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THE QUADRATIC WEIGHTED GEOMETRIC MEAN FOR BOUNDED LINEAR OPERATORS IN HILBERT SPACES

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Abstract. In this paper we introduce the *quadratic weighted geometric mean*

$$T ⓢ_\nu V := \left| \left| VT^{-1} \right|^\nu T \right|^2$$

for bounded linear operators T, V in the Hilbert space H with T invertible and $\nu \in [0,1]$. Some fundamental inequalities concerning this mean under various assumptions for the operators involved are also provided.

1. Introduction

Assume that A, B are positive operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. The *weighted operator arithmetic mean* for the pair (A, B) is defined by

$$A \nabla_\nu B := (1-\nu) A + \nu B.$$

In 1980, Kubo & Ando, [9] introduced the *weighted operator geometric mean* for the pair (A, B) with A positive and invertible and B positive by

$$A \sharp_\nu B := A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^\nu A^{1/2}, \ \nu \in [0,1].$$

If A, B are positive invertible operators then $A \sharp_\nu B$ can be extended for any real ν. We can also consider the *weighted operator harmonic mean* defined by (see for instance [9])

$$A !_\nu B := \left((1-\nu) A^{-1} + \nu B^{-1} \right)^{-1}, \ \nu \in [0,1].$$

We have the following fundamental operator means inequalities

$$A !_\nu B \le A \sharp_\nu B \le A \nabla_\nu B, \ \nu \in [0,1] \tag{1.1}$$

for any A, B positive invertible operators. For $\nu = \frac{1}{2}$, we denote the above means by $A \nabla B$, $A \sharp B$ and $A ! B$.

The weighted operator geometric mean enjoys the following important properties (see for instance [12, p. 146])

$$(tA + (1-t) C) \sharp_\nu (tB + (1-t) D) \ge tA \sharp_\nu B + (1-t) C \sharp_\nu D \tag{1.2}$$

and

$$(A + C) \sharp_\nu (B + D) \ge A \sharp_\nu B + C \sharp_\nu D \tag{1.3}$$

for any positive invertible operators A, B, C, D and ν, $t \in [0,1]$. These mean that the mapping

$$\sharp_\nu : (A, B) \to A \sharp_\nu B$$

is *operator concave* and *superadditive* in the pair (A, B).

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For positive invertible operators A, B, C, D such that $A \geq C$ and $B \geq D$, we also have

$$A\sharp_\nu B \geq C\sharp_\nu D. \tag{1.4}$$

This means that the mapping $\sharp_\nu$ is *operator monotone* in the pair (A, B).

If $KC \geq A \geq kC$ and $KD \geq B \geq kD$ for some positive constants k, K then we have

$$KC\sharp_\nu D \geq A\sharp_\nu B \geq kC\sharp_\nu D. \tag{1.5}$$

If A, B are positive invertible operators we have

$$B\sharp_{1-\nu}A = A\sharp_\nu B$$

for any real ν.

For a bounded linear operator T we define the modulus of it by $|T| := (T^*T)^{1/2}$ where T^* is the adjoint operator.

Since B is positive, then

$$\begin{aligned} A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2} &= A^{1/2}\left(A^{-1/2}B^{1/2}B^{1/2}A^{-1/2}\right)^\nu A^{1/2} \\ &= A^{1/2}\left|B^{1/2}A^{-1/2}\right|^{2\nu} A^{1/2} = \left|\left|B^{1/2}A^{-1/2}\right|^\nu A^{1/2}\right|^2 \end{aligned}$$

and we can express the weighted geometric mean in terms of modulus as

$$A\sharp_\nu B = \left|\left|B^{1/2}A^{-1/2}\right|^\nu A^{1/2}\right|^2.$$

Now, if T, $V \in \mathcal{B}(H)$, the Banach algebra of bounded linear operators on H, with T invertible, then we have

$$|T|\sharp_\nu |V| = \left|\left||V|^{1/2}|T|^{-1/2}\right|^\nu |T|^{1/2}\right|^2$$

and

$$|T|^2\sharp_\nu |V|^2 = \left|\left||V||T|^{-1}\right|^\nu |T|\right|^2 = |T|\left||V||T|^{-1}\right|^{2\nu}|T| = |T|\left(|T|^{-1}|V|^2|T|^{-1}\right)^\nu |T|$$

and by the operator means inequalities (1.5) we then have the modulus inequalities

$$\begin{aligned} (1-\nu)|T| + \nu|V| &\geq \left|\left||V|^{1/2}|T|^{-1/2}\right|^\nu |T|^{1/2}\right|^2 \\ &\geq \left((1-\nu)|T|^{-1} + \nu|V|^{-1}\right)^{-1} \end{aligned} \tag{1.6}$$

and

$$(1-\nu)|T|^2 + \nu|V|^2 \geq \left|\left||V||T|^{-1}\right|^\nu |T|\right|^2 \geq \left((1-\nu)|T|^{-2} + \nu|V|^{-2}\right)^{-1} \tag{1.7}$$

for any $\nu \in [0,1]$. For the last inequalities in (1.6) and (1.7) we need to assume that both T and V are invertible.

If $(T_1, ..., T_n)$ is an n-tuple of invertible bounded linear operators and $(p_1, ..., p_n)$ a probability distribution then by (1.6) we have

$$(1-\nu)|T_i| + \nu|T_j| \geq \left|\left||T_j|^{1/2}|T_i|^{-1/2}\right|^\nu |T_i|^{1/2}\right|^2 \geq \left((1-\nu)|T_i|^{-1} + \nu|T_j|^{-1}\right)^{-1}$$

for any $i,\ j\in\{1,...,n\}$ and by multiplying with $p_i,\ p_j\geq 0$ and summing over $i,\ j$ from 1 to n we get

$$\begin{aligned}(1.8)\qquad \sum_{i=1}^{n}p_i\left|T_i\right| &\geq \sum_{i,j=1}^{n}p_ip_j\left|\left||T_j|^{1/2}\left|T_i\right|^{-1/2}\right|^{\nu}\left|T_i\right|^{1/2}\right|^2\\ &\geq \sum_{i,j=1}^{n}p_ip_j\left((1-\nu)\left|T_i\right|^{-1}+\nu\left|T_j\right|^{-1}\right)^{-1}\geq\left(\sum_{i=1}^{n}p_i\left|T_i\right|^{-1}\right)^{-1}\end{aligned}$$

for any $\nu\in[0,1]$. We notice that, we used the operator convexity of the function $f(t)=t^{-1}$, $t>0$ and Jensen's inequality to get the last inequality in (1.8). This improves the known inequality between the first and last term.

