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The Damascus Inequality

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1 The problem formulation

In 2016 Prof. Fozi M. Dannan from Damascus, Syria proposed the next inequality

$$\frac{x-1}{x^2-x+1}+\frac{y-1}{y^2-y+1}+\frac{z-1}{z^2-z+1}\le 0, \tag{1}$$

providing that $xyz=1$ for $x,y,z>0$. It became widely known but was not proved yet in spite of elementary formulation.

An obvious generalization is the next inequality

$$\sum_{k=1}^{n}\frac{x_k-1}{x_k^2-x_k+1}\le 0, \tag{2}$$

providing that $x_1\cdot x_2\ldots\cdot x_n=1$ for $x_k\ge 0, 1\le k\le n$.

It is obvious that (2) is true for $n=1$, it is easy to prove it also for $n=2$ directly. But it is not true for $n=4$ as follows from an example with $x_1=x_2=x_3=2, x_4=\frac{1}{8}$, then (2) is reducing to $1-\frac{56}{57}\le 0$ which is untrue.

As a consequence (2) is also untrue for any $n\ge 4$ due to an example with $x_1=x_2=x_3=2, x_4=\frac{1}{8}, x_5=\ldots=x_n=1$. So the only non–trivial case in (2) is $n=3$.

In this paper we prove inequality (1) together with similiar ones

$$\frac{1}{x^2-x+1}+\frac{1}{y^2-y+1}+\frac{1}{z^2-z+1}\le 3 \tag{3}$$

$$\frac{x}{x^2-x+1}+\frac{y}{y^2-y+1}+\frac{z}{z^2-z+1}\le 3 \tag{4}$$

$$\frac{x-1}{x^2+x+1}+\frac{y-1}{y^2+y+1}+\frac{z-1}{z^2+z+1}\le 0 \tag{5}$$

$$\frac{1}{x^2+x+1}+\frac{1}{y^2+y+1}+\frac{1}{z^2+z+1}\ge 1 \tag{6}$$

$$\frac{x}{x^2+x+1}+\frac{y}{y^2+y+1}+\frac{z}{z^2+z+1}\le 1 \tag{7}$$

$$\frac{x+1}{x^2+x+1}+\frac{y+1}{y^2+y+1}+\frac{z+1}{z^2+z+1}\le 2. \tag{8}$$

Also some generalizations will be considered.

2 Proof of the main inequality (1)

Theorem 1. An inequality (1) holds true providing that $xyz = 1$ for $x, y, z > 0$.

For the proof we need an auxiliary inequality that seems to be very interesting by itself.

Lemma 1. Let x, y, z be positive numbers such that $xyz = 1$. Then

$$x^2 + y^2 + z^2 - 3\left(x + y + z\right) + 6 \geq 0 \tag{9}$$

holds true.

Note that inequality (9) is not a consequence of well–known family of Klamkin–type inequalities for symmetric functions [1]. So (9) is a new quadratic Klamkin–type inequality in three variables under restriction $xyz = 1$. Due to its importance we give three proofs to it based on different ideas.

First proof of Lemma 1.

To prove (9) let introduce the Lagrange function

$$L(x, y, z, \lambda) = x^2 + y^2 + z^2 - 3(x + y + z) + 6 - \lambda(xyz - 1).$$

On differentiating it follows

$$\lambda = x^2 - 2x = y^2 - 2y = z^2 - 2z.$$

It follows that at the minimum (it obviously exists) $x = y$, so three variables at the minimum are $x, y = x, z = 1/x^2$. From $x^2 - 2x = z^2 - 2z$ we derive the equation in x:

$$2x^2 - 3x = 2/x^4 - 3/x^2, f(x) = 2x^6 - 3x^5 + 3x^2 - 2 = 0.$$

One root is obvious $x = 1$. Let us prove that there are no other roots for $x \geq 0$. Check that derivative is positive

$$f'(x) = 12x^5 - 15x^4 + 6x = 3x(4x^4 - 5x^3 + 2) \geq 0, x \geq 0.$$

Define a function $g(x) = 4x^4 - 5x^3 + 2$, its derivative $g'(x)$ has one zero for $x \geq 0$ at $x = 15/16$ and the function $g(x)$ is positive at this zero at its minimum $g(15/16) = 15893/16384 > 0$. So $g(x)$ is positive, $f(x)$ is strictly increasing on $x \geq 0$, so $f(1) = 0$ is its only zero.

