Received 04/09/16

REPRESENTATION OF OPERATOR PERSPECTIVES FOR CONTINUOUSLY DIFFERENTIABLE FUNCTIONS IN TERMS OF WEIGHTED MEANS WITH APPLICATIONS

S. S. DRAGOMIR[1,2]

Abstract. In this paper we obtain some representations of operator perspectives for continuously differentiable functions in terms of weighted means. Applications for weighted operator geometric mean and relative operator entropy are also provided.

1. Introduction

Let f be a continuous function defined on the interval I of real numbers, B a selfadjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right)\subset \mathring{I}$, the interior of I. Then by using the *continuous functional calculus*, we can define the *perspective* $\mathcal{P}_f(B,A)$ by setting

$$\mathcal{P}_f(B,A):=A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

If A and B are commutative, then

$$\mathcal{P}_f(B,A)=Af\left(BA^{-1}\right)$$

provided $\operatorname{Sp}\left(BA^{-1}\right)\subset \mathring{I}$.

It is well known that (see [9] and [8] or [10]), if f is an *operator convex function* defined in the positive half-line, then the mapping

$$(B,A)\mapsto \mathcal{P}_f(B,A)$$

defined in pairs of positive definite operators, is operator convex.

In the recent paper [2] we established the following reverse inequality for the perspective $\mathcal{P}_f(B,A)$.

Let $f:[m,M]\to\mathbb{R}$ be a *convex function* on the real interval $[m,M]$, A a positive invertible operator and B a selfadjoint operator such that

$$mA\le B\le MA, \tag{1.1}$$

1991 *Mathematics Subject Classification.* 47A63, 47A30, 15A60, 26D15, 26D10.

Key words and phrases. Operator noncommutative perspectives, Young's inequality, Relative operator entropy, Arithmetic mean-Geometric mean inequality.

then we have

$$0 \leq \frac{1}{M-m}\left[f(m)(MA-B)+f(M)(B-mA)\right]-\mathcal{P}_f(B,A) \tag{1.2}$$
$$\leq \frac{f'_-(M)-f'_+(m)}{M-m}\left(MA^{1/2}-BA^{-1/2}\right)\left(A^{-1/2}B-mA^{1/2}\right)$$
$$\leq \frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]A.$$

Let $f : J \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{J}$, the interior of J. Suppose that there exists the constants d, D such that

$$d \leq f''(t) \leq D \text{ for any } t \in \mathring{J}. \tag{1.3}$$

If A is a positive invertible operator and B a selfadjoint operator such that the condition (1.1) is valid with $[m, M] \subset \mathring{J}$, then we have the following result as well [3]

$$\frac{1}{2}d\left(MA^{1/2}-BA^{-1/2}\right)\left(A^{-1/2}B-mA^{1/2}\right) \tag{1.4}$$
$$\leq \frac{1}{M-m}\left[f(m)(MA-B)+f(M)(B-mA)\right]-\mathcal{P}_f(B,A)$$
$$\leq \frac{1}{2}D\left(MA^{1/2}-BA^{-1/2}\right)\left(A^{-1/2}B-mA^{1/2}\right).$$

If $d > 0$, then the first inequality in (1.6) is better than the same inequality in (1.5).

In order to provide some new inequalities for the $\mathcal{P}_f(B,A)$ in terms of integral and discrete means, we recall some representation results for absolutely continuous functions as follow.

Let L be a *linear class* of real-valued functions, $g : E \to \mathbb{R}$ having the properties

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L$, i.e., if $f(t) = 1$, $t \in E$, then $f \in L$.

An *isotonic linear functional* $\Phi : L \to \mathbb{R}$ is a functional satisfying the properties:

(A1) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$;
(A2) If $f \in L$ and $f \geq 0$, then $\Phi(f) \geq 0$.

The mapping Φ is said to be *normalised* if

(A3) $\Phi(\mathbf{1}) = 1$.

Usual examples of isotonic linear functional that are normalised are the following ones

$$\Phi(f) := \frac{1}{\mu(X)}\int_X f(x)\,d\mu(x), \text{ if } \mu(X) < \infty$$

or

$$\Phi_w(f) := \frac{1}{\int_X w(x)\,d\mu(x)}\int_X w(x) f(x)\,d\mu(x),$$

where $w(x) \geq 0$, $\int_X w(x)\,d\mu(x) > 0$, X is a measurable space and μ is a positive measure on X.

In particular, for $\bar{x} := (x_1, \ldots, x_n)$, $\bar{w} := (w_1, \ldots, w_n) \in \mathbb{R}^n$ with $w_i \geq 0$, $W_n := \sum_{i=1}^n w_i > 0$ we have

$$\Phi(\bar{x}) := \frac{1}{n}\sum_{i=1}^n x_i \text{ and } \Phi_{\bar{w}}(\bar{x}) := \frac{1}{W_n}\sum_{i=1}^n w_i x_i,$$

are normalised isotonic linear functionals on $\mathbb{R}^n$.

In 2002, we obtained the following representation result for absolutely continuous functions.

Theorem 1 (Dragomir, 2002, [1])**.** *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ and define $e(t)=t$, $t\in[a,b]$, $g(t,x)=\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda$, $t\in[a,b]$ and $x\in[a,b]$. If $\Phi:L\to\mathbb{R}$ is a normalised linear functional on a linear class L of absolutely continuous functions defined on $[a,b]$ and $(x-e)g(\cdot,x)\in L$, then we have the representation*

$$f(x)=\Phi(f)+\Phi[(x-e)g(\cdot,x)], \tag{1.5}$$

for $x\in[a,b]$.

The following particular cases are of interest:

Corollary 1. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then we have the representation:*

$$\begin{aligned} f(x)=&\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)f(t)\,dt \\ &+\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)(x-t)\left(\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda\right)dt \end{aligned}\tag{1.6}$$

for any $x\in[a,b]$, where $w:[a,b]\to\mathbb{R}$ is a Lebesgue integrable function with $\int_a^b w(t)\,dt\neq 0$.

In particular, we have

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{b-a}\int_a^b (x-t)\left(\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda\right)dt \tag{1.7}$$

for each $x\in[a,b]$.

The proof is obvious by Theorem 1 applied for the normalised linear functionals

$$\Phi_w(f):=\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)f(t)\,dt,\ \Phi(f):=\frac{1}{b-a}\int_a^b f(t)\,dt$$

defined on

$$L:=\{f:[a,b]\to\mathbb{R},\ f \text{ is absolutely continuous on } [a,b]\}.$$

The following discrete case also holds.

Corollary 2. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then we have the representation:*

$$f(x)=\frac{1}{W_n}\sum_{i=1}^n w_i f(x_i)+\frac{1}{W_n}\sum_{i=1}^n w_i(x-x_i)\int_0^1 f'[(1-\lambda)x+\lambda x_i]\,d\lambda \tag{1.8}$$

for any $x\in[a,b]$, where $x_i\in[a,b]$, $w_i\in\mathbb{R}$, $i=\{1,\dots,n\}$ with $W_n:=\sum_{i=1}^n w_i\neq 0$.

In particular, we have

$$f(x)=\frac{1}{n}\sum_{i=1}^n f(x_i)+\frac{1}{n}\sum_{i=1}^n (x-x_i)\int_0^1 f'[(1-\lambda)x+\lambda x_i]\,d\lambda \tag{1.9}$$

for any $x\in[a,b]$.

It is obvious that, with the same argument, the above identities also hold for complex valued functions f, $w : [a,b] \to \mathbb{C}$.

Motivated by the above results, we obtain in this paper some representations of operator perspectives for continuously differentiable functions in terms of weighted means. Applications for weighted operator geometric mean and relative operator entropy are also provided.

2. Representation Results

We have the following perturbed identity:

Lemma 1. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$ and $w : [a,b] \to \mathbb{C}$ a Lebesgue integrable function with $\int_a^b w(t)\,dt \neq 0$. Then for any continuous function $\mu : [a,b] \to \mathbb{C}$ and $x \in [a,b]$ we have*

$$f(x) = \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) f(t)\,dt + \left(x - \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,tdt\right)\mu(x) + \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)(x-t)\left(\int_0^1 \left(f'[(1-\lambda)x + \lambda t] - \mu(x)\right)d\lambda\right)dt \tag{2.1}$$

and, in particular

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \mu\left(x - \frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^b (x-t)\left(\int_0^1 \left(f'[(1-\lambda)x + \lambda t] - \mu(x)\right)d\lambda\right)dt. \tag{2.2}$$

The proof follows by the equality (1.6) by calculating the integral

$$\int_a^b w(t)(x-t)\left(\int_0^1 \left(f'[(1-\lambda)x + \lambda t] - \mu(x)\right)d\lambda\right)dt$$

for $x \in [a,b]$.

