In particular, we have for $\mu=0$ that

$$\begin{aligned}P_f(B,A)=&\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(a)\mathcal{P}_{f_{a,k}}(B,A)\\&+\frac{1}{n!}\mathcal{P}_{f_{a,n+1}}(B,A)A^{-1}\int_0^1(1-s)^n\mathcal{P}_{f^{(n+1)}}((aA)\nabla_sB,A)\,ds.\end{aligned}\tag{2.4}$$

Proof. By taking $u \in I$ and by using the change of variable $[0,1] \ni t \longmapsto s = (1-t)\, a + tu$ in Taylor's representation (2.2) we have

$$
\begin{aligned}
f(u) &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(u-a)^k \\
&+ \frac{1}{n!}(u-a)^{n+1} \int_0^1 f^{(n+1)}((1-s)\, a + su)(1-s)^n\, ds \\
&= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(u-a)^k \\
&+ \frac{1}{n!}(u-a)^{n+1} \int_0^1 \left[f^{(n+1)}((1-s)\, a + su) - \mu \right](1-s)^n\, ds \\
&+ \frac{1}{n!}(u-a)^{n+1} \mu \int_0^1 (1-s)^n\, ds \\
&= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(u-a)^k + \frac{1}{(n+1)!}(u-a)^{n+1}\mu \\
&+ \frac{1}{n!}(u-a)^{n+1} \int_0^1 \left[f^{(n+1)}((1-s)\, a + su) - \mu \right](1-s)^n\, ds
\end{aligned} \tag{2.5}
$$

for any $\mu \in \mathbb{C}$.

Let T be a selfadjoint operator with $\operatorname{Sp}(T) \subset \mathring{I}$. Take the real numbers m, M such that $\operatorname{Sp}(T) \subseteq [m, M] \subset \mathring{I}$. If $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ is the spectral family of the operator T, then by the *spectral representation theorem* (SRT) we have

$$
f(T) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda := \lim_{\varepsilon \to 0+} \int_{m-\varepsilon}^{M} f(\lambda)\, dE_\lambda,
$$

where the integral is taken in the Riemann-Stieltjes sense.

Let $\varepsilon > 0$ small enough such that $[m-\varepsilon, M] \subset \mathring{I}$, then by integrating the equality (2.5) on the interval $[m-\varepsilon, M]$ and using the Fubini theorem, we have

$$
\begin{aligned}
&\int_{m-\varepsilon}^{M} f(\lambda)\, dE_\lambda \\
&= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a) \int_{m-\varepsilon}^{M} (\lambda-a)^k\, dE_\lambda + \frac{1}{(n+1)!}\mu \int_{m-\varepsilon}^{M} (\lambda-a)^{n+1}\, dE_\lambda \\
&+ \frac{1}{n!} \int_{m-\varepsilon}^{M} \left[(\lambda-a)^{n+1} \int_0^1 \left[f^{(n+1)}((1-s)\, a + s\lambda) - \mu \right](1-s)^n\, ds \right] dE_\lambda \\
&= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a) \int_{m-\varepsilon}^{M} (\lambda-a)^k\, dE_\lambda + \frac{1}{(n+1)!}\mu \int_{m-\varepsilon}^{M} (\lambda-a)^{n+1}\, dE_\lambda \\
&+ \frac{1}{n!} \int_0^1 \left[(1-s)^n \int_{m-\varepsilon}^{M} \left[(\lambda-a)^{n+1} \left[f^{(n+1)}((1-s)\, a + s\lambda) - \mu \right] \right] dE_\lambda \right] ds
\end{aligned} \tag{2.6}
$$

for any $a \in \mathring{I}$.

Taking the limit over $\varepsilon \to 0+$ in (2.6) we get by SRT that

$$f(T) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(T - a1_H)^k + \frac{1}{(n+1)!}\mu (T - a1_H)^{n+1}$$
$$+ \frac{1}{n!}\int_0^1 (1-s)^n (T - a1_H)^{n+1} \left[f^{(n+1)}((1-s)a1_H + sT) - \mu 1_H\right] ds$$

which can be written in an equivalent form as

$$f(T) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(T - a1_H)^k + \frac{1}{(n+1)!}\mu (T - a1_H)^{n+1}$$
$$+ \frac{1}{n!}(T - a1_H)^{n+1}\int_0^1 (1-s)^n \left[f^{(n+1)}((1-s)a1_H + sT) - \mu 1_H\right] ds. \tag{2.7}$$

This result is of interest in itself.