Second proof of Lemma 1.

Consider the function

$$f\left(x, y\right) = x^2 + y^2 + \frac{1}{x^2 y^2} - 3\left(x + y + \frac{1}{xy}\right) + 6 \quad ,$$

where x, y, z are positive numbers. We show that $f(x, y)$ attains its minimum 0 at $x = 1, y = 1$ using partial derivative test.

Calculate

$$\frac{\partial f}{\partial x} = 2x - \frac{2}{x^3 y^2} - 3 + \frac{3}{x^2 y} = 0, \tag{10}$$

$$\frac{\partial f}{\partial y} = 2y - \frac{2}{x^2 y^3} - 3 + \frac{3}{x y^2} = 0. \tag{11}$$

Now multiplying (10) and (11) respectively by x and $-y$ and adding to obtain

$$(x-y)(2x+2y-3)=0 \quad .$$

Here we have two cases.

Case I. $x=y$, which implies from equation (10) that

$$2x^6-2-3x^5+3x^2=0$$

or

$$\left(x^3-1\right)\left(2x^3-3x^2+2\right)=0. \tag{12}$$

Equation (12) has only one positive root $x=1$ and consequently $y=1$. Notice that the equation

$$2x^3-3x^2+2=0$$

does not have positive roots because for $x\geq 0$ the function $u(x)=2x^3-3x^2+2$ satisfies the following properties : $(i)\, u(0)=2, (ii)\ \min u(x)=1$ at $x=1$, $(iii)\, u(\infty)=\infty$. Therefore $f(x,y)$ attains its maximum or minimum at $x=1,\ y=1$.

Case II. $2x+2y=3$. Adding (10) and (11) we get

$$2(x+y)-6-2\left(\frac{1}{x^3y^2}+\frac{1}{x^2y^3}\right)+\ 3\left(\frac{1}{x^2y}+\frac{1}{xy^2}\right)=0$$

or

$$-3-\frac{3}{x^3y^3}+\frac{3}{2x^2y^2}=0$$

and

$$-6x^3y^3+3xy-6=0 \quad .$$

Putting $t=xy$ we obtain

$$2t^3-t+2=0.$$

In fact this equation does not have positive root (notice that $t=xy$ should be positive). This is because the function $u=2t^3-t+2$ satisfies the following properties :

$$(i)\quad u(0)=2,$$

$$(ii)\ \text{ for }\ t>0, \min u(t)=u\left(\frac{1}{\sqrt{6}}\right)>0,\ (iii)\quad u(\infty)>0.$$

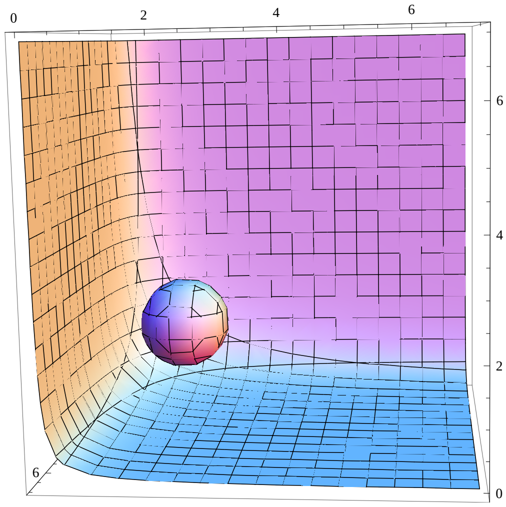
The last step is to show that

$$f(x,y)\geq f(1,1)=0.$$

It is enough to show that $f(s,t)>f(1,1)$ for at least one point $(s,t)\neq(1,1)$. Take for example $f(2,3)=\frac{7}{2}+\frac{1}{36}$.