The discrete case is as follows:

Lemma 2. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $x_i \in [a,b]$, $w_i \in \mathbb{R}$, $i = \{1,\dots,n\}$ with $W_n := \sum_{i=1}^n w_i \neq 0$. Then for any continuous function $\mu : [a,b] \to \mathbb{C}$ and $x \in [a,b]$ we have*

$$f(x) = \frac{1}{W_n}\sum_{i=1}^n w_i f(x_i) + \left(x - \frac{1}{W_n}\sum_{i=1}^n w_i x_i\right)\mu(x) + \frac{1}{W_n}\sum_{i=1}^n w_i (x-x_i)\int_0^1 \left(f'[(1-\lambda)x + \lambda x_i] - \mu(x)\right)d\lambda \tag{2.3}$$

and, in particular

$$f(x) = \frac{1}{n}\sum_{i=1}^n f(x_i) + \left(x - \frac{1}{n}\sum_{i=1}^n x_i\right)\mu(x) + \frac{1}{n}\sum_{i=1}^n (x-x_i)\int_0^1 \left(f'[(1-\lambda)x + \lambda x_i] - \mu(x)\right)d\lambda. \tag{2.4}$$

We have:

Theorem 2. *Let $f : I \to \mathbb{C}$ be a continuously differentiable function on $\mathring{I}$. If T is a selfadjoint operator such that the spectrum $\operatorname{Sp}(T) \subseteq [m, M] \subset \mathring{I}$, for some real numbers m, M with $m < M$ and for any continuous function $\mu : [m, M] \to \mathbb{C}$ we have*

$$\begin{aligned} f(T) &= \frac{1}{\int_m^M w(t)\,dt} \int_m^M w(t) f(t)\,dt 1_H \\ &+ \left(T - \frac{1}{\int_m^M w(t)\,dt} \int_m^M w(t)\,t dt 1_H\right) \mu(T) \\ &+ \frac{1}{\int_m^M w(t)\,dt} \\ &\times \int_m^M w(t)(T - t1_H) \left(\int_0^1 (f'[(1-s)T + st1_H] - \mu(T))\,ds\right) dt, \end{aligned} \tag{2.5}$$

where $w : [m, M] \to \mathbb{C}$ is a Lebesgue integrable function with $\int_m^M w(t)\,dt \neq 0$.

In particular, we have

$$\begin{aligned} f(T) = \frac{1}{M-m} \int_m^M f(t)\,dt 1_H + \left(T - \frac{m+M}{2} 1_H\right) \mu(T) \\ + \frac{1}{M-m} \int_m^M (T - t1_H) \left(\int_0^1 (f'[(1-s)T + st1_H] - \mu(T))\,ds\right) dt. \end{aligned} \tag{2.6}$$

Proof. If $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the operator T, then by the spectral representation theorem (SRT) [15, p. 263-p.266] we have

$$f(T) = \int_{m-0}^M f(\lambda)\,dE_\lambda := \lim_{\varepsilon\to 0+} \int_{m-\varepsilon}^M f(\lambda)\,dE_\lambda,$$

where the integral is taken in the Riemann-Stieltjes sense.

Let $\varepsilon > 0$ small enough such that $[m - \varepsilon, M] \subset \mathring{I}$, then by integrating the equality (2.1) written for $a = m$ and $b = M$ on the interval $[m - \varepsilon, M]$ we have

$$\begin{aligned} &\int_{m-\varepsilon}^M f(\lambda)\,dE_\lambda \\ &= \frac{1}{\int_m^M w(t)\,dt} \int_m^M w(t) f(t)\,dt \int_{m-\varepsilon}^M dE_\lambda \\ &+ \left(\int_{m-\varepsilon}^M \lambda dE_\lambda - \frac{1}{\int_m^M w(t)\,dt} \int_m^M w(t)\,t dt \int_{m-\varepsilon}^M dE_\lambda\right) \int_{m-\varepsilon}^M \mu(\lambda)\,dE_\lambda \\ &+ \frac{1}{\int_m^M w(t)\,dt} \int_{m-\varepsilon}^M \\ &\times \left(\int_m^M w(t)(\lambda - t) \left(\int_0^1 (f'[(1-s)\lambda + st] - \mu(\lambda))\,ds\right) dt\right) dE_\lambda. \end{aligned} \tag{2.7}$$

Using Fubini's theorem, we can interchange the integration and then we have

$$\int_{m-\varepsilon}^{M}\left(\int_{m}^{M} w(t)(\lambda-t)\left(\int_{0}^{1}\left(f'[(1-s)\lambda+st]-\mu(\lambda)\right)ds\right)dt\right)dE_{\lambda}$$
$$=\int_{m}^{M} w(t)\left(\int_{0}^{1}\left(\int_{m-\varepsilon}^{M}(\lambda-t)\left(f'[(1-s)\lambda+st]-\mu(\lambda)\right)dE_{\lambda}\right)ds\right)dt,$$

which gives, by (2.7) that

$$\begin{aligned}(2.8)\quad &\int_{m-\varepsilon}^{M} f(\lambda)\,dE_{\lambda}\\ &=\frac{1}{\int_{m}^{M} w(t)\,dt}\int_{m}^{M} w(t)\,f(t)\,dt\int_{m-\varepsilon}^{M} dE_{\lambda}\\ &+\left(\int_{m-\varepsilon}^{M}\lambda dE_{\lambda}-\frac{1}{\int_{m}^{M} w(t)\,dt}\int_{m}^{M} w(t)\,tdt\int_{m-\varepsilon}^{M} dE_{\lambda}\right)\int_{m-\varepsilon}^{M}\mu(\lambda)\,dE_{\lambda}\\ &+\frac{1}{\int_{m}^{M} w(t)\,dt}\\ &\times\int_{m}^{M} w(t)\left(\int_{0}^{1}\left(\int_{m-\varepsilon}^{M}(\lambda-t)\left(f'[(1-s)\lambda+st]-\mu(\lambda)\right)dE_{\lambda}\right)ds\right)dt\end{aligned}$$

for small $\varepsilon>0$ such that $[m-\varepsilon,M]\subset\mathring{I}$.

Since, by SRT we have

$$\lim_{\varepsilon\to 0+}\int_{m-\varepsilon}^{M} f(\lambda)\,dE_{\lambda}=f(T),\quad \lim_{\varepsilon\to 0+}\int_{m-\varepsilon}^{M} dE_{\lambda}=1_{H},$$
$$\lim_{\varepsilon\to 0+}\int_{m-\varepsilon}^{M}\lambda dE_{\lambda}=T$$

and

$$\begin{aligned}(2.9)\qquad &\int_{m-\varepsilon}^{M}(\lambda-t)\left(f'[(1-s)\lambda+st]-\mu(\lambda)\right)dE_{\lambda}\\ &=(T-t1_{H})\left(f'[(1-s)T+st1_{H}]-\mu(T)\right)\end{aligned}$$

then by taking the limit over $\varepsilon\to 0+$ in (2.8) and using the properties of integrals, we obtain the desired result (2.5). □

We also have:

Theorem 3. *Let $f:I\to\mathbb{C}$ be a continuously differentiable function on $\mathring{I}$ the interior of I. If T is a selfadjoint operator such that the spectrum $\operatorname{Sp}(T)\subseteq[m,M]\subset\mathring{I}$, for some real numbers m, M with $m<M$ and $x_i\in[m,M]$, $w_i\in\mathbb{R}$, $i=\{1,\dots,n\}$ with $W_n:=\sum_{i=1}^{n}w_i\neq 0$ and for any continuous function $\mu:[m,M]\to\mathbb{C}$ we have*

$$\begin{aligned}(2.10)\qquad f(T)=&\frac{1}{W_n}\sum_{i=1}^{n}w_i f(x_i)\,1_H+\left(T-\frac{1}{W_n}\sum_{i=1}^{n}w_i x_i 1_H\right)\mu(T)\\ &+\frac{1}{W_n}\sum_{i=1}^{n}w_i(T-x_i1_H)\int_{0}^{1}\left(f'[(1-\lambda)T+\lambda x_i1_H]-\mu(T)\right)d\lambda\end{aligned}$$

and, in particular

$$f(T)=\frac{1}{n}\sum_{i=1}^{n}f(x_i)1_H+\left(T-\frac{1}{n}\sum_{i=1}^{n}x_i1_H\right)\mu(T)+\frac{1}{n}\sum_{i=1}^{n}(T-x_i1_H)\int_0^1\left(f'\left[(1-\lambda)T+\lambda x_i1_H\right]-\mu(T)\right)d\lambda. \tag{2.11}$$