For $\mu = 0$ we also have the simpler representation

$$f(T) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(T - a1_H)^k$$
$$+ \frac{1}{n!}(T - a1_H)^{n+1}\int_0^1 (1-s)^n \left[f^{(n+1)}((1-s)a1_H + sT)\right] ds. \tag{2.8}$$

If we take $T = A^{-1/2}BA^{-1/2}$ in (2.7), then we have

$$f\left(A^{-1/2}BA^{-1/2}\right) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)\left(A^{-1/2}BA^{-1/2} - a1_H\right)^k$$
$$+ \frac{1}{(n+1)!}\mu\left(A^{-1/2}BA^{-1/2} - a1_H\right)^{n+1}$$
$$+ \frac{1}{n!}\left(A^{-1/2}BA^{-1/2} - a1_H\right)^{n+1}$$
$$\times \int_0^1 (1-s)^n \left[f^{(n+1)}\left((1-s)a1_H + sA^{-1/2}BA^{-1/2}\right) - \mu 1_H\right] ds.$$

If we multiply both sides by $A^{1/2}$ then we get

$$A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right) A^{1/2}$$
$$= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a) A^{1/2}\left(A^{-1/2}BA^{-1/2} - a1_H\right)^k A^{1/2}$$
$$+ \frac{1}{(n+1)!}\mu A^{1/2}\left(A^{-1/2}BA^{-1/2} - a1_H\right)^{n+1} A^{1/2}$$
$$+ \frac{1}{n!} A^{1/2}\left(A^{-1/2}BA^{-1/2} - a1_H\right)^{n+1}$$
$$\times \int_0^1 (1-s)^n \left[f^{(n+1)}\left((1-s)a1_H + sA^{-1/2}BA^{-1/2}\right) - \mu 1_H\right] A^{1/2} ds. \tag{2.9}$$

Observe that

$$A^{1/2}\left(A^{-1/2}BA^{-1/2} - a1_H\right)^k A^{1/2} = \mathcal{P}_{f_{a,k}}(B, A),$$

$$A^{1/2}\left(A^{-1/2}BA^{-1/2}-a1_H\right)^{n+1}A^{1/2}=\mathcal{P}_{f_{a,n+1}}(B,A)$$

and

$$\begin{aligned}&A^{1/2}\left(A^{-1/2}BA^{-1/2}-a1_H\right)^{n+1}\\&\times\int_0^1(1-s)^n\left[f^{(n+1)}\left((1-s)a1_H+sA^{-1/2}BA^{-1/2}\right)-\mu 1_H\right]A^{1/2}ds\\&=A^{1/2}\left(A^{-1/2}BA^{-1/2}-a1_H\right)^{n+1}A^{1/2}A^{-1}\\&\times\int_0^1(1-s)^nA^{1/2}\left[f^{(n+1)}\left(A^{-1/2}((1-s)aA+sB)A^{-1/2}\right)-\mu 1_H\right]A^{1/2}ds\\&=\mathcal{P}_{f_{a,n+1}}(B,A)A^{-1}\int_0^1(1-s)^n\mathcal{P}_{f^{(n+1)}-\mu}((1-s)aA+sB,A)\,ds\end{aligned}$$

for $n\geq 1$ and $a\in I$.

Using the equality (2.9) we deduce the desired result (2.3). □

Corollary 1. *Let A be a positive invertible operator, B a selfadjoint operator such that*

$$mA\leq B\leq MA \tag{2.10}$$

for some real numbers m, M with $[m,M]\subset \mathring{I}$ and $f:I\to\mathbb{C}$ be a n-time differentiable function on $\mathring{I}$ with the derivative $f^{(n+1)}$ continuous on $\mathring{I}$. Then for any $\mu\in\mathbb{C}$ we have

$$\begin{aligned}P_f(B,A)&=\sum_{k=0}^n\frac{1}{k!}f^{(k)}\left(\frac{m+M}{2}\right)\mathcal{P}_{f_{\frac{m+M}{2},k}}(B,A)\\&+\frac{1}{(n+1)!}\mu\mathcal{P}_{f_{\frac{m+M}{2},n+1}}(B,A)+\frac{1}{n!}\mathcal{P}_{f_{\frac{m+M}{2},n+1}}(B,A)A^{-1}\\&\times\int_0^1(1-s)^n\mathcal{P}_{f^{(n+1)}-\mu}\left(\left(\frac{m+M}{2}A\right)\nabla_sB,A\right)ds.\end{aligned}\tag{2.11}$$

In particular, we have for $\mu=0$ that

$$\begin{aligned}P_f(B,A)&=\sum_{k=0}^n\frac{1}{k!}f^{(k)}\left(\frac{m+M}{2}\right)\mathcal{P}_{f_{a,k}}(B,A)\\&+\frac{1}{n!}\mathcal{P}_{f_{\frac{m+M}{2},n+1}}(B,A)A^{-1}\\&\times\int_0^1(1-s)^n\mathcal{P}_{f^{(n+1)}}\left(\left(\frac{m+M}{2}A\right)\nabla_sB,A\right)ds.\end{aligned}\tag{2.12}$$

From the condition (2.10) we have by multiplying both sides with $A^{-1/2}$ that $m1_H\leq A^{-1/2}BA^{-1/2}\leq M1_H$. Therefore $\mathrm{Sp}\left(A^{-1/2}BA^{-1/2}\right)\subset\mathring{I}$ and by taking $a=\frac{m+M}{2}$ in Theorem 3 we get (2.11) and (2.12).