Third proof of lemma 1 (Geometrical Method).

Geometrically it is enough to prove that the surface $xyz=1$ lies outside the sphere $(x-3/2)^2+(y-3/2)^2+(z-3/2)^2=3/4$ except the only intersection point (1,1,1) as it is shown on the next graph:



Let M and S be surfaces defined by
$M: \quad xyz = 1$ and $S: \ (x-\frac{3}{2})^2 + (y-\frac{3}{2})^2 + (z-\frac{3}{2})^2 - \frac{3}{4} \ = 0$
1. If $(z-\frac{3}{2})^2 - \frac{3}{4} \geq 0$ then

$$(x-\frac{3}{2})^2 + (y-\frac{3}{2})^2 + (z-\frac{3}{2})^2 - \frac{3}{4} \geq 0$$

and equivalently

$$x^2 + y^2 + z^2 - 3\,(x+y+z) + 6 \geq 0.$$

2. If $(z-\frac{3}{2})^2 - \frac{3}{4} \leq 0$ then $\frac{3\,-\,\sqrt{3}}{2} \leq z \leq \frac{3\,+\,\sqrt{3}}{2}$.
3. We take horizontal sections for both M and so get for any plane

$$\frac{3\ -\ \sqrt{3}}{2} \leq z = k \leq \frac{3\ +\ \sqrt{3}}{2}$$

two curves: equilateral hyperbola $H(k)$ with vertex $(\frac{1}{\sqrt{k}}, \frac{1}{\sqrt{k}})$ and a circle $C\,(k)$ which radius is given by

$$r^2\,(k) = \frac{3}{4} - (k-\frac{3}{2})^2 = -k^2 + 3k - \frac{3}{2}$$

with center at $(\frac{3}{2}, \frac{3}{2}, k)$.
4. For $z = 1$, we have the hyperbola $xy = 1$ and the circle

$$(x-\frac{3}{2})^2 + (y-\frac{3}{2})^2 = \frac{1}{2}.$$

5. We show that the distance $d(v,c)$ between the vertex of the hyperbola and the center c of the circle is always greater than or equal to the radius of the circle. The distance $d(v,c)$ is given by

$$d^2\,(v,c) = \left(\frac{1}{\sqrt{k}} - \frac{3}{2}\right)^2 + \left(\frac{1}{\sqrt{k}} - \frac{3}{2}\right)^2 = 2\left(\frac{1}{\sqrt{k}} - \frac{3}{2}\right)^2.$$

The radius is given by

$$r^2\,(k) = \frac{3}{4} - (k - \frac{3}{2})^2 = -k^2 + 3k - \frac{3}{2} \quad .$$

We need to show that the vertex is always outside the circle i.e. $d^2\,(v,c) \geq r^2\,(k)$ for all

$$\frac{3\;-\;\sqrt{3}}{2} \leq k \leq \frac{3\;+\;\sqrt{3}}{2} \quad .$$

Clearly that $d\,(v,c) = r$ for $k = 1$ and the hyperbola tangents the circle at the point $(1,1,1)$.

For $\frac{3\,-\,\sqrt{3}}{2} \leq k < 1$, as k decreases from 1 to $\frac{3\,-\,\sqrt{3}}{2}$, the radius of the circle becomes smaller . From the other side the vertex $(\frac{1}{\sqrt{k}}, \frac{1}{\sqrt{k}}, k)$ moves away from $(1,1,1)$ towards a point $(0,0,k)$. This follows from the distance function of the vertex

$$Ov = \frac{\sqrt{2}}{\sqrt{k}}, \left(0 < k_1 \leq k_2 < 1 \rightarrow \frac{\sqrt{2}}{\sqrt{k_2}} < \frac{\sqrt{2}}{\sqrt{k_1}} \right).$$

6. For $1 < k \leq \frac{3\,+\,\sqrt{3}}{2}$, we show that

$$d^2\,(v,c) = g\,(k) = 2\,\left(\frac{1}{\sqrt{k}} - \frac{3}{2}\right)^2 > r^2 = \frac{3}{4} - (k - \frac{3}{2})^2 = h\,(k)\,.$$