We have:

Theorem 4. *Let $f:I\to\mathbb{C}$ be a continuously differentiable function on $\mathring{I}$. Assume that A is a positive invertible operator and B a selfadjoint operator such that the condition (1.1) holds for some real numbers $m<M$ with the property that $[m,M]\subset\mathring{I}$. Then for any continuous function $\mu:[m,M]\to\mathbb{C}$ and any $w:[m,M]\to\mathbb{C}$ a Lebesgue integrable function with $\int_m^M w(t)\,dt\neq 0$, we have*

$$\begin{aligned}\mathcal{P}_f(B,A)&=\frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)f(t)\,dtA\\&+\left(BA^{-1}-\frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\,tdt1_H\right)\mathcal{P}_\mu(B,A)\\&+\frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\left(BA^{-1}-t1_H\right)\\&\times\left(\int_0^1\left[\mathcal{P}_{f'}(B\nabla_s(tA),A)-\mathcal{P}_\mu(B,A)\right]ds\right)dt.\end{aligned} \tag{2.12}$$

If $x_i\in[m,M]$ and $w_i\in\mathbb{R}$, $i=\{1,\dots,n\}$ with $W_n:=\sum_{i=1}^n w_i\neq 0$, then we also have

$$\begin{aligned}\mathcal{P}_f(B,A)&=\frac{1}{W_n}\sum_{i=1}^{n}w_if(x_i)A\\&+\left(BA^{-1}-\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i1_H\right)\mathcal{P}_\mu(B,A)\\&+\frac{1}{W_n}\sum_{i=1}^{n}w_i\left(BA^{-1}-x_i1_H\right)\\&\times\int_0^1\left[\mathcal{P}_{f'}(B\nabla_s(x_iA),A)-\mathcal{P}_\mu(B,A)\right]ds.\end{aligned} \tag{2.13}$$

Proof. If we take $T = A^{-1/2}BA^{-1/2}$, then $\operatorname{Sp}(T) \subseteq [m, M] \subset \mathring{I}$ and by (2.1) we have

$$f\left(A^{-1/2}BA^{-1/2}\right) = \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t) f(t)\,dt 1_H + \left(A^{-1/2}BA^{-1/2} - \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\,tdt1_H\right)\mu\left(A^{-1/2}BA^{-1/2}\right) + \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\left(A^{-1/2}BA^{-1/2} - t1_H\right) \times \left(\int_0^1 \left(f'\left[(1-s)A^{-1/2}BA^{-1/2} + st1_H\right] - \mu\left(A^{-1/2}BA^{-1/2}\right)\right)ds\right)dt. \tag{2.14}$$

If we multiply both sides of (2.14) by $A^{1/2}$ then we get

$$A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} = \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t) f(t)\,dtA + A^{1/2}\left(A^{-1/2}BA^{-1/2} - \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\,tdt1_H\right) \times \mu\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} + \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\,A^{1/2}\left(A^{-1/2}BA^{-1/2} - t1_H\right) \times \int_0^1 \left(f'\left[(1-s)A^{-1/2}BA^{-1/2} + st1_H\right] - \mu\left(A^{-1/2}BA^{-1/2}\right)\right)dsA^{1/2}dt. \tag{2.15}$$

Since

$$\begin{aligned}
&A^{1/2}\left(A^{-1/2}BA^{-1/2} - \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\,tdt1_H\right)\mu\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} \\
&= A^{1/2}\left(A^{-1/2}BA^{-1/2} - \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\,tdt1_H\right)A^{-1/2} \\
&\times A^{1/2}\mu\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} \\
&= \left(BA^{-1} - \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\,tdt1_H\right)A^{1/2}\mu\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} \\
&= \left(BA^{-1} - \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\,tdt1_H\right)\mathcal{P}_\mu(B, A)
\end{aligned}$$

and

$$\begin{aligned}
&\int_m^M w(t)\, A^{1/2}\left(A^{-1/2}BA^{-1/2} - t1_H\right) \\
&\times \left(\int_0^1 \left(f'\left[(1-s)\, A^{-1/2}BA^{-1/2} + st1_H\right] - \mu\left(A^{-1/2}BA^{-1/2}\right)\right) ds\right) A^{1/2} dt \\
&= \int_m^M w(t)\, A^{1/2}\left(A^{-1/2}BA^{-1/2} - t1_H\right) A^{-1/2} \\
&\times A^{1/2}\left(\int_0^1 \left(f'\left[A^{-1/2}\left((1-s)\, B + stA\right) A^{-1/2}\right] - \mu\left(A^{-1/2}BA^{-1/2}\right)\right) ds\right) A^{1/2} dt \\
&= \int_m^M w(t)\, A^{1/2}\left(A^{-1/2}BA^{-1/2} - t1_H\right) A^{-1/2} \\
&\times A^{1/2}\left(\int_0^1 \left(f'\left[A^{-1/2}\left(B\nabla_s(tA)\right) A^{-1/2}\right] - \mu\left(A^{-1/2}BA^{-1/2}\right)\right) ds\right) A^{1/2} dt \\
&= \int_m^M w(t)\left(BA^{-1} - t1_H\right)\left(\int_0^1 \left[\mathcal{P}_{f'}\left(B\nabla_s(tA), A\right) - \mathcal{P}_\mu(B, A)\right] ds\right) dt,
\end{aligned}$$

then by (2.15) we obtain the desired result (2.12). □

If we take in (2.12) $w(t) = 1$, $t \in [m, M]$, then we get the simpler representation

$$\begin{aligned}
\mathcal{P}_f(B, A) &= \frac{1}{M-m}\int_m^M f(t)\, dtA + \left(BA^{-1} - \frac{m+M}{2}1_H\right)\mathcal{P}_\mu(B, A) \\
&+ \frac{1}{M-m}\int_m^M \left(BA^{-1} - t1_H\right) \\
&\times \left(\int_0^1 \left[\mathcal{P}_{f'}\left(B\nabla_s(tA), A\right) - \mathcal{P}_\mu(B, A)\right] ds\right) dt,
\end{aligned} \tag{2.16}$$

for any continuous function $\mu : [m, M] \to \mathbb{C}$.

The unweighted discrete case that follows by (2.13) produces the representation

$$\begin{aligned}
\mathcal{P}_f(B, A) &= \frac{1}{n}\sum_{i=1}^n f(x_i)\, A + \left(BA^{-1} - \frac{1}{n}\sum_{i=1}^n x_i 1_H\right)\mathcal{P}_\mu(B, A) \\
&+ \frac{1}{n}\sum_{i=1}^n \left(BA^{-1} - x_i 1_H\right) \\
&\times \int_0^1 \left[\mathcal{P}_{f'}\left(B\nabla_s(x_iA), A\right) - \mathcal{P}_\mu(B, A)\right] ds,
\end{aligned} \tag{2.17}$$

for any continuous function $\mu : [m, M] \to \mathbb{C}$.