Corollary 2. *With the assumptions of Theorem 3 we have for any $x \in H$, $x \neq 0$ that*

$$P_f(B,A) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right) \mathcal{P}_{f_{\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle},k}}(B,A) + \frac{1}{(n+1)!}\mu \mathcal{P}_{f_{\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle},n+1}}(B,A) + \frac{1}{n!}\mathcal{P}_{f_{\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle},n+1}}(B,A)A^{-1} \times \int_0^1 (1-s)^n \mathcal{P}_{f^{(n+1)}-\mu}\left(\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}A\right)\nabla_s B, A\right) ds \tag{2.13}$$

and

$$\langle P_f(B,A)x,x\rangle = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right) \left\langle \mathcal{P}_{f_{\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle},k}}(B,A)x,x\right\rangle + \frac{1}{(n+1)!}\mu \left\langle \mathcal{P}_{f_{\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle},n+1}}(B,A)x,x\right\rangle + \frac{1}{n!}\int_0^1 (1-s)^n \times \left\langle \mathcal{P}_{f^{(n+1)}-\mu}\left(\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}A\right)\nabla_s B, A\right)x, \mathcal{P}_{f_{\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle},n+1}}(B,A)A^{-1}x\right\rangle ds. \tag{2.14}$$

Proof. Since $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right) \subset \mathring{I}$ then there exists some real numbers m, M such that $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right) \subseteq [m,M] \subset \mathring{I}$.

Let $x \in H$, $x \neq 0$ and put

$$a = \frac{\langle Bx,x\rangle}{\langle Ax,x\rangle} = \frac{\left\langle A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}x,x\right\rangle}{\left\langle A^{1/2}x, A^{1/2}x\right\rangle} = \frac{\left\langle \left(A^{-1/2}BA^{-1/2}\right)A^{1/2}x, A^{1/2}x\right\rangle}{\left\|A^{1/2}x\right\|^2} = \left\langle \left(A^{-1/2}BA^{-1/2}\right)u,u\right\rangle \in [m,M],$$

where $u = \frac{A^{1/2}x}{\left\|A^{1/2}x\right\|^2} \neq 0$ and $\|u\| = 1$.

Now, by taking $a = \frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}$ in (2.3) we get (2.13).

The equality (2.14) follows by (2.13) on taking the inner product $\langle P_f(B,A)x,x\rangle$ and doing the appropriate calculation in the right side. The details are omitted. □

Remark 1. *If we take $\mu = 0$ in (2.13) and (2.14), then we get the simpler relations*

$$P_f(B,A) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right) \mathcal{P}_{f_{\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle},k}}(B,A) + \frac{1}{n!}\mathcal{P}_{f_{\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle},n+1}}(B,A)A^{-1} \times \int_0^1 (1-s)^n \mathcal{P}_{f^{(n+1)}}\left(\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}A\right)\nabla_s B, A\right) ds \tag{2.15}$$

and

$$\begin{aligned}(2.16)\qquad &\langle P_f(B,A)x,x\rangle\\ &=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)\left\langle \mathcal{P}_{f_{\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle},k}}(B,A)x,x\right\rangle\\ &+\frac{1}{n!}\int_0^1(1-s)^n\\ &\times\left\langle \mathcal{P}_{f^{(n+1)}}\left(\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}A\right)\nabla_s B,A\right)x,\mathcal{P}_{f_{\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle},n+1}}(B,A)A^{-1}x\right\rangle ds,\end{aligned}$$

for any $x\in H$, $x\neq 0$.

3. Inequalities for Bounded Derivatives

Now, for ϕ, $\Phi\in\mathbb{C}$ and I an interval of real numbers, define the sets of complex-valued functions (see for instance [7])

$$\begin{aligned}&\bar{U}_I(\phi,\Phi)\\ &:=\left\{g:I\to\mathbb{C}|\operatorname{Re}\left[(\Phi-g(t))\left(\overline{g(t)}-\overline{\phi}\right)\right]\geq 0\text{ for almost every }t\in I\right\}\end{aligned}$$

and

$$\bar{\Delta}_I(\phi,\Phi):=\left\{g:I\to\mathbb{C}|\ \left|g(t)-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi|\ \text{ for a.e. } t\in I\right\}.$$

The following representation result may be stated.