In fact, $h\,(k)$ is a concave down parabola and has its maximum at $k = 1$, i.e. $\max\, h\,(k) = \frac{1}{2}$ and $h\,(k)$ is decreasing for $k > 1$. Also, $g\,(1) = \frac{1}{2}$ and $g(k)$ is increasing for $k > 1$ because $g'(k) = 4\left(-\frac{1}{k\sqrt{k}}\right)\left(\frac{1}{\sqrt{k}} - \frac{3}{2}\right) > 0$ for $k \geq 1$ (notice that $\frac{1}{\sqrt{k}} < 1$). Therefore $g\,(k) > h\,(k)$ for $k > 1$. Eventually we conclude that

$$(x - \frac{3}{2})^2 + (y - \frac{3}{2})^2 + (z - \frac{3}{2})^2 - \frac{3}{4} \geq 0$$

and consequently

$$x^2 + y^2 + z^2 - 3\,(x + y + z) + 6 \geq 0$$

for all values of (x,y,z) that satisfy $xyz = 1$.

Proof of the theorem 1.

Now consider the inequality to prove (1). After simplifying with the use of Wolfram Mathematica it reduces to

$$\begin{aligned} &-3 + 3x - 2x^2 + 3y - 3xy + 2x^2y - 2y^2 + 2xy^2 - x^2y^2 + 3z - 3xz + 2x^2z - \\ &\quad -3yz + 3xyz - 2x^2yz + +2y^2z - 2xy^2z + x^2y^2z - 2z^2 + 2xz^2 - x^2z^2 + \\ &\quad\quad +2yz^2 - 2xyz^2 + x^2yz^2 - y^2z^2 + xy^2z^2 \leq 0. \end{aligned}$$

Using **SymmetricReduction** function of Wolfram Mathematica we derive

$$\begin{aligned} &3 - xy - xz - yz + 3xyz - 3(x + y + z) + 2(x + y + z)^2 - xyz(xy + xz + yz) \\ &\quad - - 2(x + y + z)(xy + xz + yz) + (xy + xz + yz)^2 \geq 0. \end{aligned}$$

Using $xyz = 1$ let further simplify

$$6 - 3(x+y+z) + 2(x+y+z)^2 - 2(xy+xz+yz) -$$
$$-2(x+y+z)(xy+xz+yz) + (xy+xz+yz)^2 \geq 0.$$

In terms of elementary symmetric functions

$$S_1 = x+y+z, S_2 = xy+yz+xz$$

it is

$$S_2^2 - 2S_1S_2 - 2S_2 + 2S_1^2 - 3S_1 + 6 \geq 0. \tag{13}$$

As $S_2^2 - 2S_1S_2 + S_1^2 \geq 0$ it is enough to prove

$$S_1^2 - 2S_2 - 3S_1 + 6 \geq 0. \tag{14}$$

Expanding it again in x, y, z we derive an inequality to prove for positive variables

$$x^2 + y^2 + z^2 - 3(x+y+z) + 6 \geq 0. \tag{15}$$

But this is exactly an inequality from Lemma 1. So Theorem 1 is proved.

3 Proof of inequalities (3)–(8)

Let us start with two propositions.

Proposition 1.

For any real numbers $u,\ v,\ w$ such that

$$(1+u)(1+v)(1+w) > 0,$$

the inequality

$$\frac{1}{1+u} + \frac{1}{1+v} + \frac{1}{1+w} \leq k \qquad (\geq k)$$

is equivalent to

$$kuvw + (k-1)(uv+vw+wu) + (k-2)(u+v+w) + k - 3 \geq 0 \ \ (\leq 0).$$

Proposition 2.

For any real numbers $u\ ,\ v\ , w$ such that

$$(u-1)(v-1)(w-1) > 0$$

the inequality

$$\frac{1}{u-1} + \frac{1}{v-1} + \frac{1}{w-1} \leq k \qquad (\geq k)$$

is equivalent to

$$kuvw - (k+1)(uv+vw+wu) + (k+2)(u+v+w) - (k+3) \geq 0 \ \ (\leq 0).$$

The validity of propositions 1 and 2 can be obtained by direct expansions.

Proof that (1) $\Leftrightarrow$ (3).

In fact

$$\frac{x-1}{x^2-x+1} + \frac{y-1}{y^2-y+1} + \frac{z-1}{z^2-z+1} =$$

$$= \frac{x^2-(x^2-x+1)}{x^2-x+1} + \frac{y^2-(y^2-y+1)}{y^2-y+1} + \frac{z^2-(z^2-z+1)}{z^2-z+1} =$$

$$= -3 + \sum_{cyc} \frac{x^2}{x^2-x+1}.$$

Now if the right side is ≤ 0 then

$$\sum_{cyc} \frac{x^2}{x^2-x+1} \leq 3$$

and consequently

$$\sum_{cyc} \frac{1}{1-(1/x)+(1/x)^2} \leq 3.$$

Proof of 5.

We need to prove

$$\sum_{cyc} \frac{1}{x^2+x+1} \geq 1$$

Let $u = x^2+x$, $v = y^2+y$, $w = z^2+z$. Using Proposition 1 the required inequality can be written as follows :

$$uvw-(u+v+w)-2 \leq 0.$$

Going back to x, y, z we get

$$(x+1)(y+1)(z+1) - \left(x^2+y^2+z^2\right) - (x+y+z) - 2 \leq 0.$$

Or

$$xy+yz+zx \leq x^2+y^2+z^2$$

which is obvious.

Proof of 6.

It follows from elementary calculus that for any real number x we have

$$\frac{x}{x^2+x+1} \leq \frac{1}{3}$$

and the inequality follows directly.

Proof that $(6)+(7) \Rightarrow (5)$**.**

Really adding together (7) with (6) multiplied by -1 we derive (5).

Proof that $(6) \Rightarrow (8)$**.**

The required inequality is equivalent to

$$\sum_{cyc} \frac{x^2+x+1-x^2}{x^2+x+1} = 3 - \sum_{cyc} \frac{x^2}{x^2+x+1} \leq 2$$

or

$$\sum_{cyc} \frac{x^2}{x^2+x+1} \geq 1$$

which is true from inequality (6).

4 Modifications of original inequality

In this section we consider modifications of the original inequality (1) providing that $xyz = 1$ for $x, y, z \geq 0$.

1. An inequality (1) is equivalent to

$$\frac{x^2-1}{x^3-1} + \frac{y^2-1}{y^3-1} + \frac{z^2-1}{z^3-1} \leq 0. \tag{16}$$

This form leads to generalization with more powers, cf. below.

2. An inequality (1) is equivalent to

$$\frac{x^2}{x^2-x+1} + \frac{y^2}{y^2-y+1} + \frac{z^2}{z^2-z+1} \leq 3. \tag{17}$$

3. Let take $x \to \frac{1}{x}, y \to \frac{1}{y}, z \to \frac{1}{z}$. Then we derive another equivalent form of the inequality (1)

$$\frac{x-x^2}{x^2-x+1} + \frac{y-y^2}{y^2-y+1} + \frac{z-z^2}{z^2-z+1} \leq 0, \tag{18}$$

due to the functional equation

$$f(\frac{1}{x}) = -xf(x) \tag{19}$$

for the function

$$f(x) = \frac{x-1}{x^2-x+1}. \tag{20}$$

So it seems possible to generalize the original inequality in terms of functional equations too.

To one more similar variant leads a change of variables $x \to xy, y \to yz, z \to xz$:

$$\frac{xy-1}{x^2y^2-xy+1} + \frac{yz-1}{y^2z^2-yz+1} + \frac{xz-1}{x^2z^2-xz+1} \leq 0, \tag{21}$$

or like (18)

$$\frac{xy-x^2y^2}{x^2y^2-xy+1} + \frac{yz-y^2z^2}{y^2z^2-yz+1} + \frac{xz-x^2z^2}{x^2z^2-xz+1} \leq 0, \tag{22}$$

It is also possible to consider generalizations of (1) under the most general transformations $x \to g(x,y,z), y \to h(x,y,z), z \to \frac{1}{g(x,y,z)h(x,y,z)}$ with positive functions $g(x,y,z), h(x,y,z)$ still preserving a condition $xyz = 1$.