Remark 1. *If we assume that μ is a constant, then from (2.12) and (2.13) we get*

$$
\begin{aligned}
\mathcal{P}_f(B,A) &= \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t) f(t)\,dtA \\
&\quad + \mu\left(B - \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\,tdtA\right) \\
&\quad + \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\left(BA^{-1} - t1_H\right) \\
&\quad \times \left(\int_0^1 \left[\mathcal{P}_{f'}\left(B\nabla_s(tA), A\right) - \mu A\right] ds\right) dt
\end{aligned} \tag{2.18}
$$

and

$$
\begin{aligned}
\mathcal{P}_f(B,A) &= \frac{1}{W_n}\sum_{i=1}^n w_i f(x_i) A + \mu\left(B - \frac{1}{W_n}\sum_{i=1}^n w_i x_i A\right) \\
&\quad + \frac{1}{W_n}\sum_{i=1}^n w_i \left(BA^{-1} - x_i 1_H\right) \\
&\quad \times \int_0^1 \left[\mathcal{P}_{f'}\left(B\nabla_s(x_i A), A\right) - \mu A\right] ds,
\end{aligned} \tag{2.19}
$$

which in the particular case $\mu = 0$ reduce to the simpler representations

$$
\begin{aligned}
\mathcal{P}_f(B,A) &= \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t) f(t)\,dtA \\
&\quad + \frac{1}{\int_m^M w(t)\,dt} \\
&\quad \times \int_m^M w(t)\left(BA^{-1} - t1_H\right)\left(\int_0^1 \mathcal{P}_{f'}\left(B\nabla_s(tA), A\right) ds\right) dt
\end{aligned} \tag{2.20}
$$

and

$$
\begin{aligned}
\mathcal{P}_f(B,A) &= \frac{1}{W_n}\sum_{i=1}^n w_i f(x_i) A \\
&\quad + \frac{1}{W_n}\sum_{i=1}^n w_i \left(BA^{-1} - x_i 1_H\right)\int_0^1 \mathcal{P}_{f'}\left(B\nabla_s(x_i A), A\right) ds.
\end{aligned} \tag{2.21}
$$

From (2.20) we have for $w(t) = 1$ that

$$
\begin{aligned}
\mathcal{P}_f(B,A) &= \frac{1}{M-m}\int_m^M f(t)\,dtA \\
&\quad + \frac{1}{M-m}\int_m^M \left(BA^{-1} - t1_H\right)\left(\int_0^1 \mathcal{P}_{f'}\left(B\nabla_s(tA), A\right) ds\right) dt
\end{aligned} \tag{2.22}
$$

while from (2.21) we get for $w_i = 1$, $i \in \{1, ..., n\}$ that

$$\mathcal{P}_f(B,A) = \frac{1}{n}\sum_{i=1}^{n} f(x_i)A + \frac{1}{n}\sum_{i=1}^{n}\left(BA^{-1} - x_i 1_H\right)\int_0^1 \mathcal{P}_{f'}(B\nabla_s(x_iA), A)\,ds. \tag{2.23}$$

The following particular case is also natural to consider.

Remark 2. *Is useful to provide error bounds in the case that $\mu = f'$ with f' continuous on $[m, M]$. In this situation we get the identities of interest*

$$\begin{aligned}\mathcal{P}_f(B,A) &= \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)f(t)\,dtA \\ &+ \left(BA^{-1} - \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\,tdt1_H\right)\mathcal{P}_{f'}(B,A) \\ &+ \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\left(BA^{-1} - t1_H\right) \\ &\times \left(\int_0^1 \left[\mathcal{P}_{f'}(B\nabla_s(tA), A) - \mathcal{P}_{f'}(B,A)\right]ds\right)dt\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}\mathcal{P}_f(B,A) &= \frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i)A \\ &+ \left(BA^{-1} - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i 1_H\right)\mathcal{P}_{f'}(B,A) \\ &+ \frac{1}{W_n}\sum_{i=1}^{n} w_i\left(BA^{-1} - x_i 1_H\right) \\ &\times \int_0^1 \left[\mathcal{P}_{f'}(B\nabla_s(x_iA), A) - \mathcal{P}_{f'}(B,A)\right]ds.\end{aligned} \tag{2.25}$$

Moreover, the unweighted case for integrals can be stated as

$$\begin{aligned}\mathcal{P}_f(B,A) &= \frac{1}{M-m}\int_m^M f(t)\,dtA + \left(BA^{-1} - \frac{m+M}{2}1_H\right)\mathcal{P}_{f'}(B,A) \\ &+ \frac{1}{M-m}\int_m^M \left(BA^{-1} - t1_H\right) \\ &\times \left(\int_0^1 \left[\mathcal{P}_{f'}(B\nabla_s(tA), A) - \mathcal{P}_{f'}(B,A)\right]ds\right)dt,\end{aligned} \tag{2.26}$$

while the unweighted discrete case is as follows

$$\begin{aligned}(2.27)\qquad \mathcal{P}_f(B,A) &= \frac{1}{n}\sum_{i=1}^{n} f(x_i)\,A + \left(BA^{-1} - \frac{1}{n}\sum_{i=1}^{n} x_i 1_H\right)\mathcal{P}_{f'}(B,A)\\ &\quad + \frac{1}{n}\sum_{i=1}^{n} w_i\left(BA^{-1} - x_i 1_H\right)\\ &\quad \times \int_0^1 \left[\mathcal{P}_{f'}(B\nabla_s(x_i A),A) - \mathcal{P}_{f'}(B,A)\right] ds.\end{aligned}$$

3. Inequalities for Bounded Derivatives

Now, for ϕ, $\Phi \in \mathbb{C}$ and I an interval of real numbers, define the sets of complex-valued functions (see for instance [7])

$$\begin{aligned}&\bar{U}_I(\phi,\Phi)\\ &:= \left\{g : I \to \mathbb{C} \mid \operatorname{Re}\left[(\Phi - g(t))\left(\overline{g(t)} - \overline{\phi}\right)\right] \geq 0 \text{ for almost every } t \in I\right\}\end{aligned}$$

and

$$\bar{\Delta}_I(\phi,\Phi) := \left\{g : I \to \mathbb{C} \mid \left|g(t) - \frac{\phi+\Phi}{2}\right| \leq \frac{1}{2}\left|\Phi - \phi\right| \text{ for a.e. } t \in I\right\}.$$

The following representation result may be stated [7].

Proposition 1. *For any ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that $\bar{U}_I(\phi,\Phi)$ and $\bar{\Delta}_I(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$(3.1)\qquad \bar{U}_I(\phi,\Phi) = \bar{\Delta}_I(\phi,\Phi).$$

On making use of the complex numbers field properties we can also state that:

Corollary 3. *For any ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that*

$$\begin{aligned}(3.2)\qquad \bar{U}_I(\phi,\Phi) = \{g : I \to \mathbb{C} \mid\ &(\operatorname{Re}\Phi - \operatorname{Re} g(t))(\operatorname{Re} g(t) - \operatorname{Re}\phi)\\ &+ (\operatorname{Im}\Phi - \operatorname{Im} g(t))(\operatorname{Im} g(t) - \operatorname{Im}\phi) \geq 0 \textit{ for a.e. } t \in I\}.\end{aligned}$$

Now, if we assume that $\operatorname{Re}(\Phi) \geq \operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi) \geq \operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned}(3.3)\qquad \bar{S}_I(\phi,\Phi) := \{g : I \to \mathbb{C} \mid\ &\operatorname{Re}(\Phi) \geq \operatorname{Re} g(t) \geq \operatorname{Re}(\phi)\\ &\text{and } \operatorname{Im}(\Phi) \geq \operatorname{Im} g(t) \geq \operatorname{Im}(\phi) \text{ for a.e. } t \in I\}.\end{aligned}$$

One can easily observe that $\bar{S}_I(\phi,\Phi)$ is closed, convex and

$$(3.4)\qquad \emptyset \neq \bar{S}_I(\phi,\Phi) \subseteq \bar{U}_I(\phi,\Phi).$$

We need the following lemma [6]:

Lemma 3. *Let T be a selfadjoint operator and $A \geq 0$. Then we have*

$$(3.5)\qquad -A^{1/2}\left|T\right|A^{1/2} \leq A^{1/2}TA^{1/2} \leq A^{1/2}\left|T\right|A^{1/2}$$

in the operator order, where $|T|$ is the absolute value of T.

We also have

$$(3.6)\qquad \left\|A^{1/2}TA^{1/2}\right\| \leq \left\|A^{1/2}\left|T\right|A^{1/2}\right\|.$$

Proof. For the sake of completeness we give here a short proof.

If we use Jensen's operator inequality for the convex function $f(t) = |t|$, then we have

$$|\langle Ty, y\rangle| \leq \langle |T| y, y\rangle$$

for any $y \in H$.

If we take in this inequality $y = A^{1/2}x$, $x \in H$, then we get

$$\left|\left\langle TA^{1/2}x, A^{1/2}x\right\rangle\right| \leq \left\langle |T| A^{1/2}x, A^{1/2}x\right\rangle$$

that is equivalent to

$$\left|\left\langle A^{1/2}TA^{1/2}x, x\right\rangle\right| \leq \left\langle A^{1/2} |T| A^{1/2}x, x\right\rangle \tag{3.7}$$

or to

$$-\left\langle A^{1/2} |T| A^{1/2}x, x\right\rangle \leq \left\langle A^{1/2}TA^{1/2}x, x\right\rangle \leq \left\langle A^{1/2} |T| A^{1/2}x, x\right\rangle$$

for any $x \in H$, which proves the inequality (3.5).