Proposition 1. *For any* ϕ, $\Phi\in\mathbb{C}$, $\phi\neq\Phi$, *we have that* $\bar{U}_I(\phi,\Phi)$ *and* $\bar{\Delta}_I(\phi,\Phi)$ *are nonempty, convex and closed sets and*

$$\bar{U}_I(\phi,\Phi)=\bar{\Delta}_I(\phi,\Phi). \tag{3.1}$$

Proof. We observe that for any $z\in\mathbb{C}$ we have the equivalence

$$\left|z-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi|$$

if and only if

$$\operatorname{Re}[(\Phi-z)(\bar{z}-\phi)]\geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Phi-\phi|^2-\left|z-\frac{\phi+\Phi}{2}\right|^2=\operatorname{Re}[(\Phi-z)(\bar{z}-\phi)]$$

that holds for any $z\in\mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 3. *For any* ϕ, $\Phi\in\mathbb{C}$, $\phi\neq\Phi$, *we have that*

$$\begin{aligned}(3.2)\qquad \bar{U}_I(\phi,\Phi)=\{g:I\to\mathbb{C}\ |\ &(\operatorname{Re}\Phi-\operatorname{Re}g(t))(\operatorname{Re}g(t)-\operatorname{Re}\phi)\\ &+(\operatorname{Im}\Phi-\operatorname{Im}g(t))(\operatorname{Im}g(t)-\operatorname{Im}\phi)\geq 0\ \textit{for a.e. } t\in I\}.\end{aligned}$$

Now, if we assume that $\operatorname{Re}(\Phi) \geq \operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi) \geq \operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\bar{S}_I(\phi, \Phi) := \{g : I \to \mathbb{C} \mid \operatorname{Re}(\Phi) \geq \operatorname{Re} g(t) \geq \operatorname{Re}(\phi) \text{ and } \operatorname{Im}(\Phi) \geq \operatorname{Im} g(t) \geq \operatorname{Im}(\phi) \text{ for a.e. } t \in I\}. \tag{3.3}$$

One can easily observe that $\bar{S}_I(\phi, \Phi)$ is closed, convex and

$$\emptyset \neq \bar{S}_I(\phi, \Phi) \subseteq \bar{U}_I(\phi, \Phi). \tag{3.4}$$

We need the following lemma:

Lemma 1. *Let T be a selfadjoint operator and $A \geq 0$. Then we have*

$$-A^{1/2}|T|A^{1/2} \leq A^{1/2}TA^{1/2} \leq A^{1/2}|T|A^{1/2} \tag{3.5}$$

in the operator order, where $|T|$ is the absolute value of T.

We also have

$$\left\|A^{1/2}TA^{1/2}\right\| \leq \left\|A^{1/2}|T|A^{1/2}\right\|. \tag{3.6}$$

Proof. If we use Jensen's operator inequality for the convex function $f(t) = |t|$, then we have

$$|\langle Ty, y\rangle| \leq \langle |T| y, y\rangle$$

for any $y \in H$.

If we take in this inequality $y = A^{1/2}x$, $x \in H$, then we get

$$\left|\left\langle TA^{1/2}x, A^{1/2}x\right\rangle\right| \leq \left\langle |T| A^{1/2}x, A^{1/2}x\right\rangle$$

that is equivalent to

$$\left|\left\langle A^{1/2}TA^{1/2}x, x\right\rangle\right| \leq \left\langle A^{1/2}|T|A^{1/2}x, x\right\rangle \tag{3.7}$$

or to

$$-\left\langle A^{1/2}|T|A^{1/2}x, x\right\rangle \leq \left\langle A^{1/2}TA^{1/2}x, x\right\rangle \leq \left\langle A^{1/2}|T|A^{1/2}x, x\right\rangle$$

for any $x \in H$, which proves the inequality (3.5).

By taking the supremum over $x \in H$, $\|x\| = 1$ in (3.7) we obtain the desired inequality (3.6). □

Theorem 4. *Let A be a positive invertible operator, B a selfadjoint operator such that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right) \subset \mathring{I}$ and $f : I \to \mathbb{C}$ be a n-time differentiable function on $\mathring{I}$ with the derivative $f^{(n+1)}$ continuous on $\mathring{I}$ and such that $f^{(n+1)} \in \bar{\Delta}_{\mathring{I}}(\phi_{n+1}, \Phi_{n+1})$ for some ϕ_{n+1}, $\Phi_{n+1} \in \mathbb{C}$, $\phi_{n+1} \neq \Phi_{n+1}$. Then for any $a \in I$ we have*

$$\begin{aligned} &\left\|P_f(B, A) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a) \mathcal{P}_{f_{a,k}}(B, A) - \frac{1}{(n+1)!} \frac{\phi_{n+1} + \Phi_{n+1}}{2} \mathcal{P}_{f_{a,n+1}}(B, A)\right\| \\ &\leq \frac{1}{2(n+1)!} \left|\Phi_{n+1} - \phi_{n+1}\right| \left\|A^{-1/2}BA^{-1/2} - a1_H\right\|^{n+1} \|A\|^2 \left\|A^{-1}\right\|. \end{aligned} \tag{3.8}$$