From (1.7) we also have

$$\begin{aligned}(1.9)\qquad \sum_{i=1}^{n}p_i\left|T_i\right|^2 &\geq \sum_{i,j=1}^{n}p_ip_j\left|\left||T_j|\left|T_i\right|^{-1}\right|^{\nu}\left|T_i\right|\right|^2\\ &\geq \sum_{i,j=1}^{n}p_ip_j\left((1-\nu)\left|T_i\right|^{-2}+\nu\left|T_j\right|^{-2}\right)^{-1}\geq\left(\sum_{i=1}^{n}p_i\left|T_i\right|^{-2}\right)^{-1}\end{aligned}$$

for any $\nu\in[0,1]$. This improves the known inequality between the first and last term.

We denote by $\mathcal{B}^{-1}(H)$ the class of all bounded linear invertible operators on H. For $T\in\mathcal{B}^{-1}(H)$ and $V\in\mathcal{B}(H)$ we define the *quadratic weighted operator geometric mean* of (T,V) by

$$(1.10)\qquad T\circledS_{\nu}V:=T^{*}\left((T^{*})^{-1}V^{*}VT^{-1}\right)^{\nu}T=T^{*}\left|VT^{-1}\right|^{2\nu}T=\left|\left|VT^{-1}\right|^{\nu}T\right|^2$$

for $\nu\geq 0$. For $V\in\mathcal{B}^{-1}(H)$ we can also extend the definition (1.10) for $\nu<0$.

For $\nu=\frac{1}{2}$ we denote

$$(1.11)\qquad T\circledS V:=\left|\left|VT^{-1}\right|^{1/2}T\right|^2=T^{*}\left|VT^{-1}\right|T=T^{*}\left((T^{*})^{-1}V^{*}VT^{-1}\right)^{1/2}T.$$

In the following, we establish some inequalities between the operators $|T|^2\nabla_{\nu}|V|^2$, $T\circledS_{\nu}V$ and $|T|^2!_{\nu}|V|^2$. Upper and lower bounds for the difference

$$|T|^2\nabla_{\nu}|V|^2-T\circledS_{\nu}V$$

as well as multiplicative inequalities of the form

$$\phi T\circledS_{\nu}V\leq|T|^2\nabla_{\nu}|V|^2\leq\Phi T\circledS_{\nu}V$$

with appropriate positive constants ϕ and Φ and under various assumptions for the operators involved, are also given.

2. Fundamental Facts

We have the following representation for the quadratic weighted operator geometric mean:

Lemma 1. *For any $T,\ V\in B^{-1}(H)$ any real λ we have that*

$$(2.1)\qquad T\circledS_{\lambda}V=|T|^2\sharp_{\lambda}|V|^2.$$

Proof. We use the following equality obtained by Furuta in [5] for any positive invertible operator A and any invertible operator B

$$(BAB^{*})^{\lambda}=BA^{1/2}\left(A^{1/2}B^{*}BA^{1/2}\right)^{\lambda-1}A^{1/2}B^{*}. \tag{2.2}$$

His proof is based on the polar decomposition of the invertible operator $BA^{1/2}$.

If we take $B=(T^{*})^{-1}$ and $A=|V|^{2}:=V^{*}V$ in (2.2), then we get

$$\begin{aligned}\left((T^{*})^{-1}|V|^{2}T^{-1}\right)^{\lambda}&=(T^{*})^{-1}|V|\left(|V|T^{-1}(T^{*})^{-1}|V|\right)^{\lambda-1}|V|T^{-1}\\&=(T^{*})^{-1}|V|\left(|V|(T^{*}T)^{-1}|V|\right)^{\lambda-1}|V|T^{-1}\\&=(T^{*})^{-1}|V|\left(|V||T|^{-2}|V|\right)^{\lambda-1}|V|T^{-1}.\end{aligned}$$

If we multiply at left by T^{*} and at right by T, we get

$$T^{*}\left((T^{*})^{-1}|V|^{2}T^{-1}\right)^{\lambda}T=|V|\left(|V||T|^{-2}|V|\right)^{\lambda-1}|V|=|V|\left(|V|^{-1}|T|^{2}|V|^{-1}\right)^{1-\lambda}|V|,$$

which means that

$$Tⓢ_{\lambda}V=|V|^{2}\sharp_{1-\lambda}|T|^{2}.$$

Since, from the theory of extended operator geometric mean, we have

$$|V|^{2}\sharp_{1-\lambda}|T|^{2}=|T|^{2}\sharp_{\lambda}|V|^{2},$$

the desired representation (2.1) is thus obtained. $\square$

We have the following fundamental inequalities:

Theorem 1. *For $T,\ V\in\mathcal{B}^{-1}(H)$ we have for $\nu\in[0,1]$ that*

$$|T|^{2}\nabla_{\nu}|V|^{2}\geq Tⓢ_{\nu}V\geq|T|^{2}!_{\nu}|V|^{2}. \tag{2.3}$$

In particular, we have

$$|T|^{2}\nabla|V|^{2}\geq TⓢV\geq|T|^{2}!|V|^{2} \tag{2.4}$$

for $T,\ V\in\mathcal{B}^{-1}(H)$.

Proof. 1. Follows by Lemma 1 and the inequalities for operator means

$$|T|^{2}\nabla_{\nu}|V|^{2}\geq|T|^{2}\sharp_{\lambda}|V|^{2}\geq|T|^{2}!_{\nu}|V|^{2},$$

where $T,\ V\in\mathcal{B}^{-1}(H)$ and $\nu\in[0,1]$.

2. A direct proof is as follows.

For $x>0$ and $\nu\in[0,1]$ we have the *scalar arithmetic mean-geometric mean inequality*

$$1-\nu+\nu x\geq x^{\nu}.$$

Using the continuous functional calculus for the selfadjoint operator $X\geq0$, we have

$$(1-\nu)1_{H}+\nu X\geq X^{\nu}. \tag{2.5}$$

If $T,\ V\in\mathcal{B}^{-1}(H)$ then the operator $X=\left|VT^{-1}\right|^{2}$ is selfadjoint and positive and by (2.5) we have

$$(1-\nu)1_{H}+\nu\left|VT^{-1}\right|^{2}\geq\left(\left|VT^{-1}\right|^{2}\right)^{\nu} \tag{2.6}$$

for $\nu\in[0,1]$.