By taking the supremum over $x \in H$, $\|x\| = 1$ in (3.7) we obtain the desired inequality (3.6). □

Theorem 5. *Assume that A is a positive invertible operator and B a selfadjoint operator such that the condition (1.1) holds for some real numbers $m < M$ with the property that $[m, M] \subset \mathring{I}$ and $f : I \to \mathbb{C}$ is a continuously differentiable function on $\mathring{I}$ and such that $f' \in \bar{\Delta}_{[m,M]}(\phi, \Phi)$ for some ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$. Then for any $w : [m, M] \to \mathbb{C}$ a Lebesgue integrable function with $\int_m^M w(t)\,dt \neq 0$, we have*

$$\begin{aligned}\Bigg\| \mathcal{P}_f(B, A) &- \frac{1}{\int_m^M w(t)\,dt} \int_m^M w(t) f(t)\,dtA \\ &- \frac{\phi + \Phi}{2}\left(B - \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\,tdtA\right)\Bigg\| \\ &\leq \frac{1}{2}|\Phi - \phi| \|A\| \frac{1}{\left|\int_m^M w(t)\,dt\right|}\int_m^M |w(t)| \left\|BA^{-1} - t1_H\right\| dt.\end{aligned} \tag{3.8}$$

If $x_i \in [m, M]$ and $w_i \in \mathbb{C}$, $i = \{1, \ldots, n\}$ with $W_n := \sum_{i=1}^n w_i \neq 0$, then we also have

$$\begin{aligned}\left\| \mathcal{P}_f(B, A) - \frac{1}{W_n}\sum_{i=1}^n w_i f(x_i) A - \frac{\phi + \Phi}{2}\left(B - \frac{1}{W_n}\sum_{i=1}^n w_i x_i A\right)\right\| \\ \leq \frac{1}{2}|\Phi - \phi| \|A\| \frac{1}{|W_n|}\sum_{i=1}^n |w_i| \left\|BA^{-1} - x_i 1_H\right\|.\end{aligned} \tag{3.9}$$

Proof. Assume that f is such that $f' \in \bar{\Delta}_{\mathring{I}}(\phi, \Phi)$. If we use the identity (2.18) for the constant $\frac{\phi+\Phi}{2}$, then we have

$$\begin{aligned}
&\mathcal{P}_f(B, A) - \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t) f(t)\,dtA \\
&- \frac{\phi+\Phi}{2}\left(B - \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\,tdtA\right) \\
&= \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\left(BA^{-1} - t1_H\right)\int_0^1 \left(\mathcal{P}_{f'}(B\nabla_s(tA), A) - \frac{\phi+\Phi}{2}A\right) dsdt.
\end{aligned}$$

By taking the operator norm in this inequality and using the properties of integral and norm, we have

$$\begin{aligned}
&\left\|\mathcal{P}_f(B, A) - \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t) f(t)\,dtA\right. \\
&\left.-\frac{\phi+\Phi}{2}\left(B - \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\,tdtA\right)\right\| \\
&\leq \frac{1}{\left|\int_m^M w(t)\,dt\right|}\int_m^M |w(t)|\left\|BA^{-1} - t1_H\right\| \\
&\times\left(\int_0^1 \left\|\mathcal{P}_{f'}(B\nabla_s(tA), A) - \frac{\phi+\Phi}{2}A\right\| ds\right) dt.
\end{aligned} \tag{3.10}$$

Observe that

$$\operatorname{Sp}\left(A^{-1/2}(B\nabla_s(tA))A^{-1/2}\right) = \operatorname{Sp}\left((1-s)A^{-1/2}BA^{-1/2} + st1_H\right) \subset [m, M]$$

for any $t \in [m, M]$ and $s \in [0, 1]$.

Since $f' \in \bar{\Delta}_{[m,M]}(\phi, \Phi)$, then for any $u \in [m, M]$ we have

$$\left|f'(u) - \frac{\phi+\Phi}{2}\right| \leq \frac{1}{2}|\Phi - \phi| \tag{3.11}$$

and by the continuous functional calculus we get from (3.11) that

$$\left|f'\left(A^{-1/2}(B\nabla_s(tA))A^{-1/2}\right) - \frac{\phi+\Phi}{2}1_H\right| \leq \frac{1}{2}|\Phi - \phi|\, 1_H \tag{3.12}$$

for any $t \in [m, M]$ and $s \in [0, 1]$.

Now, multiplying both sides of (3.12) by $A^{1/2}$, we get

$$A^{1/2}\left|f'\left(A^{-1/2}(B\nabla_s(tA))A^{-1/2}\right) - \frac{\phi+\Phi}{2}1_H\right|A^{1/2} \leq \frac{1}{2}|\Phi - \phi|\, A$$

and by taking the norm in this inequality, we get

$$\left\|A^{1/2}\left|f'\left(A^{-1/2}(B\nabla_s(tA))A^{-1/2}\right) - \frac{\phi+\Phi}{2}1_H\right|A^{1/2}\right\| \leq \frac{1}{2}|\Phi - \phi|\,\|A\| \tag{3.13}$$

for any $t \in [m, M]$ and $s \in [0, 1]$.

Using Lemma 3 we get

$$\begin{aligned}
&\left\|\mathcal{P}_{f'}\left(B\nabla_s\left(tA\right),A\right)-\frac{\phi+\Phi}{2}A\right\| \\
&=\left\|A^{1/2}\left(f'\left(A^{-1/2}\left(B\nabla_s\left(tA\right)\right)A^{-1/2}\right)-\frac{\phi+\Phi}{2}1_H\right)A^{1/2}\right\| \\
&\leq\left\|A^{1/2}\left|f'\left(A^{-1/2}\left(B\nabla_s\left(tA\right)\right)A^{-1/2}\right)-\frac{\phi+\Phi}{2}1_H\right|A^{1/2}\right\|\leq\frac{1}{2}\left|\Phi-\phi\right|\left\|A\right\|
\end{aligned}$$

for any $t\in[m,M]$ and $s\in[0,1]$.

Therefore

$$\begin{aligned}
&\frac{1}{\left|\int_m^M w\left(t\right)dt\right|}\int_m^M\left|w\left(t\right)\right|\left\|BA^{-1}-t1_H\right\| \\
&\times\left(\int_0^1\left\|\mathcal{P}_{f'}\left(B\nabla_s\left(tA\right),A\right)-\frac{\phi+\Phi}{2}A\right\|ds\right)dt \\
&\leq\frac{1}{2}\left|\Phi-\phi\right|\left\|A\right\|\frac{1}{\left|\int_m^M w\left(t\right)dt\right|}\int_m^M\left|w\left(t\right)\right|\left\|BA^{-1}-t1_H\right\|dt
\end{aligned}$$

and by (3.10) we deduce the desired result (3.8).

The discrete inequality (3.9) follows in a similar way and we omit the details. □

We observe that if $f:[m,M]\to\mathbb{R}$ is a *convex function* and if $f'_+(m)$ and $f'_+(M)$ are finite, then from the above inequalities we can state the following inequalities that provide a large number of examples:

$$\begin{aligned}
&\left\|\mathcal{P}_f\left(B,A\right)-\frac{1}{\int_m^M w\left(t\right)dt}\int_m^M w\left(t\right)f\left(t\right)dtA\right. \\
&\left.-\frac{f'_+\left(m\right)+f'_+\left(M\right)}{2}\left(B-\frac{1}{\int_m^M w\left(t\right)dt}\int_m^M w\left(t\right)tdtA\right)\right\| \\
&\leq\frac{1}{2}\left[f'_+\left(M\right)-f'_+\left(m\right)\right]\left\|A\right\|\frac{1}{\left|\int_m^M w\left(t\right)dt\right|}\int_m^M\left|w\left(t\right)\right|\left\|BA^{-1}-t1_H\right\|dt.
\end{aligned}\tag{3.14}$$

If $x_i\in[m,M]$ and $w_i\in\mathbb{C}$, $i=\{1,\ldots,n\}$ with $W_n:=\sum_{i=1}^n w_i\neq0$, then we also have

$$\begin{aligned}
&\left\|\mathcal{P}_f\left(B,A\right)-\frac{1}{W_n}\sum_{i=1}^n w_if\left(x_i\right)A\right. \\
&\left.-\frac{f'_+\left(m\right)+f'_+\left(M\right)}{2}\left(B-\frac{1}{W_n}\sum_{i=1}^n w_ix_iA\right)\right\| \\
&\leq\frac{1}{2}\left[f'_+\left(M\right)-f'_+\left(m\right)\right]\left\|A\right\|\frac{1}{\left|W_n\right|}\sum_{i=1}^n\left|w_i\right|\left\|BA^{-1}-x_i1_H\right\|.
\end{aligned}\tag{3.15}$$