Proof. We use the identity (2.3) for $\mu = \frac{\phi_{n+1}+\Phi_{n+1}}{2}$ in the form

$$\begin{aligned} (3.9)\quad & P_f(B,A) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a) \mathcal{P}_{f_{a,k}}(B,A) \\ & - \frac{1}{(n+1)!} \frac{\phi_{n+1}+\Phi_{n+1}}{2} \mathcal{P}_{f_{a,n+1}}(B,A) \\ & = \frac{1}{n!} A^{1/2} \left(A^{-1/2}BA^{-1/2} - a1_H\right)^{n+1} A^{1/2}A^{-1} \\ & \times \int_0^1 (1-s)^n A^{1/2} \\ & \times \left[f^{(n+1)}\left(A^{-1/2}((1-s)aA + sB)A^{-1/2}\right) - \frac{\phi_{n+1}+\Phi_{n+1}}{2} 1_H \right] A^{1/2} ds, \end{aligned}$$

for any $a \in I$.

Taking the operator norm and using its properties we get

$$\begin{aligned} (3.10)\quad & \left\| P_f(B,A) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a) \mathcal{P}_{f_{a,k}}(B,A) \right. \\ & \qquad \left. - \frac{1}{(n+1)!} \frac{\phi_{n+1}+\Phi_{n+1}}{2} \mathcal{P}_{f_{a,n+1}}(B,A) \right\| \\ & \le \frac{1}{n!} \left\| A^{-1/2}BA^{-1/2} - a1_H \right\|^{n+1} \|A\| \left\|A^{-1}\right\| \int_0^1 (1-s)^n \\ & \times \left\| A^{1/2} \left[f^{(n+1)}\left(A^{-1/2}((1-s)aA + sB)A^{-1/2}\right) - \frac{\phi_{n+1}+\Phi_{n+1}}{2} 1_H \right] A^{1/2} \right\| ds, \end{aligned}$$

for any $a \in I$.

We observe that

$$\operatorname{Sp}\left(A^{-1/2}[(1-s)aA + sB]A^{-1/2}\right) = \operatorname{Sp}\left((1-s)a1_H + sA^{-1/2}BA^{-1/2}\right) \subset \mathring{I}$$

for any $a \in \mathring{I}$ and $s \in [0,1]$, and by the continuous functional calculus for $f^{(n+1)} \in \bar{\Delta}_{\mathring{I}}(\phi_{n+1}, \Phi_{n+1})$ we have

$$\begin{aligned} (3.11)\quad & \left| f^{(n+1)}\left(A^{-1/2}[(1-s)aA + sB]A^{-1/2}\right) - \frac{\phi_{n+1}+\Phi_{n+1}}{2} 1_H \right| \\ & \le \frac{1}{2} \left|\Phi_{n+1} - \phi_{n+1}\right| 1_H \end{aligned}$$

for any $a \in \mathring{I}$ and $s \in [0,1]$.

Now, on multiplying both sides of (3.11) by $A^{1/2}$, we get

$$\begin{aligned} & A^{1/2} \left| f^{(n+1)}\left(A^{-1/2}[(1-s)aA + sB]A^{-1/2}\right) - \frac{\phi_{n+1}+\Phi_{n+1}}{2} 1_H \right| A^{1/2} \\ & \le \frac{1}{2} \left|\Phi_{n+1} - \phi_{n+1}\right| A \end{aligned}$$

for any $a \in \mathring{I}$ and $s \in [0,1]$.

By taking the norm in this inequality, we get

$$\left\|A^{1/2}\left|f^{(n+1)}\left(A^{-1/2}\left[(1-s)aA+sB\right]A^{-1/2}\right)-\frac{\phi_{n+1}+\Phi_{n+1}}{2}1_H\right|A^{1/2}\right\| \le \frac{1}{2}\left|\Phi_{n+1}-\phi_{n+1}\right|\|A\|$$

for any $a\in\mathring{I}$ and $s\in[0,1]$.

Using Lemma 1 we get

$$\left\|A^{1/2}\left[f^{(n+1)}\left(A^{-1/2}\left((1-s)aA+sB\right)A^{-1/2}\right)-\frac{\phi_{n+1}+\Phi_{n+1}}{2}1_H\right]A^{1/2}\right\|$$
$$\left\|A^{1/2}\left|f^{(n+1)}\left(A^{-1/2}\left[(1-s)aA+sB\right]A^{-1/2}\right)-\frac{\phi_{n+1}+\Phi_{n+1}}{2}1_H\right|A^{1/2}\right\| \le \frac{1}{2}\left|\Phi_{n+1}-\phi_{n+1}\right|\|A\|$$

for any $a\in\mathring{I}$ and $s\in[0,1]$.