It is well know that, if $P \geq 0$ then by multiplying at left with T^* and at right with T where $T \in \mathcal{B}(H)$ we have that $T^*PT \geq 0$. If A, B are selfadjoint operators with $A \geq B$ then for any $T \in \mathcal{B}(H)$ we have $T^*AT \geq T^*BT$.

Therefore, by (2.6) we get

$$T^*\left[(1-\nu)1_H + \nu\left|VT^{-1}\right|^2\right]T \geq T^*\left(\left|VT^{-1}\right|^2\right)^\nu T \tag{2.7}$$

for $\nu \in [0,1]$.

Since

$$\begin{aligned} T^*\left[(1-\nu)1_H + \nu\left|VT^{-1}\right|^2\right]T &= (1-\nu)T^*T + \nu T^*\left|VT^{-1}\right|^2 T \\ &= (1-\nu)T^*T + \nu T^*(T^*)^{-1}V^*VT^{-1}T \\ &= (1-\nu)|T|^2 + \nu|V|^2 = |T|^2\nabla_\nu|V|^2 \end{aligned}$$

and

$$T^*\left(\left|VT^{-1}\right|^2\right)^\nu T = T^*\left(\left|VT^{-1}\right|\right)^{2\nu}T = Tⓢ_\nu V,$$

then by (2.7) we get the first inequality in (2.3).

For $x > 0$ we have the *geometric mean-harmonic mean inequality*

$$x^\nu \geq \left(1-\nu+\nu x^{-1}\right)^{-1}$$

for $\nu \in [0,1]$.

Using the continuous functional calculus for the invertible positive operator X, we have

$$X^\nu \geq \left(1-\nu+\nu(X)^{-1}\right)^{-1} \tag{2.8}$$

for $\nu \in [0,1]$.

If T, $V \in \mathcal{B}^{-1}(H)$ then $X = \left|VT^{-1}\right|^2 \in \mathcal{B}^{-1}(H)$ and

$$X^{-1} = \left(\left|VT^{-1}\right|^2\right)^{-1} = \left((T^*)^{-1}|V|^2T^{-1}\right)^{-1} = T|V|^{-2}T^*.$$

Therefore

$$\begin{aligned} \left(1-\nu+\nu T|V|^{-2}T^*\right)^{-1} &= \left((1-\nu)TT^{-1}(T^*)^{-1}T^* + \nu T|V|^{-2}T^*\right)^{-1} \\ &= \left((1-\nu)T|T|^{-2}T^* + \nu T|V|^{-2}T^*\right)^{-1} \\ &= \left(T\left[(1-\nu)|T|^{-2} + \nu|V|^{-2}\right]T^*\right)^{-1} \\ &= (T^*)^{-1}\left((1-\nu)|T|^{-2} + \nu|V|^{-2}\right)^{-1}T^{-1} \end{aligned}$$

and by (2.8) for $X = \left|VT^{-1}\right|^2$ we get

$$\left|VT^{-1}\right|^{2\nu} \geq (T^*)^{-1}\left((1-\nu)|T|^{-2} + \nu|V|^{-2}\right)^{-1}T^{-1} \tag{2.9}$$

for $\nu \in [0,1]$.

By multiplying the inequality (2.9) at left with T^* and at right with T we get the second inequality in (2.3). □

Remark 1. *We observe that, by using the argument from Proof 2 above, we can state the first inequality in (2.3) also for any* $V \in \mathcal{B}(H)$ *and* $T \in \mathcal{B}^{-1}(H)$. *If we take the inner product in* $|T|^2 \nabla_\nu |V|^2 \geq T \circledS_\nu V$, *for* $V \in \mathcal{B}(H)$ *and* $T \in \mathcal{B}^{-1}(H)$, *then we get the vector inequality of interest*

$$(1-\nu)\|Tx\|^2 + \nu\|Vx\|^2 \geq \left\| \left|VT^{-1}\right|^\nu Tx \right\|^2 \tag{2.10}$$

for any $x \in H$ *and* $\nu \in [0,1]$ *and in particular*

$$\frac{1}{2}\left(\|Tx\|^2 + \|Vx\|^2\right) \geq \left\| \left|VT^{-1}\right|^{1/2} Tx \right\|^2. \tag{2.11}$$

The following norm inequality should also be noticed,

$$\left\|(1-\nu)|T|^2 + \nu|V|^2\right\| \geq \left\| \left|VT^{-1}\right|^\nu T \right\|^2 \tag{2.12}$$

for $V \in \mathcal{B}(H)$, $T \in \mathcal{B}^{-1}(H)$ *and* $\nu \in [0,1]$.

We can also define the following weighted operator means for $\nu \in [0,1]$ and the operators T, V as above by

$$T \circledS_\nu^{1/2} V \ : \ = (T \circledS_\nu V)^{1/2} = \left| \left|VT^{-1}\right|^{1/2} T \right|,$$

$$T \nabla_\nu^{1/2} V \ : \ = \left(|T|^2 \nabla_\nu |V|^2\right)^{1/2} = \left((1-\nu)|T|^2 + \nu|V|^2\right)^{1/2}$$

and

$$T !_\nu^{1/2} V := \left(|T|^2 !_\nu |V|^2\right)^{1/2} = \left((1-\nu)|T|^{-2} + \nu|V|^{-2}\right)^{-1/2}.$$

Then by taking the square root in (2.3) we get

$$T \nabla_\nu^{1/2} V \geq T \circledS_\nu^{1/2} V \geq T !_\nu^{1/2} V \tag{2.13}$$

for any T, $V \in \mathcal{B}^{-1}(H)$ and $\nu \in [0,1]$.