If we take in (3.14) $w(t) = 1$, then we get

$$\left\| \mathcal{P}_f(B,A) - \frac{1}{M-m}\int_m^M f(t)\,dtA - \frac{f'_+(m)+f'_+(M)}{2}\left(B - \frac{m+M}{2}A\right)\right\| \le \frac{1}{2}\left[f'_+(M) - f'_+(m)\right]\|A\|\frac{1}{M-m}\int_m^M \left\|BA^{-1} - t1_H\right\| dt. \tag{3.16}$$

From (3.16) we get for $w_i = 1$, $i = \{1,\dots,n\}$ that

$$\left\| \mathcal{P}_f(B,A) - \frac{1}{n}\sum_{i=1}^n f(x_i)A - \frac{f'_+(m)+f'_+(M)}{2}\left(B - \frac{1}{n}\sum_{i=1}^n x_iA\right)\right\| \le \frac{1}{2}\left[f'_+(M) - f'_+(m)\right]\|A\|\frac{1}{n}\sum_{i=1}^n \left\|BA^{-1} - x_i1_H\right\|. \tag{3.17}$$

4. Inequalities for Lipschitzian Derivatives

We have:

Theorem 6. *Assume that A is a positive invertible operator and B a selfadjoint operator such that the condition (1.1) holds for some real numbers $m < M$ with the property that $[m,M] \subset \mathring{I}$ and $f : I \to \mathbb{C}$ is a continuously differentiable function on $\mathring{I}$ and such that f' is Lipschitzian on $[m,M]$ with the constant $L > 0$, i.e.*

$$|f'(t) - f'(s)| \le L|t-s| \tag{4.1}$$

for any t, $s \in [m,M]$. Then for any $w : [m,M] \to \mathbb{C}$ a Lebesgue integrable function with $\int_m^M w(t)\,dt \ne 0$, we have

$$\left\| \mathcal{P}_f(B,A) - \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)f(t)\,dtA - \left(BA^{-1} - \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\,tdt1_H\right)\mathcal{P}_{f'}(B,A)\right\| \le \frac{1}{2}\frac{L\|A\|^{3/2}\left\|A^{-1}\right\|^{1/2}}{\left|\int_m^M w(t)\,dt\right|}\int_m^M |w(t)|\left\|BA^{-1} - t1_H\right\|^2 dt. \tag{4.2}$$

If $x_i \in [m,M]$ and $w_i \in \mathbb{C}$, $i = \{1,\dots,n\}$ with $W_n := \sum_{i=1}^n w_i \ne 0$, then we also have

$$\left\| \mathcal{P}_f(B,A) - \frac{1}{W_n}\sum_{i=1}^n w_if(x_i)A - \left(BA^{-1} - \frac{1}{W_n}\sum_{i=1}^n w_ix_i1_H\right)\mathcal{P}_{f'}(B,A)\right\| \le \frac{1}{2}\frac{L\|A\|^{3/2}\left\|A^{-1}\right\|^{1/2}}{|W_n|}\sum_{i=1}^n |w_i|\left\|BA^{-1} - x_i1_H\right\|^2. \tag{4.3}$$

Proof. We use the identity (2.24) in the form

$$\begin{aligned}
&\mathcal{P}_f(B,A)-\frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\,f(t)\,dtA \\
&-\left(BA^{-1}-\frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\,tdt1_H\right)\mathcal{P}_{f'}(B,A) \\
&=\frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\left(BA^{-1}-t1_H\right) \\
&\times\left(\int_0^1\left[\mathcal{P}_{f'}(B\nabla_s(tA),A)-\mathcal{P}_{f'}(B,A)\right]ds\right)dt.
\end{aligned}\tag{4.4}$$

By taking the norm in this equality and using the properties of the integral we have

$$\begin{aligned}
&\left\|\mathcal{P}_f(B,A)-\frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\,f(t)\,dtA\right. \\
&\left.-\left(BA^{-1}-\frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\,tdt1_H\right)\mathcal{P}_{f'}(B,A)\right\| \\
&\leq\frac{1}{\left|\int_m^M w(t)\,dt\right|}\int_m^M |w(t)|\left\|BA^{-1}-t1_H\right\| \\
&\times\left(\int_0^1\left\|\mathcal{P}_{f'}(B\nabla_s(tA),A)-\mathcal{P}_{f'}(B,A)\right\|ds\right)dt.
\end{aligned}\tag{4.5}$$

By the fact that f' is Lipschitzian we have

$$|f'[(1-s)x+st1_H]-f'(x)|\leq Ls|x-t|$$

for any $x,\, t\in[m,M]$ and $s\in[0,1]$.

Using the continuous functional calculus for the selfadjoint operator X with $\operatorname{Sp}(T)\subseteq[m,M]$ we have

$$|f'[(1-s)X+st1_H]-f'(X)|\leq Ls|X-t1_H|$$

for any $t\in[m,M]$ and $s\in[0,1]$.

By taking in this inequality $X=A^{-1/2}BA^{-1/2}$ we get

$$\left|f'\left[(1-s)A^{-1/2}BA^{-1/2}+st1_H\right]-f'\left(A^{-1/2}BA^{-1/2}\right)\right|\leq Ls\left|A^{-1/2}BA^{-1/2}-t1_H\right|$$

for any $t\in[m,M]$ and $s\in[0,1]$.

If we multiply both sides of this inequality by $A^{1/2}$ we get

$$\begin{aligned}
&A^{1/2}\left|f'\left[(1-s)A^{-1/2}BA^{-1/2}+st1_H\right]-f'\left(A^{-1/2}BA^{-1/2}\right)\right|A^{1/2} \\
&\leq LsA^{1/2}\left|A^{-1/2}BA^{-1/2}-t1_H\right|A^{1/2},
\end{aligned}$$

for any $t\in[m,M]$ and $s\in[0,1]$.

If we take the norm in this inequality, we get

$$\begin{aligned}
(4.6)\quad & \left\| A^{1/2}\left| f'\left[(1-s)\, A^{-1/2}BA^{-1/2} + st1_H \right] - f'\left(A^{-1/2}BA^{-1/2} \right) \right| A^{1/2} \right\| \\
& \leq Ls \left\| A^{1/2}\left| A^{-1/2}BA^{-1/2} - t1_H \right| A^{1/2} \right\| \\
& = Ls \left\| A^{1/2}\left| A^{-1/2}BA^{-1/2} - t1_H \right| A^{1/2} \right\| \\
& = Ls \left\| A^{1/2}\left| A^{-1/2}\left(BA^{-1} - t1_H \right) A^{1/2} \right| A^{1/2} \right\| \\
& \leq Ls \left\| A^{1/2} \right\| \left\| \left| A^{-1/2}\left(BA^{-1} - t1_H \right) A^{1/2} \right| \right\| \left\| A^{1/2} \right\| \\
& = Ls \left\| A^{1/2} \right\| \left\| A^{-1/2}\left(BA^{-1} - t1_H \right) A^{1/2} \right\| \left\| A^{1/2} \right\| \\
& \leq Ls \left\| A \right\|^{1/2} \left\| A^{-1} \right\|^{1/2} \left\| BA^{-1} - t1_H \right\| \left\| A \right\|^{1/2} \left\| A \right\|^{1/2} \\
& = Ls \left\| A \right\|^{3/2} \left\| A^{-1} \right\|^{1/2} \left\| BA^{-1} - t1_H \right\|
\end{aligned}$$

for any $t \in [m, M]$ and $s \in [0, 1]$.

By Lemma 3 and by (4.6) we get

$$\begin{aligned}
& \left\| \mathcal{P}_{f'}\left(B\nabla_s (tA), A \right) - \mathcal{P}_{f'}(B, A) \right\| \\
& = \left\| A^{1/2}\left[f'\left[(1-s)\, A^{-1/2}BA^{-1/2} + st1_H \right] - f'\left(A^{-1/2}BA^{-1/2} \right) \right] A^{1/2} \right\| \\
& \leq \left\| A^{1/2}\left| f'\left[(1-s)\, A^{-1/2}BA^{-1/2} + st1_H \right] - f'\left(A^{-1/2}BA^{-1/2} \right) \right| A^{1/2} \right\| \\
& \leq Ls \left\| A \right\|^{3/2} \left\| A^{-1} \right\|^{1/2} \left\| BA^{-1} - t1_H \right\|
\end{aligned}$$

for any $t \in [m, M]$ and $s \in [0, 1]$.