By multiplying this inequality by $(1-s)^n$ and integrating over $s\in[0,1]$, we get by (3.10) the desired result (3.8). □

Corollary 4. *Let A be a positive invertible operator, B a selfadjoint operator such that the condition (2.10) for some real numbers m, M with $[m,M]\subset\mathring{I}$ and $f:I\to\mathbb{C}$ be a n-time differentiable function on $\mathring{I}$ and such that $f^{(n+1)}\in\bar{\Delta}_{\mathring{I}}\left(\phi_{n+1},\Phi_{n+1}\right)$ for some ϕ_{n+1}, $\Phi_{n+1}\in\mathbb{C}$, $\phi_{n+1}\neq\Phi_{n+1}$. Then we have*

$$\begin{aligned}(3.12)\quad &\left\|P_f(B,A)-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}\left(\frac{m+M}{2}\right)\mathcal{P}_{f_{\frac{m+M}{2},k}}(B,A)\right.\\ &\left.-\frac{1}{(n+1)!}\frac{\phi_{n+1}+\Phi_{n+1}}{2}\mathcal{P}_{f_{\frac{m+M}{2},n+1}}(B,A)\right\|\\ &\le\frac{1}{2(n+1)!}\left|\Phi_{n+1}-\phi_{n+1}\right|\left\|A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right\|^{n+1}\|A\|^2\left\|A^{-1}\right\|\\ &\le\frac{1}{2^{n+2}(n+1)!}\left|\Phi_{n+1}-\phi_{n+1}\right|(M-m)^{n+1}\|A\|^2\left\|A^{-1}\right\|.\end{aligned}$$

Remark 2. *Let A be a positive invertible operator, B a selfadjoint operator such that the condition (2.10) holds for some real numbers m, M with $[m,M]\subset\mathring{I}$. If $f:I\to\mathbb{R}$ is a n-time differentiable function on $\mathring{I}$ and such that $f^{(n+1)}$ is continuous and monotonic nondecreasing on $[m,M]$, then we can take $\phi_{n+1}=f^{(n+1)}(m)$ and $\Phi_{n+1}=f^{(n+1)}(M)$ and by (3.12) we get the following result that can provide many examples, as shown below*

$$\begin{aligned}(3.13)\quad &\left\|P_f(B,A)-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}\left(\frac{m+M}{2}\right)\mathcal{P}_{f_{\frac{m+M}{2},k}}(B,A)\right.\\ &\left.-\frac{1}{(n+1)!}\frac{f^{(n+1)}(m)+f^{(n+1)}(M)}{2}\mathcal{P}_{f_{\frac{m+M}{2},n+1}}(B,A)\right\|\\ &\le\frac{1}{2^{n+2}(n+1)!}\left[f^{(n+1)}(M)-f^{(n+1)}(m)\right](M-m)^{n+1}\|A\|^2\left\|A^{-1}\right\|.\end{aligned}$$

4. Applications for Relative Operator Entropy

Consider the logarithmic function ln . Then the relative operator entropy can be interpreted as the perspective of ln, namely

$$\mathcal{P}_{\ln}(B, A) = S(A|B),$$

for positive invertible operators A, B.

For some recent results on relative operator entropy see [4]-[5], [19]-[20] and [22]-[23].

Consider the function $f : (0, \infty) \longrightarrow \mathbb{R}$, $f(x) = \ln x$, then,

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n}, \ n \geq 1, \ x > 0,$$

and

$$\mathcal{P}_{f^{(n+1)}}((aA)\nabla_s B, A) = (-1)^n n! A^{1/2}\left(A^{1/2}((aA)\nabla_s B)A^{1/2}\right)^{-n-1} A^{1/2}$$

for $a > 0$ and $s \in [0, 1]$. Using the representation (2.4) we have

$$\begin{aligned} S(A|B) &= (\ln a) A + \sum_{k=1}^{n} \frac{1}{k}(-1)^{k-1} a^{-k}\mathcal{P}_{f_{a,k}}(B, A) \\ &\quad + (-1)^n \mathcal{P}_{f_{a,n+1}}(B, A) A^{-1/2} \\ &\quad \times \int_0^1 (1-s)^n \left(A^{1/2}((aA)\nabla_s B)A^{1/2}\right)^{-n-1} A^{1/2} ds \end{aligned} \tag{4.1}$$

for positive invertible operators A, B, $a > 0$ and $n \geq 1$.