4. A number of cyclic inequalities follow from previous ones by a substitution

$$x = \frac{a}{b}, y = \frac{b}{c}, z = \frac{c}{a}, xyz = 1.$$

On this way we derive from (1), (3)–(8) the next cyclic inequalities:

$$\frac{ab-b^2}{a^2-ab+b^2}+\frac{bc-c^2}{b^2-bc+c^2}+\frac{ca-a^2}{c^2-ca+a^2}\le 0, \tag{23}$$

$$\frac{b^2}{a^2-ab+b^2}+\frac{c^2}{b^2-bc+c^2}+\frac{a^2}{c^2-ca+a^2}\le 3 \tag{24}$$

$$\frac{ab}{a^2-ab+b^2}+\frac{bc}{b^2-bc+c^2}+\frac{ca}{c^2-ca+a^2}\le 3 \tag{25}$$

$$\frac{ab-b^2}{a^2+ab+b^2}+\frac{bc-c^2}{b^2+bc+c^2}+\frac{ca-a^2}{c^2+ca+a^2}\le 0 \tag{26}$$

$$\frac{b^2}{a^2+ab+b^2}+\frac{c^2}{b^2+bc+c^2}+\frac{a^2}{c^2+ca+a^2}\ge 1 \tag{27}$$

$$\frac{ab}{a^2+ab+b^2}+\frac{bc}{b^2+bc+c^2}+\frac{ca}{c^2+ca+a^2}\le 1 \tag{28}$$

$$\frac{ab+b^2}{a^2+ab+b^2}+\frac{bc+c^2}{b^2+bc+c^2}+\frac{ca+a^2}{c^2+ca+a^2}\le 2. \tag{29}$$

On cyclic inequalities among which Schur, Nessbit and Shapiro ones are the most well–known cf. [1]–[3].

5. Some geometrical quantities connected with trigonometric functions and triangle geometry satisfy a condition $xyz=1$, cf. [4]–[6]. For example, we may use in standard notations for triangular geometry values:

$$x=\frac{a}{4p},\quad y=\frac{b}{R},\quad z=\frac{c}{r};$$

$$x=\frac{a+b}{2},\quad y=\frac{b+c}{p},\quad z=\frac{a+c}{p^2+r^2+2rR};$$

$$x=Rh_a,\quad y=\frac{h_b}{2p^2},\quad z=\frac{h_c}{r^2};$$

$$x=2R^2\sin(\alpha),\quad y=\frac{\sin(\beta)}{r},\quad z=\frac{\sin(\gamma)}{p};$$

$$x=(p^2-4R^2-4rR-r^2)\tan(\alpha),\quad y=\frac{\tan(\beta)}{2p},\quad z=\frac{\tan(\gamma)}{r};$$

$$x=\frac{\tan(\alpha)}{\tan(\alpha)+\tan(\beta)+\tan(\gamma)},y=\frac{\tan(\beta)}{\tan(\alpha)+\tan(\beta)+\tan(\gamma)},z=\frac{\tan(\gamma)}{\tan(\alpha)+\tan(\beta)+\tan(\gamma)};$$

$$x=\tan(\alpha/2),\quad y=p\tan(\beta/2),\quad z=\frac{\tan(\gamma/2)}{r};$$

$$x=\frac{a}{4(p-a)},\quad y=\frac{b}{R(p-b)},\quad z=r\frac{c}{p-c};$$

$$x=4R\sin(\alpha/2),\quad y=\sin(\beta/2),\quad z=\frac{\sin(\gamma/2)}{r};$$

$$x=4R\cos(\alpha/2),\quad y=\cos(\beta/2),\quad z=\frac{\cos(\gamma/2)}{p}.$$

6. The above geometrical identities of the type $x\,y\,z=1$ which we use for applications of considered inequalities are mostly consequences of Vieta's formulas [5]. It is interesting to use these formulas for cubic equation directly.

Theorem 2. Let x, y, z be positive roots of the cubic equation with any real a, b

$$t^3 + at^2 + bt - 1 = 0.$$

The for these roots x, y, z all inequalities of this paper are valid.

7. We can generalize inequalities (3), (6)–(8) for more general powers. For this aim we use Bernoulli's inequalities [1]–[2] : for $u > 0$ the following inequalities hold true

$$u^\alpha - \alpha u + \alpha - 1 \geq 0 \ , \quad (\ \alpha > 1 \ \text{ or } \alpha < 0),$$

$$u^\alpha - \alpha u + \alpha - 1 \leq 0 \ , \quad (\ 0 < \alpha < 1 \ \ ) \ .$$

Lemma 2. Assume that x, y, z are positive numbers such that $xyz = 1$. Then the following inequality holds true :

$$\left(\frac{1}{x^2 - x + 1}\right)^\alpha + \left(\frac{1}{y^2 - y + 1}\right)^\alpha + \left(\frac{1}{z^2 - z + 1}\right)^\alpha \leq 3$$

for $0 < \alpha < 1$.

Proof. Let

$$X = x^2 - x + 1 \ , \ Y = y^2 - y + 1 \ , \ Z = \ z^2 - z + 1 \ \ .$$

Then we have

$$\left(\frac{1}{x^2 - x + 1}\right)^\alpha + \left(\frac{1}{y^2 - y + 1}\right)^\alpha + \left(\frac{1}{z^2 - z + 1}\right)^\alpha \leq$$

$$\leq \alpha \left(\frac{1}{X} + \frac{1}{Y} + \frac{1}{Z}\right) + 3\,(1 - \alpha) \leq 3 \quad .$$

Similarly we have from (7) that

$$\left(\frac{x}{x^2 + x + 1}\right)^\alpha + \left(\frac{y}{y^2 + y + 1}\right)^\alpha + \left(\frac{z}{z^2 + z + 1}\right)^\alpha \leq 3 - 2\alpha$$

and from (8) we have

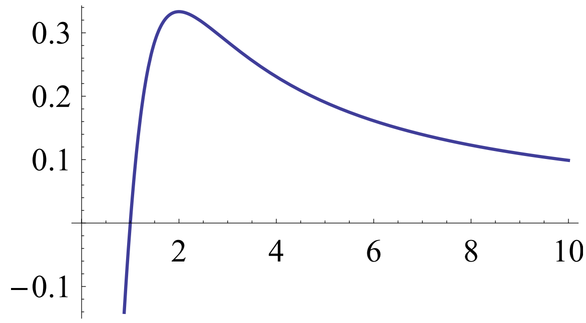
$$\left(\frac{x + 1}{x^2 + x + 1}\right)^\alpha + \left(\frac{y + 1}{y^2 + y + 1}\right)^\alpha + \left(\frac{z + 1}{z^2 + z + 1}\right)^\alpha \leq 3 - \alpha \ .$$

For $\alpha > 1$ or $\alpha < 0$ we have from (6)

$$\left(\frac{1}{x^2 + x + 1}\right)^\alpha + \left(\frac{1}{y^2 + y + 1}\right)^\alpha + \left(\frac{1}{z^2 + z + 1}\right)^\alpha \geq 3 - 2\alpha.$$

5 Generalizations of original inequality to ones with a set of restrictions on symmetric functions

It is easy to show that the maximum of the function (20) is attained for $x \geq 0$ at $x = 2$ and equals to $1/3$.



So the next unconditional inequality holds

$$\sum_{k=1}^{k=n} \frac{x_k - 1}{x_k^2 - x_k + 1} \le \frac{n}{3}; \quad x_k \ge 0 \tag{30}$$

Consider symmetric functions

$$S_1 = \sum_{k=1}^{k=n} x_k, S_2 = \sum_{\substack{k,m=1,\\ k\neq m}}^{n} x_k \cdot x_m, \ldots, S_n = x_1 x_2 \cdots x_n.$$

The generalized Damascus inequality

Prove an inequality

$$\sum_{k=1}^{k=n} \frac{x_k - 1}{x_k^2 - x_k + 1} \le \frac{n}{3} - C(a_1, a_2, \cdots, a_n); \quad x_k \ge 0 \tag{31}$$

and find the best positive constant in it under conditions on symmetric functions

$$S_1 = a_1, S_2 = a_2, \cdots, S_n = a_n \tag{32}$$

with may be some restrictions in (32) omitted.

The unconditional constant for positive numbers in (31) is $C = 0$ and the original inequality gives $C = \frac{n}{3}$ in case $n = 3$ and a single restriction $S_3 = 1$ in the list (32).

It seems that a problem to find the sharp constant in the inequality (31) under general conditions (32) is a difficult problem.

For three numbers so more inequalities of the type (31) may be considered, e.g.

1. Prove inequality (31) for positive numbers under condition $S_1 = 1$ and find the best constant for this case.

2. Prove inequality (31) for positive numbers under condition $S_2 = 1$ and find the best constant for this case.

Also combined conditions may be considered.

3. Prove inequality (31) for positive numbers under conditions $S_1 = a, S_2 = b$ and find the best constant $C(a, b)$ in (31) for this case.

6 Symmetricity of symmetric inequalities

There are many inequalities that are written in terms of symmetric functions as $F(p,q) \le 0 \quad (\ge 0)$, where

$$p = S_1 = x+y+z,\ q = S_2 = xy+yz+zx,\quad r = S_3 = xyz = 1.$$

The following Lemma enlarge the amount of inequalities that one can obtain as a series of very complicated inequalities.

Lemma 3. If the inequality

$$F(p,q) \le 0 \quad (\ge 0)$$

holds true , then the following inequalities are satisfied :

$$(i) \qquad F(q,p) \le 0 \quad (\ge 0),$$

and

$$(ii) \quad F\left(q^2-2p\ ,\ p^2-2q\right) \le 0 \qquad (\ge 0)\ .$$

Proof. (i). Assume that

$$F(p,q) = F(x+y+z, xy+yz+zx) \ge 0.$$

Using transformations

$$x \to xy, y \to yz, z \to zx$$

we obtain

$$F(p,q) = F(xy+yz+zx,\ xyyz+yzzx+zxxy) =$$

$$= F(xy+yz+zx,\ x+y+z) = F(q,p) \ge 0.$$

Notice that we can also use transformations

$$x \to \frac{1}{x}, y \to \frac{1}{y}, z \to \frac{1}{z}.$$

(ii). Now assume that

$$F(p,q) = F(x+y+z, xy+yz+zx) \ge 0.$$

Using transformations

$$x \to \frac{xy}{z}, y \to \frac{yz}{x}, z \to \frac{zx}{y}$$

we derive

$$\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} = x^2y^2+y^2z^2+z^2x^2 =$$

$$= (xy+yz+zx)^2 - 2(x+y+z) = q^2-2p\ .$$

Also it follows

$$\frac{xy}{z}\frac{yz}{x} + \frac{yz}{x}\frac{zx}{y} + \frac{zx}{y}\frac{xy}{z} =$$

$$= \frac{y}{zx} + \frac{z}{xy} + \frac{x}{yz} = x^2+y^2+z^2$$

$$= (x+y+z)^2 - 2\,(xy+yz+zx\ ) = p^2 - 2q\ \ .$$

The proof is complete.

At the end we propose an unsolved problem.

Problem. Find all possible non–negative values of four variables x_1, x_2, x_3, x_4 with restriction $x_1 \cdot x_2 \cdot x_3 \cdot x_4 = 1$ for which the next inequality holds

$$\sum_{k=1}^{4} \frac{x_k - 1}{x_k^2 - x_k + 1} \le 0, \tag{33}$$

As we know from the example at the beginning of the paper the inequality (33) is not true for all such values, e.g. it fails for $x_1 = x_2 = x_3 = 2, x_4 = 1/8$.

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