Therefore,

$$\begin{aligned}
& \frac{1}{\left| \int_m^M w(t)\, dt \right|} \int_m^M |w(t)| \left\| BA^{-1} - t1_H \right\| \\
& \times \left(\int_0^1 \left\| \mathcal{P}_{f'}\left(B\nabla_s (tA), A \right) - \mathcal{P}_{f'}(B, A) \right\| ds \right) dt \\
& \leq \frac{1}{\left| \int_m^M w(t)\, dt \right|} \int_m^M |w(t)| \left\| BA^{-1} - t1_H \right\| \\
& \times \left(\int_0^1 s \left\| A \right\|^{3/2} \left\| A^{-1} \right\|^{1/2} \left\| BA^{-1} - t1_H \right\| ds \right) dt \\
& = \frac{L \left\| A \right\|^{3/2} \left\| A^{-1} \right\|^{1/2}}{2 \left| \int_m^M w(t)\, dt \right|} \int_m^M |w(t)| \left\| BA^{-1} - t1_H \right\|^2 dt
\end{aligned}$$

and by (4.5) we get the desired result (4.2).

The discrete inequality (4.3) can be proved in a similar way, however the details are omitted. □

If we take in (4.2) $w(t) = 1$, then we get

$$\left\| \mathcal{P}_f(B,A) - \frac{1}{M-m} f(t)\,dtA - \left(BA^{-1} - \frac{M+m}{2} 1_H \right) \mathcal{P}_{f'}(B,A) \right\| \tag{4.7}$$
$$\leq \frac{1}{2} \frac{L\|A\|^{3/2}\left\|A^{-1}\right\|^{1/2}}{M-m} \int_m^M \left\|BA^{-1} - t1_H\right\|^2 dt,$$

while from (4.3) we get

$$\left\| \mathcal{P}_f(B,A) - \frac{1}{n}\sum_{i=1}^n f(x_i)\,A - \left(BA^{-1} - \frac{1}{n}\sum_{i=1}^n x_i 1_H \right) \mathcal{P}_{f'}(B,A) \right\| \tag{4.8}$$
$$\leq \frac{1}{2} \frac{L\|A\|^{3/2}\left\|A^{-1}\right\|^{1/2}}{n} \sum_{i=1}^n \left\|BA^{-1} - x_i 1_H\right\|^2 .$$

5. Applications for Relative Operator Entropy

Kamei and Fujii [11], [12] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B) := A^{\frac{1}{2}} \left(\ln \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) \right) A^{\frac{1}{2}}, \tag{5.1}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [26].

In general, we can define for positive operators A, B

$$S(A|B) := s\text{-}\lim_{\varepsilon \to 0+} S(A + \varepsilon 1_H | B)$$

if it exists, here 1_H is the identity operator.

Consider the logarithmic function $\ln$. Then the relative operator entropy can be interpreted as the perspective of ln, namely

$$\mathcal{P}_{\ln}(B,A) = S(A|B),$$

for positive invertible operators A, B.

For some recent results on relative operator entropy see [4]-[5], [19]-[20] and [22]-[23].

If we use the inequality (3.14) for the convex function $f = -\ln t$ and the positive invertible operators A, B that satisfy condition (1.1) with $M > m > 0$ we have

$$\left\| S(A|B) - \frac{1}{\int_m^M w(t)\,dt} \int_m^M w(t) \ln t dtA \right. \tag{5.2}$$
$$\left. - \frac{m+M}{2mM} \left(B - \frac{1}{\int_m^M w(t)\,dt} \int_m^M w(t)\, t dtA \right) \right\|$$
$$\leq \frac{M-m}{2mM} \|A\| \frac{1}{\left|\int_m^M w(t)\,dt\right|} \int_m^M |w(t)| \left\|BA^{-1} - t1_H\right\| dt,$$

for any $w : [m,M] \to \mathbb{C}$ a Lebesgue integrable function with $\int_m^M w(t)\,dt \neq 0$.

Define the *identric mean* of a, $b>0$ by

$$I(a,b):=\begin{cases} a & \text{if } a=b, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a\neq b \end{cases} \quad a,\ b>0.$$

We observe that

$$I(a,b)=\frac{1}{b-a}\int_a^b \ln t dt \text{ for } a\neq b.$$

If we take in (5.2) $w(t)=1$ for $t\in[m,M]$, then we get the norm inequality

$$\left\|S(A|B)-I(m,M)A-\frac{m+M}{2mM}\left(B-\frac{m+M}{2}A\right)\right\| \le \frac{1}{2mM}\|A\|\int_m^M \left\|BA^{-1}-t1_H\right\|dt, \tag{5.3}$$

for any positive invertible operators A, B that satisfy condition (1.1) with $M>m>0$.

We define the *logarithmic mean* as

$$L(a,b):=\begin{cases} a & \text{if } a=b \\ \frac{b-a}{\ln b-\ln a} & \text{if } a\neq b \end{cases} \quad a,\ b>0$$

and the *geometric mean* as $G(a,b):=\sqrt{ab}$.

If we take in (5.2) $w(t)=\frac{1}{t}$, $t>0$, and since

$$\frac{1}{\int_m^M \frac{1}{t}dt}\int_m^M \frac{\ln t}{t}dt=\frac{\ln^2 M-\ln^2 m}{2(\ln M-\ln m)}=\ln G(m,M),$$

then we get the norm inequality

$$\left\|S(A|B)-\ln G(m,M)A-\frac{m+M}{2mM}(B-L(m,M)A)\right\| \le \frac{L(m,M)}{2mM}\|A\|\int_m^M \frac{1}{t}\left\|BA^{-1}-t1_H\right\|dt, \tag{5.4}$$

for any positive invertible operators A, B that satisfy condition (1.1) with $M>m>0$.

If we use the inequality (3.15) for the convex function $f=-\ln t$ and positive invertible operators A, B that satisfy condition (1.1) with $M>m>0$, we have for $x_i\in[m,M]$, $w_i>0$ with $i\in\{1,...,n\}$ that

$$\left\|S(A|B)-\ln G_n(\bar{x},\bar{w})A-\frac{m+M}{2mM}(B-A_n(\bar{x},\bar{w})A)\right\| \le \frac{1}{2}\frac{M-m}{mM}\|A\|\frac{1}{W_n}\sum_{i=1}^n w_i\left\|BA^{-1}-x_i1_H\right\|, \tag{5.5}$$

where

$$A_n(\bar{x},\bar{w}):=\frac{1}{W_n}\sum_{i=1}^n w_ix_iA$$

is the *weighted arithmetic mean* and

$$G_n(\bar{x},\bar{w}) := \left(\prod_{i=1}^{n} x_i^{w_i}\right)^{\frac{1}{W_n}}$$

is the *weighted geometric mean.*

6. Applications for Operator Geometric Mean

Assume that A, B are positive invertible operators on a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. We use the following notations for operators [21]

$$A\nabla_\nu B := (1-\nu)A + \nu B,$$

the *weighted operator arithmetic mean* and

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2},$$

the *weighted operator geometric mean,* where $\nu \in [0,1]$. When $\nu = \frac{1}{2}$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

The definition $A\sharp_\nu B$ can be extended accordingly for any real number ν.

The following inequality is well known as the operator *Young inequality* or operator *ν-weighted arithmetic-geometric mean inequality*:

$$A\sharp_\nu B \le A\nabla_\nu B \text{ for all } \nu \in [0,1]. \tag{6.1}$$

For recent results on operator Young inequality see [13]-[17], [18] and [27]-[28].