For $a = 1$ we have for $k \geq 1$ and positive invertible operators A, B that

$$\mathcal{P}_{f_{1,k}}(B, A) = \left(BA^{-1} - 1_H\right)^k A$$

and the representation (4.1) can be written as

$$\begin{aligned} S(A|B) &= \sum_{k=1}^{n} \frac{1}{k}(-1)^{k-1}\left(BA^{-1} - 1_H\right)^k A \\ &\quad + (-1)^n \left(BA^{-1} - 1_H\right)^{n+1} \\ &\quad \times \int_0^1 (1-s)^n A^{1/2}\left(A^{1/2}(A\nabla_s B)A^{1/2}\right)^{-n-1} A^{1/2} ds \end{aligned} \tag{4.2}$$

for positive invertible operators A, B and for any $n \geq 1$.

Let $n = 2\ell + 1$ with $\ell \geq 0$. Then

$$f^{(2\ell+2)}(x) = -(2\ell+1)!x^{-2\ell-2}, \ x > 0$$

which is an increasing function and if $x \in [m, M] \subset (0, \infty)$, then

$$-(2\ell+1)!m^{-2\ell-2} \leq f^{(2\ell+2)}(x) \leq -(2\ell+1)!M^{-2\ell-2}.$$

Making use of the inequality (3.13) and assuming that positive invertible operators A, B satisfy the condition (2.10), then we get

$$\begin{aligned}(4.3)\quad &\left\| S\left(A|B\right) - \ln\left(\frac{m+M}{2}\right)A - \sum_{k=1}^{2\ell+1}\frac{\left(-1\right)^{k-1}}{k}\left(\frac{m+M}{2}\right)^{-k}\mathcal{P}_{f_{\frac{m+M}{2},k}}\left(B,A\right)\right. \\ &\left. +\frac{1}{2\ell+2}\frac{M^{-2\ell-2}+m^{-2\ell-2}}{2}\mathcal{P}_{f_{\frac{m+M}{2},2\ell+2}}\left(B,A\right)\right\| \\ &\leq \frac{1}{2^{n+2}\left(2\ell+2\right)}\left(m^{-2\ell-2}-M^{-2\ell-2}\right)\left(M-m\right)^{2\ell+2}\left\|A\right\|^{2}\left\|A^{-1}\right\|,\end{aligned}$$

for any $\ell \geq 0$.

5. Applications for Operator Geometric Mean

If we consider the continuous function $f_\nu : [0,\infty) \to [0,\infty)$, $f_\nu(t) = t^\nu$, $\nu \in [0,1]$, then the operator *ν-weighted arithmetic-geometric mean* can be interpreted as the perspective $\mathcal{P}_{f_\nu}(B,A)$, namely

$$\mathcal{P}_{f_\nu}\left(B,A\right) = A\sharp_\nu B.$$

For recent results on operator Young inequality see [13]-[17], [18] and [26]-[27].

We have

$$f_\nu'\left(t\right) = \nu t^{\nu-1},\ f_\nu''\left(t\right) = \nu\left(\nu-1\right)t^{\nu-2}, ...,\ f_\nu^{(k)}\left(t\right) = \nu\left(\nu-1\right)...\left(\nu-k+1\right)t^{\nu-k}$$

for $k \geq 1$.

If, for convenience, we denote

$$\left(\nu\right)_k := \nu\left(\nu-1\right)...\left(\nu-k+1\right) \text{ for } \nu\in\mathbb{R} \text{ and } k\geq 1$$

then we can write

$$f_\nu^{(k)}\left(t\right) = \left(\nu\right)_k t^{\nu-k} \text{ for } \nu\in[0,1] \text{ and } k\geq 1.$$

Observe that

$$\left(\nu\right)_{k+1} := \nu\left(\nu-1\right)...\left(\nu-k+1\right)\left(\nu-k\right) = \nu\left(\nu-1\right)_k \text{ for } \nu\in\mathbb{R} \text{ and } k\geq 1.$$

We have for $n \geq 1$ and s, $\nu \in [0,1]$ that

$$\begin{aligned}&\mathcal{P}_{f^{(n+1)}-\nu}\left(A\nabla_s B, A\right) \\ &= A^{1/2}\left[\left(\nu\right)_{n+1}\left(\left(A^{-1/2}\left(A\nabla_s B\right)A^{-1/2}\right)\right)^{\nu-n-1} - \nu 1_H\right]A^{1/2} \\ &= \nu A^{1/2}\left[\left(\nu-1\right)_n\left(\left(A^{-1/2}\left(A\nabla_s B\right)A^{-1/2}\right)\right)^{\nu-n-1} - 1_H\right]A^{1/2}.\end{aligned}$$

By using (2.3) we get for $a=1$ and $\mu=\nu\in[0,1]$ that

$$
\begin{aligned}
A\sharp_{\nu}B &= A\nabla_{\nu}B+\sum_{k=2}^{n}\frac{1}{k!}(\nu)_{k}\left(BA^{-1}-1_{H}\right)^{k}A \\
&+\frac{1}{(n+1)!}\nu\left(BA^{-1}-1_{H}\right)^{n+1}A \\
&+\frac{1}{n!}\nu\left(BA^{-1}-1_{H}\right)^{n+1}\int_{0}^{1}(1-s)^{n} \\
&\times A^{1/2}\left[(\nu-1)_{n}\left(A^{-1/2}\left(A\nabla_{s}B\right)A^{-1/2}\right)^{\nu-n-1}-1_{H}\right]A^{1/2}ds,
\end{aligned}
\tag{5.1}
$$

for positive invertible operators A, B and for any $n\geq 2$.

If the positive invertible operators A, B satisfy the condition (2.10), then one can get various inequalities similar to the one from (4.3).

6. Applications for Some Exponential Perspectives

For $\alpha\neq 0$ and $a\in\mathbb{R}$ we consider the family of functions $E_{\alpha,a}:\mathbb{R}\to\mathbb{R}$ defined by

$$E_{\alpha,a}(t)=\exp(\alpha(t-a)).$$

We have that

$$E_{\alpha,a}^{(k)}(t)=\alpha^{k}\exp(\alpha(t-a)) \text{ for any } k\geq 0$$

and

$$E_{\alpha,a}^{(k)}(a)=\alpha^{k} \text{ for any } k\geq 0.$$

For two positive invertible operators A, B and $\alpha\neq 0$ and $a\in\mathbb{R}$ we define the (α,a)-*exponential perspective* by

$$
\begin{aligned}
E_{\alpha,a}(B,A) &:= A^{1/2}\left[\exp\left(\alpha\left(A^{-1/2}BA^{-1/2}-a1_{H}\right)\right)\right]A^{1/2} \\
&= A^{1/2}\left[\exp\left(\alpha A^{-1/2}(B-aA)A^{-1/2}\right)\right]A^{1/2}.
\end{aligned}
\tag{6.1}
$$

Observe that

$$
\begin{aligned}
&\mathcal{P}_{E_{\alpha,a}^{(n+1)}}\left((aA)\nabla_{s}B,A\right) \\
&=\alpha^{n+1}E_{\alpha,a}\left((aA)\nabla_{s}B,A\right) \\
&=\alpha^{n+1}A^{1/2}\left[\exp\left(\alpha A^{-1/2}\left((aA)\nabla_{s}B-aA\right)A^{-1/2}\right)\right]A^{1/2} \\
&=\alpha^{n+1}A^{1/2}\left[\exp\left(\alpha A^{-1/2}\left((1-s)(aA)+sB-aA\right)A^{-1/2}\right)\right]A^{1/2} \\
&=\alpha^{n+1}A^{1/2}\left[\exp\left(\alpha sA^{-1/2}(B-aA)A^{-1/2}\right)\right]A^{1/2}
\end{aligned}
$$

for positive invertible operators A, B, $\alpha\neq 0$ and $a\in\mathbb{R}$.

If we write the identity (2.4) for $f = E_{\alpha,a}$, then we get the identity

$$E_{\alpha,a}(B,A) = A + \sum_{k=1}^{n} \frac{1}{k!}\alpha^k \mathcal{P}_{f_{a,k}}(B,A) + \frac{1}{n!}\alpha^{n+1}\mathcal{P}_{f_{a,n+1}}(B,A) \times \int_0^1 (1-s)^n A^{-1/2}\left[\exp\left(\alpha s A^{-1/2}(B-aA)A^{-1/2}\right)\right] A^{1/2} ds. \tag{6.2}$$

For $\alpha = 1$ and $a = 0$, we consider the *exponential perspective* defined by

$$E(B,A) := E_{1,a}(B,A) = A^{1/2}\left[\exp\left(A^{-1/2}BA^{-1/2}\right)\right] A^{1/2}. \tag{6.3}$$

Since for $a = 0$ we have

$$\mathcal{P}_{f_{0,k}}(B,A) = \left(BA^{-1}\right)^k A,$$

then by (6.2) we get the following simple representation

$$E(B,A) = A + \sum_{k=1}^{n} \frac{1}{k!}\left(BA^{-1}\right)^k A + \frac{1}{n!}\left(BA^{-1}\right)^{n+1} \int_0^1 (1-s)^n A^{1/2}\left[\exp\left(sA^{-1/2}BA^{-1/2}\right)\right] A^{1/2} ds \tag{6.4}$$

for positive invertible operators A, B and $n \geq 1$.

If the positive invertible operators A, B satisfy the condition (2.10), then one can get various inequalities similar to the one from (4.3). The details are omitted.

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