Corollary 1. *If* $(T_1, ..., T_n)$ *is an* n*-tuple of invertible bounded linear operators and* $(p_1, ..., p_n)$ *a probability distribution, then for any* $\nu \in [0,1]$ *we have*

$$\begin{aligned} \sum_{i=1}^n p_i |T_i|^2 &\geq \sum_{i,j=1}^n p_i p_j \left| \left|T_j T_i^{-1}\right|^\nu T_i \right|^2 \\ &\geq \sum_{i,j=1}^n p_i p_j \left((1-\nu)|T_i|^{-2} + \nu|T_j|^{-2}\right)^{-1} \geq \left(\sum_{i=1}^n p_i |T_i|^{-2}\right)^{-1}. \end{aligned} \tag{2.14}$$

In particular, we have

$$\begin{aligned} \sum_{i=1}^n p_i |T_i|^2 &\geq \sum_{i,j=1}^n p_i p_j \left| \left|T_j T_i^{-1}\right|^{1/2} T_i \right|^2 \\ &\geq \sum_{i,j=1}^n p_i p_j \left((1-\nu)|T_i|^{-2} + \nu|T_j|^{-2}\right)^{-1} \geq \left(\sum_{i=1}^n p_i |T_i|^{-2}\right)^{-1}. \end{aligned} \tag{2.15}$$

Proof. Follows from (1.9) and Lemma 1. □

Remark 2. *If we take the inner product in the first inequality in (2.14) we get*

$$\sum_{i=1}^{n} p_i \left\| T_i x \right\|^2 \geq \sum_{i,j=1}^{n} p_i p_j \left\| \left| T_j T_i^{-1} \right|^{\nu} T_i x \right\|^2 \tag{2.16}$$

for any $x \in H$ and $\nu \in [0,1]$. In particular, we have

$$\sum_{i=1}^{n} p_i \left\| T_i x \right\|^2 \geq \sum_{i,j=1}^{n} p_i p_j \left\| \left| T_j T_i^{-1} \right|^{1/2} T_i x \right\|^2 \tag{2.17}$$

for any $x \in H$ and $\nu \in [0,1]$.

Using the Cauchy-Bunyakovsky-Schwarz inequality and the generalized triangle inequality we have

$$\sum_{i,j=1}^{n} p_i p_j \left\| \left| T_j T_i^{-1} \right|^{\nu} T_i x \right\|^2 \geq \left(\sum_{i,j=1}^{n} p_i p_j \left\| \left| T_j T_i^{-1} \right|^{\nu} T_i x \right\| \right)^2 \geq \left\| \sum_{i,j=1}^{n} p_i p_j \left| T_j T_i^{-1} \right|^{\nu} T_i x \right\|^2,$$

which by (2.16) produces the vector inequality

$$\sum_{i=1}^{n} p_i \left\| T_i x \right\|^2 \geq \left\| \sum_{i,j=1}^{n} p_i p_j \left| T_j T_i^{-1} \right|^{\nu} T_i x \right\|^2, \tag{2.18}$$

for any $x \in H$ and $\nu \in [0,1]$.

By taking the supremum in this inequality over $x \in H$, $\|x\| = 1$ we also get the operator norm inequality

$$\left\| \sum_{i=1}^{n} p_i \left| T_i \right|^2 \right\| \geq \left\| \sum_{i,j=1}^{n} p_i p_j \left| T_j T_i^{-1} \right|^{\nu} T_i \right\|^2 \tag{2.19}$$

for any $\nu \in [0,1]$.

3. Improvements and Refinements

Jensen's inequality for convex function is one of the most known and extensively used inequality in various filed of Modern Mathematics. It is a source of many classical inequalities including the generalized triangle inequality, the arithmetic mean-geometric mean-harmonic mean inequality, the positivity of *relative entropy* in Information Theory, Schannon's inequality, Ky Fan's inequality, Levinson's inequality and other results. For classical and contemporary developments related to the Jensen inequality, see [2], [11], [13] and [4] where further references are provided.

To be more specific, we recall that, if X is a linear space and $C \subseteq X$ a convex subset in X, then for any convex function $f : C \to \mathbb{R}$ and any $z_i \in C, r_i \geq 0$ for $i \in \{1, ..., k\}, k \geq 2$ with $\sum_{i=1}^{k} r_i = R_k > 0$ one has the *weighted Jensen's inequality:*

$$\frac{1}{R_k} \sum_{i=1}^{k} r_i f(z_i) \geq f\left(\frac{1}{R_k} \sum_{i=1}^{k} r_i z_i \right). \tag{J}$$

If $f : C \to \mathbb{R}$ is strictly convex and $r_i > 0$ for $i \in \{1, ..., k\}$ then the equality case hods in (J) if and only if $z_1 = ... = z_n$.

By $\mathcal{P}_n$ we denote the set of all nonnegative n-tuples $(p_1, ..., p_n)$ with the property that $\sum_{i=1}^{n} p_i = 1$. Consider the *normalised Jensen functional*

$$\mathcal{J}_n(f, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \geq 0,$$

where $f : C \to \mathbb{R}$ be a convex function on the convex set C and $\mathbf{x} = (x_1, ..., x_n) \in C^n$ and $\mathbf{p} \in \mathcal{P}_n$.

The following result holds [3]:

Lemma 2. *If $\mathbf{p}$, $\mathbf{q} \in \mathcal{P}_n$, $q_i > 0$ for each $i \in \{1, ..., n\}$ then*

$$(0 \leq) \min_{1 \leq i \leq n} \left\{\frac{p_i}{q_i}\right\} \mathcal{J}_n(f, \mathbf{x}, \mathbf{q}) \leq \mathcal{J}_n(f, \mathbf{x}, \mathbf{p}) \leq \max_{1 \leq i \leq n} \left\{\frac{p_i}{q_i}\right\} \mathcal{J}_n(f, \mathbf{x}, \mathbf{q}). \tag{3.1}$$

In the case $n = 2$, if we put $p_1 = 1 - p$, $p_2 = p$, $q_1 = 1 - q$ and $q_2 = q$ with $p \in [0, 1]$ and $q \in (0, 1)$ then by (3.1) we get

$$\begin{aligned} &\min\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\} [(1-q) f(x) + q f(y) - f((1-q)x + qy)] \\ &\leq (1-p) f(x) + p f(y) - f((1-p)x + py) \\ &\leq \max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\} [(1-q) f(x) + q f(y) - f((1-q)x + qy)] \end{aligned} \tag{3.2}$$

for any x, $y \in C$.

If we take $q = \frac{1}{2}$ in (3.2), then we get

$$\begin{aligned} &2 \min\{t, 1-t\} \left[\frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right)\right] \\ &\leq (1-t) f(x) + t f(y) - f((1-t)x + ty) \\ &\leq 2 \max\{t, 1-t\} \left[\frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right)\right] \end{aligned} \tag{3.3}$$

for any x, $y \in C$ and $t \in [0, 1]$.

We consider the scalar weighted arithmetic, geometric and harmonic means defined by

$$A_\nu(a, b) := (1-\nu) a + \nu b,\ G_\nu(a, b) := a^{1-\nu} b^\nu \text{ and } H_\nu(a, b) = A_\nu^{-1}\left(a^{-1}, b^{-1}\right)$$

where a, $b > 0$ and $\nu \in [0, 1]$.

If we take the convex function $f : \mathbb{R} \to (0, \infty)$, $f(x) = \exp(\alpha x)$, with $\alpha \neq 0$, then we have from (3.2) that

$$\begin{aligned} &\min\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\} [A_q(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A_q(a, b))] \\ &\leq A_p(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A_p(a, b)) \\ &\leq \max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\} [A_q(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A_q(a, b))] \end{aligned} \tag{3.4}$$

for any $p \in [0, 1]$ and $q \in (0, 1)$ and any x, $y \in \mathbb{R}$.

For $q=\frac{1}{2}$ we have by (3.4) that

$$\begin{aligned} (3.5)\qquad & 2\min\{p,1-p\}\left[A\left(\exp\left(\alpha x\right),\exp\left(\alpha y\right)\right)-\exp\left(\alpha A\left(a,b\right)\right)\right] \\ & \leq A_p\left(\exp\left(\alpha x\right),\exp\left(\alpha y\right)\right)-\exp\left(\alpha A_p\left(a,b\right)\right) \\ & \leq 2\max\{p,1-p\}\left[A\left(\exp\left(\alpha x\right),\exp\left(\alpha y\right)\right)-\exp\left(\alpha A\left(a,b\right)\right)\right] \end{aligned}$$

for any $p\in[0,1]$ and any x, $y\in\mathbb{R}$.

If we take $x=\ln a$ and $y=\ln b$ in (3.4), then we get

$$\begin{aligned} (3.6)\qquad & \min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left[A_q\left(a^\alpha,b^\alpha\right)-G_q^\alpha\left(a,b\right)\right] \\ & \leq A_p\left(a^\alpha,b^\alpha\right)-G_p^\alpha\left(a,b\right) \\ & \leq \max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left[A_q\left(a^\alpha,b^\alpha\right)-G_q^\alpha\left(a,b\right)\right] \end{aligned}$$

for any a, $b>0$, for any $p\in[0,1]$, $q\in(0,1)$ and $\alpha\neq0$.

For $q=\frac{1}{2}$ we have by (3.6) that

$$\begin{aligned} (3.7)\qquad \min\{p,1-p\}\left(b^{\frac{\alpha}{2}}-a^{\frac{\alpha}{2}}\right)^2 & \leq A_p\left(a^\alpha,b^\alpha\right)-G_p^\alpha\left(a,b\right) \\ & \leq \max\{p,1-p\}\left(b^{\frac{\alpha}{2}}-a^{\frac{\alpha}{2}}\right)^2 \end{aligned}$$

for any a, $b>0$, for any $p\in[0,1]$ and $\alpha\neq0$.

For $\alpha=1$ we get from (3.7) that

$$\begin{aligned} (3.8)\qquad \min\{p,1-p\}\left(\sqrt{b}-\sqrt{a}\right)^2 & \leq A_p\left(a,b\right)-G_p\left(a,b\right) \\ & \leq \max\{p,1-p\}\left(\sqrt{b}-\sqrt{a}\right)^2 \end{aligned}$$

for any a, $b>0$ and for any $p\in[0,1]$, which are the inequalities obtained by Kittaneh and Manasrah in [7] and [8].

For $\alpha=1$ in (3.6) we obtain

$$\begin{aligned} (3.9)\qquad & \min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left[A_q\left(a,b\right)-G_q\left(a,b\right)\right] \\ & \leq A_p\left(a,b\right)-G_p\left(a,b\right) \\ & \leq \max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left[A_q\left(a,b\right)-G_q\left(a,b\right)\right], \end{aligned}$$

for any a, $b>0$, for any $p\in[0,1]$, which is the inequality (2.1) from [1] in the particular case $\lambda=1$ in a slightly more general form for the weights p, q.

We have the following refinement and reverse for the inequality (2.3):

Theorem 2. *If $T\in\mathcal{B}^{-1}(H)$ and $V\in\mathcal{B}(H)$, we have for $p\in[0,1]$ and $q\in(0,1)$ that*

$$\begin{aligned} (3.10)\qquad & \min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left(\left|T\right|^2\nabla_q\left|V\right|^2-T\circledS_qV\right) \\ & \leq \left|T\right|^2\nabla_p\left|V\right|^2-T\circledS_pV \\ & \leq \max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left(\left|T\right|^2\nabla_q\left|V\right|^2-T\circledS_qV\right). \end{aligned}$$

In particular, we have

$$2\min\{p,1-p\}\left(|T|^2\nabla|V|^2-TⓢV\right)\le|T|^2\nabla_p|V|^2-Tⓢ_pV\le2\max\{p,1-p\}\left(|T|^2\nabla|V|^2-TⓢV\right), \tag{3.11}$$

for any $p\in[0,1]$.

Proof. From the inequality (3.9) for $a=1$ and $b=x\ge0$ we have

$$\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}(1-q+qx-x^q)\le1-p+px-x^p\le\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}(1-q+qx-x^q), \tag{3.12}$$

where $p\in[0,1]$ and $q\in(0,1)$.

Using the continuous functional calculus for nonnegative operator X we have

$$\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}((1-q)1_H+qX-X^q)\le(1-p)1_H+pX-X^p\le\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}((1-q)1_H+qX-X^q), \tag{3.13}$$

where $p\in[0,1]$ and $q\in(0,1)$.

If $T\in\mathcal{B}^{-1}(H)$ and $V\in\mathcal{B}(H)$ then the operator $X=\left|VT^{-1}\right|^2$ is selfadjoint and nonnegative and by (3.13) we have

$$\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left((1-q)1_H+q\left|VT^{-1}\right|^2-\left(\left|VT^{-1}\right|^2\right)^q\right)\le(1-p)1_H+p\left|VT^{-1}\right|^2-\left(\left|VT^{-1}\right|^2\right)^p\le\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left((1-q)1_H+q\left|VT^{-1}\right|^2-\left(\left|VT^{-1}\right|^2\right)^q\right), \tag{3.14}$$

where $p\in[0,1]$ and $q\in(0,1)$.

By multiplying the inequality (3.14) at left with T^* and at right with T we get the desired result (3.10). □

We observe that, by taking the inner product in (3.10) we have the vector inequalities

$$\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left((1-q)\|Tx\|^2+q\|Vx\|^2-\left\|\left|VT^{-1}\right|^q Tx\right\|^2\right)\le(1-p)\|Tx\|^2+p\|Vx\|^2-\left\|\left|VT^{-1}\right|^p Tx\right\|^2\le\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left((1-q)\|Tx\|^2+q\|Vx\|^2-\left\|\left|VT^{-1}\right|^q Tx\right\|^2\right) \tag{3.15}$$

for any $p\in[0,1]$, $q\in(0,1)$ and $x\in H$.

In particular, we have

$$\begin{aligned}(3.16)\qquad & 2\min\{p,1-p\}\left(\frac{1}{2}\left(\|Tx\|^2+\|Vx\|^2\right)-\left\|\left|VT^{-1}\right|^{1/2}Tx\right\|^2\right)\\ &\le(1-p)\|Tx\|^2+p\|Vx\|^2-\left\|\left|VT^{-1}\right|^{p}Tx\right\|^2\\ &\le 2\max\{p,1-p\}\left(\frac{1}{2}\left(\|Tx\|^2+\|Vx\|^2\right)-\left\|\left|VT^{-1}\right|^{1/2}Tx\right\|^2\right),\end{aligned}$$

for any $p\in[0,1]$ and $x\in H$.

Remark 3. *If A is positive and invertible and B is positive, then by taking $T=A^{1/2}$ and $V=B^{1/2}$ in (3.10) and (3.11) we get*

$$\begin{aligned}(3.17)\qquad \min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left(A\nabla_q B-A\sharp_q B\right) &\le A\nabla_p B-A\sharp_p B\\ &\le \max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left(A\nabla_q B-A\sharp_q B\right),\end{aligned}$$

for any $p\in[0,1]$ and $q\in(0,1)$.

In particular, for $q=1/2$ we have

$$\begin{aligned}(3.18)\qquad 2\min\{p,1-p\}\left(A\nabla B-A\sharp B\right) &\le A\nabla_p B-A\sharp_p B\\ &\le 2\max\{p,1-p\}\left(A\nabla B-A\sharp B\right),\end{aligned}$$

for any $p\in[0,1]$. The inequality (3.18) has been obtained in [6].

Corollary 2. *If $(T_1,...,T_n)$ is an n-tuple of invertible bounded linear operators and $(p_1,...,p_n)$ a probability distribution then for any $p\in[0,1]$ and $q\in(0,1)$ we have*

$$\begin{aligned}(3.19)\qquad & \min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left(\sum_{i=1}^n p_i|T_i|^2-\sum_{i,j=1}^n p_ip_j\left|\left|V_jT_i^{-1}\right|^q T_i\right|^2\right)\\ &\le\sum_{i=1}^n p_i|T_i|^2-\sum_{i,j=1}^n p_ip_j\left|\left|V_jT_i^{-1}\right|^p T_i\right|^2\\ &\le \max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left(\sum_{i=1}^n p_i|T_i|^2-\sum_{i,j=1}^n p_ip_j\left|\left|V_jT_i^{-1}\right|^q T_i\right|^2\right).\end{aligned}$$

In particular, we have

$$\begin{aligned}(3.20)\qquad & 2\min\{p,1-p\}\left(\sum_{i=1}^n p_i|T_i|^2-\sum_{i,j=1}^n p_ip_j\left|\left|V_jT_i^{-1}\right|^{1/2} T_i\right|^2\right)\\ &\le\sum_{i=1}^n p_i|T_i|^2-\sum_{i,j=1}^n p_ip_j\left|\left|V_jT_i^{-1}\right|^p T_i\right|^2\\ &\le 2\max\{p,1-p\}\left(\sum_{i=1}^n p_i|T_i|^2-\sum_{i,j=1}^n p_ip_j\left|\left|V_jT_i^{-1}\right|^{1/2} T_i\right|^2\right),\end{aligned}$$

for any $p\in[0,1]$.

By taking the inner product in (3.19) we also have the vector inequality

$$\begin{aligned} &\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left(\sum_{i=1}^{n}p_i\left\|T_ix\right\|^2-\sum_{i,j=1}^{n}p_ip_j\left\|\left|T_jT_i^{-1}\right|^q T_ix\right\|^2\right) \\ &\leq \sum_{i=1}^{n}p_i\left\|T_ix\right\|^2-\sum_{i,j=1}^{n}p_ip_j\left\|\left|T_jT_i^{-1}\right|^p T_ix\right\|^2 \\ &\leq \max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left(\sum_{i=1}^{n}p_i\left\|T_ix\right\|^2-\sum_{i,j=1}^{n}p_ip_j\left\|\left|T_jT_i^{-1}\right|^q T_ix\right\|^2\right), \end{aligned} \tag{3.21}$$

for any $p\in[0,1]$, $q\in(0,1)$ and $x\in H$.

In particular,

$$\begin{aligned} &2\min\{p,1-p\}\left(\sum_{i=1}^{n}p_i\left\|T_ix\right\|^2-\sum_{i,j=1}^{n}p_ip_j\left\|\left|T_jT_i^{-1}\right|^{1/2} T_ix\right\|^2\right) \\ &\leq \sum_{i=1}^{n}p_i\left\|T_ix\right\|^2-\sum_{i,j=1}^{n}p_ip_j\left\|\left|T_jT_i^{-1}\right|^p T_ix\right\|^2 \\ &\leq 2\max\{p,1-p\}\left(\sum_{i=1}^{n}p_i\left\|T_ix\right\|^2-\sum_{i,j=1}^{n}p_ip_j\left\|\left|T_jT_i^{-1}\right|^{1/2} T_ix\right\|^2\right), \end{aligned} \tag{3.22}$$

for any $p\in[0,1]$ and $x\in H$.

4. Inequalities Under Boundedness Conditions

If we take in (3.2) $f(x)=-\ln x$, then we get

$$\left(\frac{A_q(x,y)}{G_q(x,y)}\right)^{\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}}\leq\frac{A_p(x,y)}{G_p(x,y)}\leq\left(\frac{A_q(x,y)}{G_q(x,y)}\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} \tag{4.1}$$

for any x, $y>0$ and for any $p\in[0,1]$, $q\in(0,1)$.

This inequality is a particular case for $n=2$ of the inequality (4.2) from [3].

For $q=\frac{1}{2}$ we have by (4.1) (for $x=a$, $y=b$) that

$$\left(\frac{A(a,b)}{G(a,b)}\right)^{2\min\{p,1-p\}}\leq\frac{A_p(a,b)}{G_p(a,b)}\leq\left(\frac{A(a,b)}{G(a,b)}\right)^{2\max\{p,1-p\}} \tag{4.2}$$

for any a, $b>0$ and for any $p\in[0,1]$.

Recall that *Kantorovich's constant* $\mathcal{K}$ is defined by

$$\mathcal{K}(h):=\frac{(h+1)^2}{4h},\ h>0. \tag{K}$$

It is well known that $\mathcal{K}$ is *decreasing* on $(0,1)$ and *increasing* on $[1,\infty)$, $\mathcal{K}(h)\geq1$ for any $h>0$ and $\mathcal{K}(h)=\mathcal{K}\left(\frac{1}{h}\right)$ for any $h>0$.

The inequality (4.2) can be thus written as

$$\mathcal{K}^{\min\{p,1-p\}}\left(\frac{a}{b}\right)\leq\frac{A_p(a,b)}{G_p(a,b)}\leq\mathcal{K}^{\max\{p,1-p\}}\left(\frac{a}{b}\right). \tag{ZL}$$

The first inequality in (ZL) was obtained by Zou et al. in [14] while the second by Liao et al. [10].

For $q \in (0,1)$ we consider the function $f_q : (0,\infty) \to (0,\infty)$ defined by

$$f_q(h) := \frac{A_q(1,h)}{G_q(1,h)} = \frac{1-q+qh}{h^q} = (1-q)\, h^{-q} + q h^{1-q}.$$

The function f_q is differentiable and

$$f_q'(h) = (1-q)\, q h^{-q-1}(h-1),$$

which shows that the function f_q is *decreasing* on $(0,1)$ and *increasing* on $[1,\infty)$. We have $f_q(1) = 1$, $\lim_{h\to 0+} f_q(h) = +\infty$, $\lim_{h\to\infty} f_q(h) = +\infty$ and $f_q\left(\frac{1}{h}\right) = f_{1-q}(h)$ for any $h > 0$ and $q \in (0,1)$.

Therefore, by considering the 3 possible situations for the location of the interval $[\ell, L]$ and the number 1 we get

$$\max_{h\in[\ell,L]} f_q(h) = \begin{cases} f_q(\ell) \text{ if } L < 1, \\ \\ \max\{f_q(\ell), f_q(L)\} \text{ if } \ell \le 1 \le L, \\ \\ f_q(L) \text{ if } 1 < \ell, \end{cases} \tag{4.3}$$

$$= \begin{cases} \frac{A_q(1,\ell)}{G_q(1,\ell)} \text{ if } L < 1, \\ \\ \max\left\{\frac{A_q(1,\ell)}{G_q(1,\ell)}, \frac{A_q(1,L)}{G_q(1,L)}\right\} \text{ if } \ell \le 1 \le L, \\ \\ \frac{A_q(1,L)}{G_q(1,L)} \text{ if } 1 < \ell \end{cases}$$

and

$$\min_{h\in[\ell,L]} f_q(h) = \begin{cases} f_q(L) \text{ if } L < 1, \\ \\ 1 \text{ if } \ell \le 1 \le L \\ \\ f_q(\ell) \text{ if } 1 < \ell, \end{cases} = \begin{cases} \frac{A_q(1,L)}{G_q(1,L)} \text{ if } L < 1, \\ \\ 1 \text{ if } \ell \le 1 \le L, \\ \\ \frac{A_q(1,\ell)}{G_q(1,\ell)} \text{ if } 1 < \ell. \end{cases} \tag{4.4}$$

Lemma 3. *Let T, $V \in \mathcal{B}^{-1}(H)$ and $0 < m < M < \infty$. Then the following statements are equivalent:*

(i) The inequality

$$m\,\|Tx\| \le \|Vx\| \le M\,\|Tx\| \tag{4.5}$$

holds for any $x \in H$;

(ii) We have the operator inequality

$$m 1_H \le \left|VT^{-1}\right| \le M 1_H. \tag{4.6}$$

Proof. The inequality (4.5) is equivalent to

$$m^2\,\|Tx\|^2 \le \|Vx\|^2 \le M^2\,\|Tx\|^2$$

for any $x \in H$, namely

$$m^2 \langle T^*Tx, x\rangle \le \langle V^*Vx, x\rangle \le M^2 \langle T^*Tx, x\rangle$$

for any $x \in H$, which can be written in the operator order as

$$m^2 T^*T \le V^*V \le M^2 T^*T. \tag{4.7}$$

Since $T\in\mathcal{B}^{-1}(H)$, then the inequality (4.7) is equivalent to

$$m^2 1_H \le \left(T^{-1}\right)^* V^* V T^{-1} \le M^2 1_H,$$

namely

$$m^2 1_H \le \left|VT^{-1}\right|^2 \le M^2 1_H,$$

which in its turn is equivalent to (4.6). □

We have the following result for operators:

Theorem 3. *Let T, $V\in\mathcal{B}^{-1}(H)$ and $0<m<M<\infty$. Assume that the pair of operators (T,V) satisfies either the condition (4.5) or, equivalently, the condition (4.6). Then we have for $p\in[0,1]$ and $q\in(0,1)$ that*

$$\gamma_{p,q}(m,M)\,T\circledS_p V \le |T|^2 \nabla_p |V|^2 \le \Gamma_{p,q}(m,M)\,T\circledS_p V, \tag{4.8}$$

where

$$\Gamma_{p,q}(m,M):=\begin{cases}\left(\frac{A_q\left(1,m^2\right)}{G_q\left(1,m^2\right)}\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} & \text{if } M<1,\\ \max\left\{\left(\frac{A_q\left(1,m^2\right)}{G_q\left(1,m^2\right)}\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}},\left(\frac{A_q\left(1,M^2\right)}{G_q\left(1,M^2\right)}\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}}\right\} & \\ \text{if } m\le 1\le M, & \\ \left(\frac{A_q\left(1,M^2\right)}{G_q\left(1,M^2\right)}\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} & \text{if } 1<m\end{cases}$$

and

$$\gamma_{p,q}(m,M):=\begin{cases}\left(\frac{A_q\left(1,M^2\right)}{G_q\left(1,M^2\right)}\right)^{\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} & \text{if } M<1,\\ 1 \text{ if } m\le 1\le M, & \\ \left(\frac{A_q\left(1,m^2\right)}{G_q\left(1,m^2\right)}\right)^{\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} & \text{if } 1<m.\end{cases}$$

Proof. From the inequality (4.1) we have

$$\left(f_q(t)\right)^{\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} \le \frac{A_p(1,t)}{G_p(1,t)} \le \left(f_q(t)\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} \tag{4.9}$$

for any $t>0$, for $p\in[0,1]$ and $q\in(0,1)$.

If $t\in[\ell,L]$ then from (4.9) we have

$$\left(\min_{h\in[\ell,L]} f_q(h)\right)^{\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} \le \frac{A_p(1,t)}{G_p(1,t)} \le \left(\max_{h\in[\ell,L]} f_q(h)\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}}$$

that can be written as

$$\left(\min_{h\in[\ell,L]} f_q(h)\right)^{\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} t^p \le 1-p+pt \le t^p\left(\max_{h\in[\ell,L]} f_q(h)\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} \tag{4.10}$$

for any $t\in[\ell,L]$, for $p\in[0,1]$ and $q\in(0,1)$.

Using the functional calculus for the selfadjoint operator X with spectrum included in $[\ell, L]$ we have

$$\left(\min_{h\in[\ell,L]} f_q(h)\right)^{\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} X^p \le (1-p)\,1_H + pX \le X^p\left(\max_{h\in[\ell,L]} f_q(h)\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} \tag{4.11}$$

for $p \in [0,1]$ and $q \in (0,1)$.

Due to the condition (4.6) the operator $X = \left|VT^{-1}\right|^2$ has the spectrum included in $\left[m^2, M^2\right]$ and by (4.11) we have

$$\begin{aligned}&\left(\min_{h\in[m^2,M^2]} f_q(h)\right)^{\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} \left(\left|VT^{-1}\right|^2\right)^p \\ &\le (1-p)\,1_H + p\left|VT^{-1}\right|^2 \\ &\le \left(\left|VT^{-1}\right|^2\right)^p \left(\max_{h\in[m^2,M^2]} f_q(h)\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}},\end{aligned} \tag{4.12}$$

for $p \in [0,1]$ and $q \in (0,1)$.

By multiplying the inequality (4.12) at left with T^* and at right with T we get the desired result (4.8). □

For $q = 1/2$ in the above Theorem 3 we get

$$\gamma_p(m,M)\, T \circledS_p V \le |T|^2 \nabla_p |V|^2 \le \Gamma_p(m,M)\, T \circledS_p V, \tag{4.13}$$

where

$$\Gamma_p(m,M) := \begin{cases} \mathcal{K}^{\max\{p,1-p\}}\left(m^2\right) \text{ if } M < 1, \\ \max\left\{\mathcal{K}^{\max\{p,1-p\}}\left(m^2\right), \mathcal{K}^{\max\{p,1-p\}}\left(M^2\right)\right\} \\ \text{if } m \le 1 \le M, \\ \mathcal{K}^{\max\{p,1-p\}}\left(M^2\right) \text{ if } 1 < m \end{cases}$$

and

$$\gamma_p(m,M) := \begin{cases} \mathcal{K}^{\min\{p,1-p\}}\left(M^2\right) \text{ if } M < 1, \\ 1 \text{ if } m \le 1 \le M, \\ \mathcal{K}^{\min\{p,1-p\}}\left(m^2\right) \text{ if } 1 < m \end{cases}$$

for any $p \in [0,1]$, where $\mathcal{K}$ is Kantorovich's constant.

Now, if we take the inner product in (4.13) we also have the vector inequalities

$$\gamma_p(m,M)\left\|\left|VT^{-1}\right|^p Tx\right\|^2 \le (1-p)\|Tx\|^2 + p\|Vx\|^2 \le \Gamma_p(m,M)\left\|\left|VT^{-1}\right|^p Tx\right\|^2, \tag{4.14}$$

for any $x \in H$.

By taking the supremum over $x \in H$, $\|x\| = 1$ in (4.14) we get the operator norm inequality

$$\gamma_p(m,M)\left\|\left|VT^{-1}\right|^p T\right\|^2 \le \left\|(1-\nu)|T|^2+\nu|V|^2\right\| \le \Gamma_p(m,M)\left\|\left|VT^{-1}\right|^p T\right\|^2. \tag{4.15}$$

Remark 4. *Assume that the positive invertible operators A, B satisfy the condition $kA \le B \le KA$ for the constants $0 < k < K$. Then by multiplying both sides by $A^{-1/2}$ we get $k1_H \le A^{-1/2}BA^{-1/2} \le K1_H$ that can be written as $k1_H \le \left|B^{1/2}A^{-1/2}\right|^2 \le K1_H$ that is equivalent to $\sqrt{k}1_H \le \left|B^{1/2}A^{-1/2}\right| \le \sqrt{K}1_H$. Now, if we apply Theorem 3 for $T = A^{1/2}$, $V = B^{1/2}$, $m = \sqrt{k}$ and $M = \sqrt{K}$ then we get*

$$\delta_{p,q}(k,K)\,A\sharp_p B \le A\nabla_p B \le \Delta_{p,q}(k,K)\,A\sharp_p B, \tag{4.16}$$

where

$$\Delta_{p,q}(k,K) := \begin{cases} \left(\frac{A_q(1,k)}{G_q(1,k)}\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} & \text{if } K<1, \\ \max\left\{\left(\frac{A_q(1,k)}{G_q(1,k)}\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}}, \left(\frac{A_q(1,K)}{G_q(1,K)}\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}}\right\} \\ \text{if } k\le 1\le K, \\ \left(\frac{A_q(1,K)}{G_q(1,K)}\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} & \text{if } 1<k \end{cases}$$

and

$$\gamma_{p,q}(k,K) := \begin{cases} \left(\frac{A_q(1,K)}{G_q(1,K)}\right)^{\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} & \text{if } K<1, \\ 1 \text{ if } k\le 1\le K, \\ \left(\frac{A_q(1,k)}{G_q(1,k)}\right)^{\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} & \text{if } 1<k, \end{cases}$$

where $p \in [0,1]$ and $q \in (0,1)$.

In particular, we have for $q = 1/2$ that

$$\delta_p(k,K)\,A\sharp_p B \le A\nabla_p B \le \Delta_p(k,K)\,A\sharp_p B, \tag{4.17}$$

where

$$\Delta_p(k,K) := \begin{cases} \mathcal{K}^{\max\{p,1-p\}}(k) & \text{if } K<1, \\ \max\left\{\mathcal{K}^{\max\{p,1-p\}}(k), \mathcal{K}^{\max\{p,1-p\}}(K)\right\} \\ \text{if } k\le 1\le K, \\ \mathcal{K}^{\max\{p,1-p\}}(K) & \text{if } 1<k \end{cases}$$

and

$$\delta_p(k,K) := \begin{cases} \mathcal{K}^{\min\{p,1-p\}}(K) & \text{if } K<1, \\ 1 \text{ if } k\le 1\le K, \\ \mathcal{K}^{\min\{p,1-p\}}(k) & \text{if } 1<k, \end{cases}$$

where $p \in [0, 1]$.

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