If we consider the continuous function $f_\nu : [0,\infty) \to [0,\infty)$, $f_\nu(x) = x^\nu$ then the operator *ν-weighted geometric mean* can be interpreted as the perspective $\mathcal{P}_{f_\nu}(B,A)$, namely

$$\mathcal{P}_{f_\nu}(B,A) = A\sharp_\nu B.$$

Since, for $\nu \in (0,1)$, $f_\nu : [0,\infty) \to [0,\infty)$, $f_\nu(x) = x^\nu$ is operator concave and positive on $[0,\infty)$, then we have that (see [24, p. 146])

$$(tA + (1-t)C)\sharp_\nu(tB + (1-t)D) \ge tA\sharp_\nu B + (1-t)C\sharp_\nu D \tag{6.2}$$

and we also have that (see [24, p. 146])

$$(A+C)\sharp_\nu(B+D) \ge A\sharp_\nu B + C\sharp_\nu D \tag{6.3}$$

for any positive invertible operators A, B, C, D and $\nu \in [0,1]$.

For positive invertible operators A, B, C, D such that $A > C$ and $B > D$, then we have (see also [24, p. 139])

$$A\sharp_\nu B \ge C\sharp_\nu D. \tag{6.4}$$

Moreover, if $KC \ge A \ge kC$ and $KD \ge B \ge kD$ for some positive constants k, K then we also have that

$$KC\sharp_\nu D \ge A\sharp_\nu B \ge kC\sharp_\nu D. \tag{6.5}$$

If we use the inequality (3.14) for the convex function $f(x) = -x^{\nu}$, $x > 0$, $\nu \in [0,1]$, then we have

$$\begin{aligned}(6.6)\quad &\left\| A\sharp_{\nu}B - \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\,t^{\nu}dtA \right.\\ &\left. -\frac{1}{2}\nu\frac{m^{1-\nu}+M^{1-\nu}}{m^{1-\nu}M^{1-\nu}}\left(B - \frac{1}{\int_m^M w(t)\,dt}\int_m^M w(t)\,tdtA\right)\right\| \\ &\le \frac{1}{2}\nu\frac{M^{1-\nu}-m^{1-\nu}}{m^{1-\nu}M^{1-\nu}}\|A\|\frac{1}{\left|\int_m^M w(t)\,dt\right|}\int_m^M |w(t)|\left\|BA^{-1}-t1_H\right\|dt\end{aligned}$$

for positive invertible operators A, B that satisfy condition (1.1) with $M > m > 0$ and any $w : [m,M] \to \mathbb{C}$ a Lebesgue integrable function with $\int_m^M w(t)\,dt \neq 0$.

If we take in (6.6) $w(t) = 1$, then we get

$$\begin{aligned}(6.7)\quad &\left\| A\sharp_{\nu}B - \frac{M^{\nu+1}-m^{\nu+1}}{(\nu+1)(M-m)}A - \frac{1}{2}\nu\frac{m^{1-\nu}+M^{1-\nu}}{m^{1-\nu}M^{1-\nu}}\left(B-\frac{m+M}{2}A\right)\right\| \\ &\le \frac{1}{2}\nu\frac{M^{1-\nu}-m^{1-\nu}}{m^{1-\nu}M^{1-\nu}(M-m)}\|A\|\int_m^M\left\|BA^{-1}-t1_H\right\|dt\end{aligned}$$

for positive invertible operators A, B that satisfy condition (1.1) with $M > m > 0$.

If we use the inequality (3.15) for the convex function $f = -x^{\nu}$, $x > 0$, $\nu \in [0,1]$ and positive invertible operators A, B that satisfy condition (1.1) with $M > m > 0$, then we have for $x_i \in [m,M]$, $w_i > 0$ with $i \in \{1,...,n\}$ that

$$\begin{aligned}(6.8)\quad &\left\| \mathcal{P}_f(B,A) - \frac{1}{W_n}\sum_{i=1}^n w_i x_i^{\nu}A - \frac{1}{2}\nu\frac{m^{1-\nu}+M^{1-\nu}}{m^{1-\nu}M^{1-\nu}}\left(B - \frac{1}{W_n}\sum_{i=1}^n w_i x_i A\right)\right\| \\ &\le \frac{1}{2}\nu\frac{M^{1-\nu}-m^{1-\nu}}{m^{1-\nu}M^{1-\nu}}\|A\|\frac{1}{W_n}\sum_{i=1}^n w_i\left\|BA^{-1}-x_i1_H\right\|.\end{aligned}$$

References

[1] S. S. Dragomir, Ostrowski type inequalities for isotonic functionals, *J. Inequal. Pure & Appl. Math.* **3** (2002), Nr. 5, Art. 68 [Online https://www.emis.de/journals/JIPAM/article220.html?sid=220].

[2] S. S. Dragomir, Some new reverses of Young's operator inequality, *RGMIA Res. Rep. Coll.* **18** (2015), Art. 130. [Online http://rgmia.org/papers/v18/v18a130.pdf].

[3] S. S. Dragomir, On new refinements and reverses of Young's operator inequality, *RGMIA Res. Rep. Coll.* **18** (2015), Art. 135. [Online http://rgmia.org/papers/v18/v18a135.pdf].

[4] S. S. Dragomir, Some inequalities for relative operator entropy, *RGMIA Res. Rep. Coll.* **18** (2015), Art. 145. [Online http://rgmia.org/papers/v18/v18a145.pdf].

[5] S. S. Dragomir, Further inequalities for relative operator entropy, *RGMIA Res. Rep. Coll.* **18** (2015), Art. 160.[Online http://rgmia.org/papers/v18/v18a160.pdf].

[6] S. S. Dragomir, Inequalities for operator noncommutative perspectives of continuously differentiable functions with applications, *RGMIA Res. Rep. Coll.* **19** (2016), Art. 22. [Online http://rgmia.org/papers/v19/v19a22.pdf].

[7] S. S. Dragomir, M. S. Moslehian and Y. J. Cho, Some reverses of the Cauchy-Schwarz inequality for complex functions of self-adjoint operators in Hilbert spaces. *Math. Inequal. Appl.* **17** (2014), no. 4, 1365–1373. Preprint *RGMIA Res. Rep. Coll.***14** (2011), Art. 84. [Online http://rgmia.org/papers/v14/v14a84.pdf].

[8] A. Ebadian, I. Nikoufar and M. E. Gordji, Perspectives of matrix convex functions, *Proc. Natl. Acad. Sci. USA*, **108** (2011), no. 18, 7313–7314.

[9] E. G. Effros, A matrix convexity approach to some celebrated quantum inequalities, *Proc. Natl. Acad. Sci. USA* **106** (2009), 1006–1008.

[10] E. G. Effros and F. Hansen, Noncomutative perspectives, *Ann. Funct. Anal.* **5** (2014), no. 2, 74–79.

[11] J. I. Fujii and E. Kamei, Uhlmann's interpolational method for operator means. *Math. Japon.* **34** (1989), no. 4, 541–547.

[12] J. I. Fujii and E. Kamei, Relative operator entropy in noncommutative information theory. *Math. Japon.* **34** (1989), no. 3, 341–348.

[13] S. Furuichi, Refined Young inequalities with Specht's ratio, *J. Egyptian Math. Soc.* **20** (2012), 46–49.

[14] S. Furuichi, On refined Young inequalities and reverse inequalities, *J. Math. Inequal.* **5** (2011), 21-31.

[15] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc., New York, 1969.

[16] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.* **361** (2010), 262-269.

[17] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Linear Multilinear Algebra,* **59** (2011), 1031-1037.

[18] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.

[19] I. H. Kim, Operator extension of strong subadditivity of entropy, *J. Math. Phys.* 53(2012), 122204

[20] P. Kluza and M. Niezgoda, Inequalities for relative operator entropies, *Elec. J. Lin. Alg.* **27** (2014), Art. 1066.

[21] F. Kubo and T. Ando, Means of positive operators, *Math. Ann.* **264** (1980), 205–224.

[22] M. S. Moslehian, F. Mirzapour and A. Morassaei, Operator entropy inequalities. *Colloq. Math.*, **130** (2013), 159–168.

[23] I. Nikoufar, On operator inequalities of some relative operator entropies, *Adv. Math.* **259** (2014), 376-383.

[24] J. Pečarić, T. Furuta, J. Mićić Hot and Y. Seo, *Mond-Pečarić method in operator inequalities.* Inequalities for bounded selfadjoint operators on a Hilbert space. Monographs in Inequalities, 1. ELEMENT, Zagreb, 2005. xiv+262 pp.+loose errata. ISBN: 953-197-572-8

[25] A. Ostrowski, Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert, *Comment. Math. Helv.,* **10** (1938), 226-227.

[26] M. Nakamura and H. Umegaki, A note on the entropy for operator algebras. *Proc. Japan Acad.* **37** (1961) 149–154.

[27] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.H.

[28] G. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.,* **5** (2011), 551-556.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa